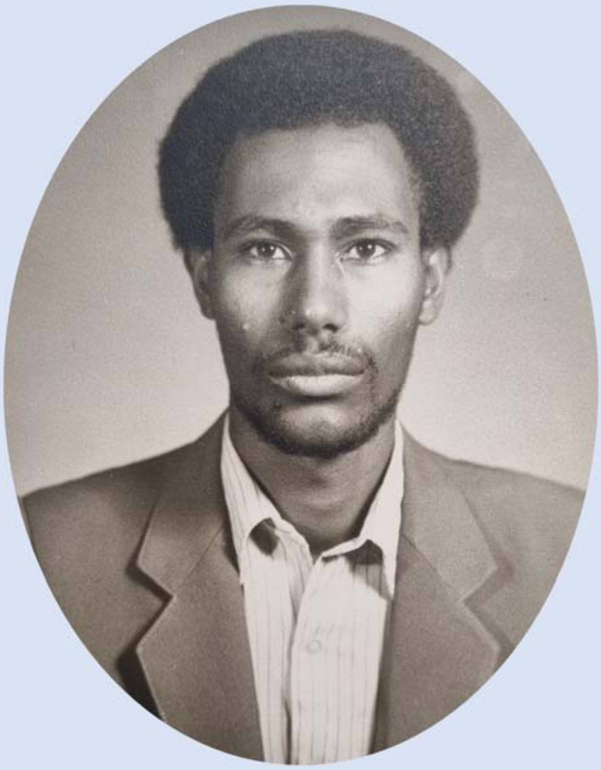

BIRU TSEGAYE'S LECTURE NOTES

Problems and Solutions in Quantum Mechanics

This document is the second part of the release of Biru Tsegaye's handwritten lecture notes which contains several worked problems in quantum mechanics

Anteneh Biru Tsegaye

19.10.2024

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antenehbiru@gmail.com

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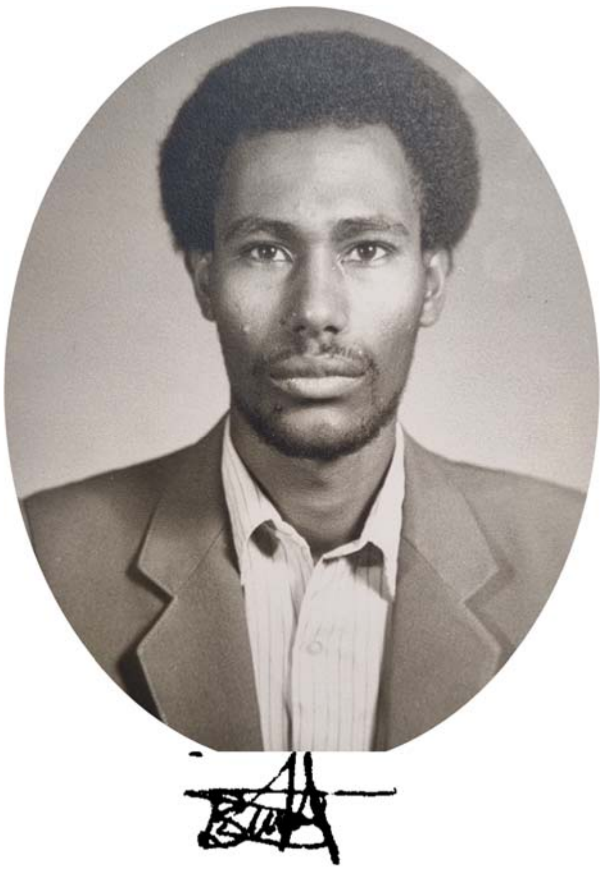
Foreword

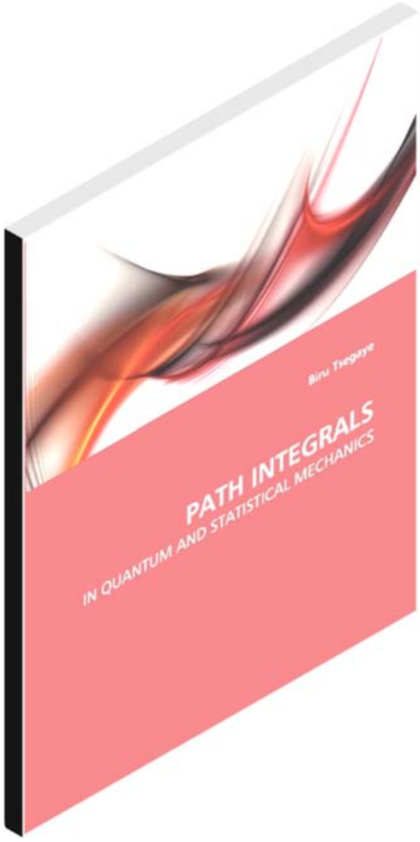
This foreword accompanies the release of the second part of **Biru Tsegaye**'s handwritten lecture notes, a project I feel privileged to share as his son. The previous release (Part 1) contained his lecture notes on Newtonian Dynamics, Electromagnetism and Thermodynamics including several examples, assignments and exam questions.



The current release contains worked problems in quantum mechanics.

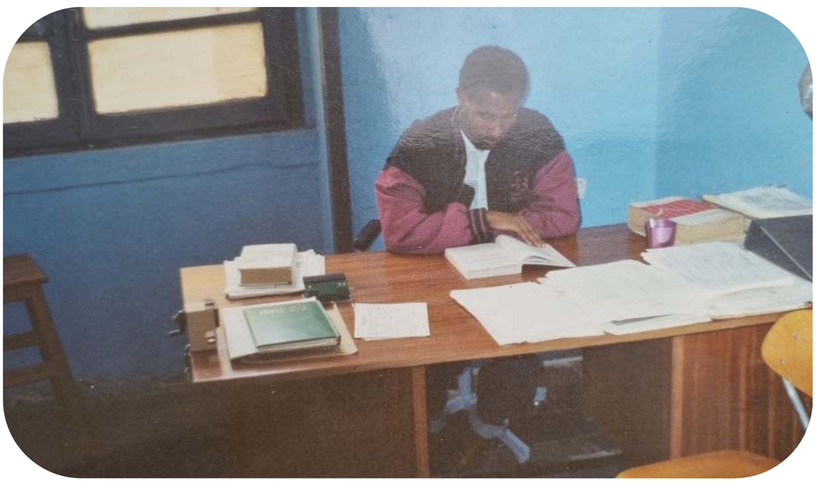
As mentioned in the foreword of the previous release, Biru wrote a thesis on "Path Integrals in Quantum and Statistical Mechanics". The board of examiners of his thesis comprised of professor N. Kumar (ecternal examiner), Dr. S.C. Chhajlany (supervisor), Dr. J. Jelen and Dr. V.N. Mal'nev (a Ukrenian expatriot.)

His thesis explores the path integral formalism of Quantum Mechanics from both Lagrangian and Hamiltonian point of view. In his work, he derived several exact propagators and applied the path integral approach to various problems in statistical mechanics through the formulation of the statistical density matrix. Additionally, Biru explored the application of the path integral method to second quantized Hamiltonians using coherent states. He also considered the variational technique, providing illustrative examples to demonstrate its effectiveness. Furthermore, Biru reviewed some of the then recent achievements of the path integral technique, offering a comprehensive overview of its applications and advancements.



In 2011, I published his thesis as a tribute to his dedication and personal integrity, and I must say it contains some of the longest equations I've ever written. Despite being well-trained in matrices and tensors, Dirac's Bra-Ket notation never felt intuitive to me, and I find myself having to double-check each detail whenever I revisit the thesis and even then, have not yet fully grasped the content.

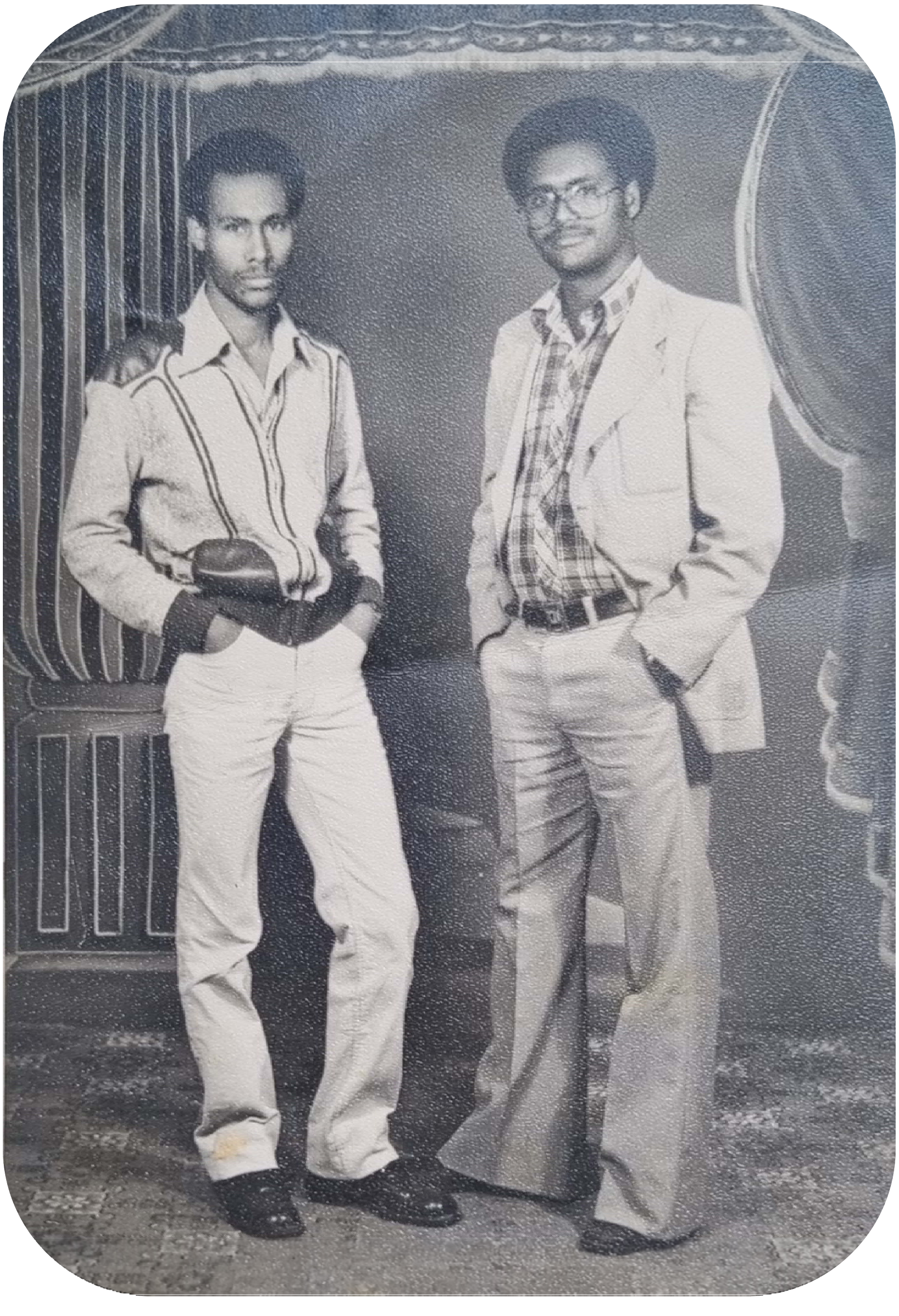
In the epilogue to his thesis, Biru noted that one area where no systematic work had yet been done was in the **quantum mechanics of compound potentials**, a domain where Schrödinger's theory already provides solutions easily. Biru expressed his intention to explore this area further in the future. While I am not sure whether this area holds significant potential for new discoveries, it certainly sounds interesting, and I would encourage physicists to investigate it further if it hasn't been thoroughly explored since.



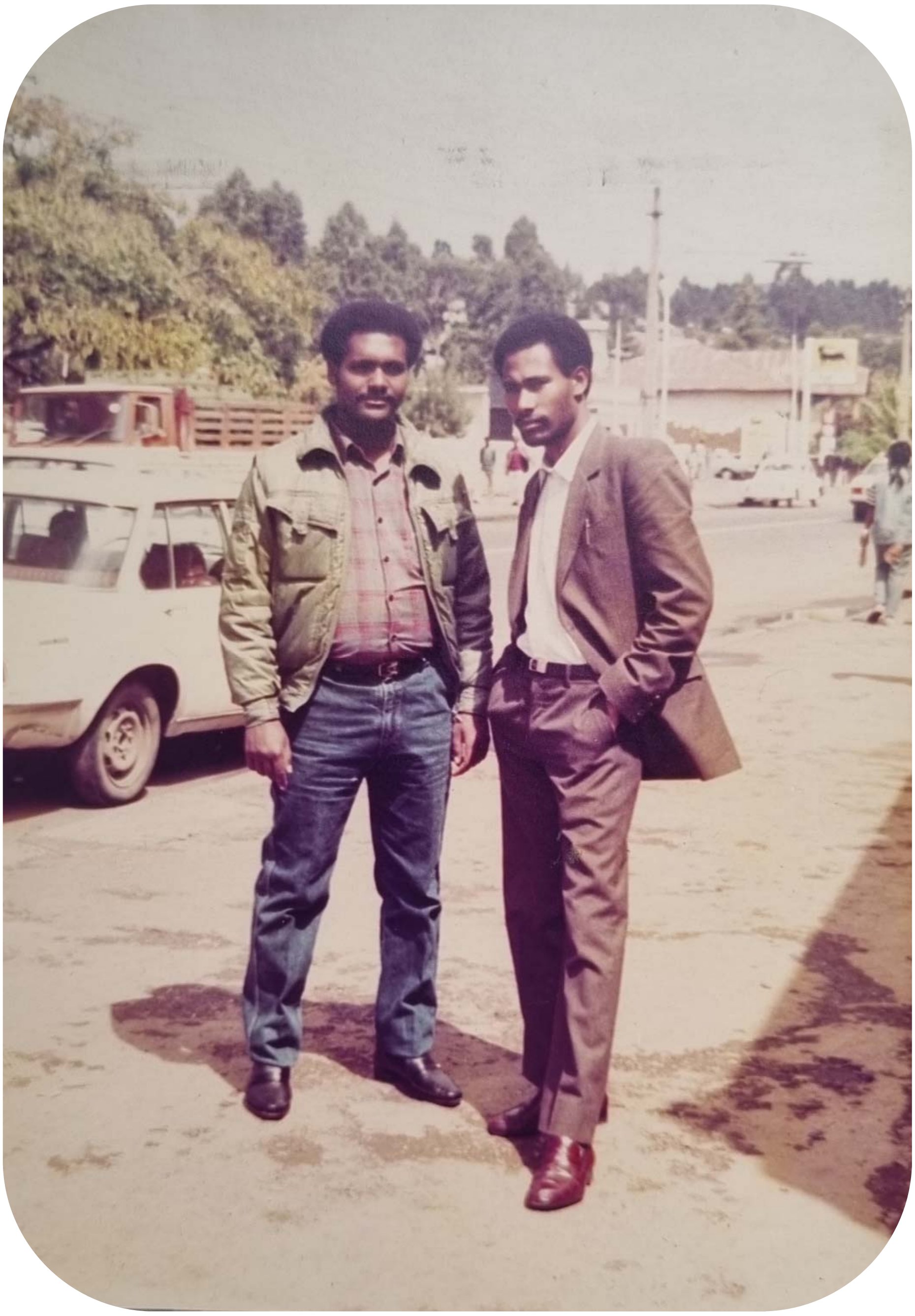
On a personal note, I was conceived when my parents were high school sweethearts, and they had no plans to settle down at the time. Biru suggested to my mother, Engidaye Belay, that she go to his mother's home to give birth, a suggestion for which I am deeply grateful. I am grateful to Biru for demonstrating such integrity and responsibility at a young age, and to Engidaye for accepting his proposal. I was raised with the loving care of my grandmother, Tiruwork Asfaw. Engidaye stayed with me for the first 40 days of my life, but then had to leave to continue her education—a difficult choice, one that I have grown to deeply appreciate as I've matured.

Anteneh Biru Tsegaye

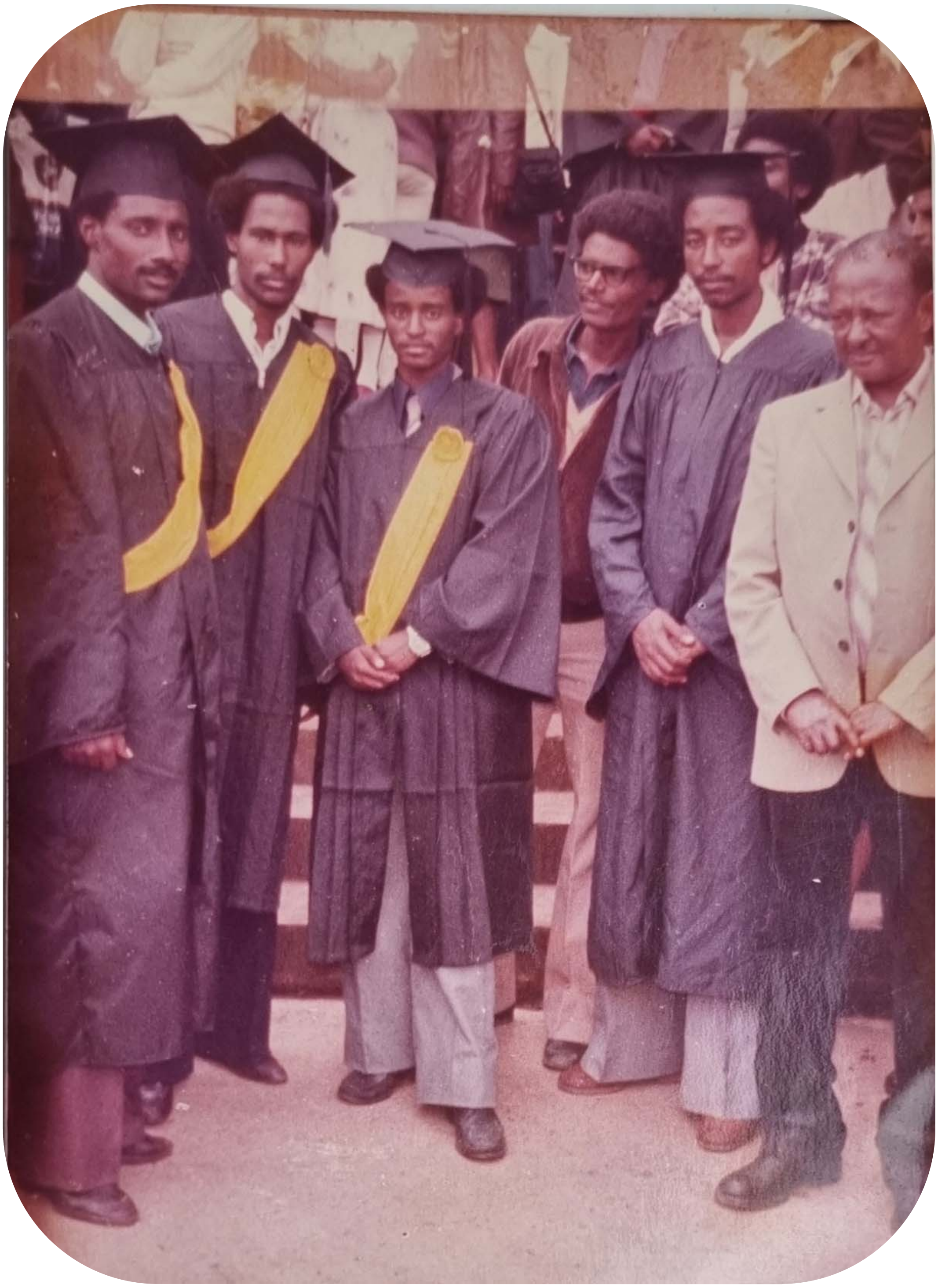
Some selected photos



Studio photo: Biru Tsegaye with his older brother, Telele Tsegaye

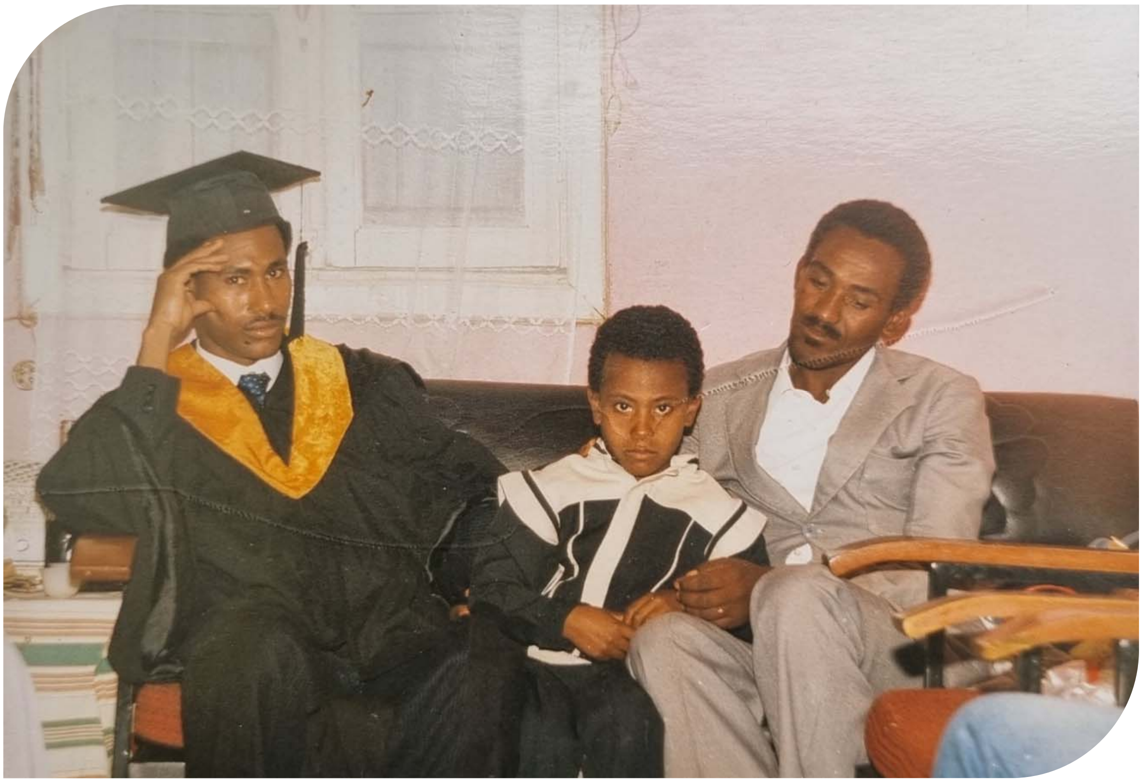


Biru Tsegaye, with his brother, Telele Tsegaye in Gondar



Biru Tsegaye, second from the left, celebrating his graduation with bachelor's degree in physics 1976 EC

Biru Tsegaye with his family, celebrating his graduation with master's degree in physics



Biru Tsegaye with his son, Anteneh Biru, and his uncle Dagne Reta

Biru Tsegaye, fifth from the right, farewell after research visit at Homerton

Contents

Some selected problems of Quantum Mechanics with solutions

1. Let B & C be two anti-commuting operators, i.e.,

$$[B, C]_+ = BC + CB = 0$$

Let ψ be an eigenstate of both B and C. What can be said about the corresponding eigenvalues? For B = baryon no. and C = charge conjugation no., the relations $[B, C]_+ = 0$ and $C^2 = 1$ hold. What does your result imply in these case?

Solution

$[B, C]_+ |\psi\rangle$

$= (BC + CB)|\psi\rangle$

$= (cb + bc)|\psi\rangle$

$= 2bc|\psi\rangle$

$= 0$

$\Rightarrow bc = 0.$

$C^2|\psi\rangle$

$= c^2|\psi\rangle$

$= |\psi\rangle$

$\Rightarrow c^2 = 1$

or $c = \pm 1$

and hence $b = 0$. Thus a state can be an eigenstate of charge conjugation only if the baryon no. is zero.

2. Three matrices M_x, M_y, M_z each with 256 rows and columns, are known to obey the commutation rules

$$[M_x, M_y] = iM_z$$

(with cyclic permutation of x, y, z). The eigenvalues of one matrix, say M_x, are ± 2, each once; $\pm 3/2$ each 8 times; ± 1 each 28 times; $\pm 1/2$ each 56 times; 0, 70 times. State the 256 eigenvalues of the matrix $M^2 = M_x^2 + M_y^2 + M_z^2$.

Solution

The matrix $\vec{M} = (M_x, M_y, M_z)$ represents an angular mom. matrix because of the commutations rules. It is evident that the matrices do not represent irreducible representations; rather they represent several irreducible representations.

If a state with spin j is represented, M_x has $(2j+1)$ eigenvalues ranging from $-j$ to $+j$, each appearing once. Hence there are no states of spin greater than 2, only one of spin 2, and eight of spin 3/2. One of the 28 entries of $+1$ is accounted for by the j = 2 state; there are, therefore 27 representations of j = 1. Similarly there are (56 − 8) = 48 representations of j = 1/2 and 42 of j = 0.

Each eigenvalue of M^2 corresponding to spin j has value $j(j+1)$; to each representation there are $(2j+1)$ such values. Then the following table is constructed.

j	j(j+1)	(2j+1)	No. of reps.	No. of entries in M^2
2	6	5	1	5
3/2	15/4	4	8	32
1	2	3	27	81
1/2	3/4	2	48	96
0	0	1	42	42

3. Prove the Thomas-Reiche-Kuhn sum rule

$$\sum_n \frac{2m}{\hbar^2} |x_{n0}|^2 (E_n - E_0) = 1$$

The sum is over the complete set eigenstates $|n\rangle$ of energy E_n of a particle of mass m, which moves in a potential; $|0\rangle$ represents a bound state.

Solution

From operator identities

$$\dot{X} = \frac{P_x}{m} = -\frac{i}{\hbar}[X,H]$$

$$[X,P_x] = i\hbar$$

$$\Rightarrow -[X,[X,H]]$$

$$= +\left[X,\frac{\hbar P_x}{mi}\right]$$

$$= \frac{\hbar}{mi}[X,P_x]$$

$$= \frac{\hbar^2}{m} \quad \cdots (1)$$

Also expanding the commutator and taking the average value in the state $|0\rangle$

$$-\langle 0|[X,[X,H]]|0\rangle = \frac{\hbar^2}{m}$$

$$= -\langle 0|(X^2H - XHX - XHX + HX^2)|0\rangle$$

$$= \langle 0|(2XHX - X^2H - HX^2)|0\rangle$$

$$= 2\langle 0|XHX|0\rangle - \langle 0|X^2H|0\rangle - \langle 0|HX^2|0\rangle$$

Now, $\langle 0|XHX|0\rangle$

$$= \sum_n \langle 0|XH|n\rangle\langle n|X|0\rangle$$

$$= \sum_n \langle 0|X|n\rangle\langle n|X|0\rangle E_n$$

$$= \sum_n X_{on} X_{no} E_n$$

$$= \sum_n X_{no}^* X_{no} E_n$$

$$= \sum_n |X_{no}|^2 E_n$$

$$\langle 0|HX^2|0\rangle$$

$$= \langle 0|X^2H|0\rangle$$

$$= \sum_n \langle 0|X|n\rangle\langle n|XH|0\rangle$$

$$= \sum_n \langle 0|X|n\rangle\langle n|X|0\rangle E_0$$

$$= \sum_n |X_{no}|^2 E_0$$

i.e., $-\langle 0|[X,[X,H]]|0\rangle$

$$= 2\sum_n |X_{no}|^2 E_n - 2\sum_n |X_{no}|^2 E_0$$

$$= \frac{\hbar^2}{m}$$

$$\Rightarrow 2\sum_n |X_{no}|^2 (E_n - E_0) = \frac{\hbar^2}{m}$$

$$\sum_n |X_{no}|^2 (E_n - E_0) = \hbar^2/2m$$

$$\Leftrightarrow \sum_n \frac{2m|X_{no}|^2}{\hbar^2}(E_n - E_0) = 1$$

4. Consider the 1-D Schrödinger eq. with

$$V(x) = \begin{cases} \frac{m}{2}\omega^2 x^2 & \text{for } x>0 \\ +\infty & \text{for } x<0 \end{cases}$$

Find the energy eigenvalues.

Solution

In the region $x>0$, ψ obeys the same differential eq as the 1-D harmonic oscillator. However, the only acceptable solutions are those that vanish at the origin. Therefore, the ~~only acceptable~~ ~~energy~~ eigenvalues are those of the ordinary harmonic oscillator belonging to W.F.S of odd parity. Now the parity of the S.H.O. WFS alternates with increasing n, starting with an even-parity ground state. Hence

$$E = ((2n+1)+\tfrac{1}{2})\hbar\omega$$
$$= \frac{(4n+3)\hbar\omega}{2}$$

with $n = 0, 1, 2, \ldots$

5- An electron in free space moves under the influence of a uniform magnetic field B. Find the energy levels. If the orbit is large, show that the magnetic flux through the electron orbit is ~~large~~ quantized. Neglect electron spin. Show also how knowledge of the energy levels found here nonrelativistically may be used to determine the relativistic corrections to the energies.

Solution

Choosing the z-axis along the magnetic field B, the Hamiltonian is

$$H = \frac{1}{2m}\left\{ \left(P_x - \frac{eA_x}{c}\right)^2 + \left(P_y - \frac{eA_y}{c}\right)^2 + P_z^2 \right\}$$

with $A_z = 0$.

We fine new variables

$$Q = \sqrt{\frac{c}{eB}}\left(P_x - \frac{e}{c}A_x\right)$$

$$P = \sqrt{\frac{c}{eB}}\left(P_y - \frac{eA_y}{c}\right)$$

one finds $[Q, P] = i\hbar$; i.e., P & Q are canonically conjugate variables. In terms of P & Q the Hamiltonian becomes

$$H = \frac{eB}{2mc}\left[P^2 + Q^2\right] + \frac{P_z^2}{2m}$$

The term in brackets represents a harmonic oscillator in Q P-space. Motion in the z-direction is not quantized. Thus the energy levels are

$$E = \frac{eB\hbar}{mc}(n+\tfrac{1}{2}) + \frac{P_z^2}{2m}$$

If the orbit is large we may use the semiclassical approach. We assume a closed orbit, and the Bohr-Sommerfeld quantization rule gives (with n = integer)

$$nh = \oint \bar{P}\cdot d\bar{r} = \oint\left(m\bar{v} + \frac{e\bar{A}}{c}\right)\cdot d\bar{r}$$

Now,
$$\oint \bar{A}\cdot d\bar{r} = \int_s \nabla\times\bar{A}\cdot d\bar{s}$$
$$= \int_s \bar{B}\cdot d\bar{s}$$
$$= \Phi, \text{ flux enclosed by the orbit.}$$

$$\oint m\bar{v}\cdot d\bar{r} = \oint m v^2 dt = -\oint m\bar{r}\cdot\frac{d\bar{v}}{dt}dt$$

because of the eq. of motion

$$m\frac{d\bar{v}}{dt} = e\frac{\bar{v}}{c}\times\bar{B}$$

becomes,

$$\oint m\bar{v}\cdot d\bar{r} = -\oint \bar{r}\cdot\left(\frac{e}{c}d\bar{r}\times\bar{B}\right)$$
$$= -\oint \frac{e}{c}(\bar{B}\times\bar{r})\cdot d\bar{r}$$
$$= -\oint \frac{e}{c}\nabla\times(\bar{B}\times\bar{r})\cdot d\bar{s}$$

for a constant field $\bar{B}$,

$$\nabla\times(\bar{B}\times\bar{r}) = 2\bar{B}.$$

Finally,

$$\oint \bar{p}\cdot d\bar{r} = \frac{e}{c}\Phi - \frac{2e}{c}\int \bar{B}\cdot d\bar{s}$$

$$= \frac{e\Phi}{c} - \frac{2e}{c}\Phi$$

$$= -\frac{e}{c}\Phi$$

$$= nh$$

or $\Phi = -n\frac{hc}{e}$ Q.E.D.

which has been predicted by F. London and confirmed through experiments in superconductor with a factor of ½.

Where as in nonrelativistic quantum mechanics the energies are given by

$$\frac{(P_x - \frac{eA_x}{c})^2}{2m} + \frac{(P_y - \frac{eA_y}{c})^2}{2m} + \frac{P_z^2}{2m}$$

$$= E_{NR}$$

They are given in relativistic quantum mechanics by

$$E_R^2 = [(P_x - \frac{eA_x}{c})^2 + (P_y - \frac{eA_y}{c})^2 + P_z^2]c^2$$

$$+ m^2c^4$$

$$= 2mc^2E_{NR} + m^2c^4$$

from which

$$E_R = mc^2\left(1 + \frac{2E_{NR}}{mc^2}\right)^{1/2}$$

6. Show that in a state ψ with a well defined value of L_z ($L_z|\psi\rangle = m|\psi\rangle$) the average value of L_x & L_y are equal to zero.

Solution

Consider the commutators

$$[L_y, L_z] = iL_x$$

$$[L_z, L_x] = iL_y$$

$$\langle\psi|[L_y, L_z]|\psi\rangle = i\langle\psi|L_x|\psi\rangle$$

$$\langle\psi|L_yL_z - L_zL_y|\psi\rangle = i\langle\psi|L_x|\psi\rangle$$

$$\langle\psi|L_yL_z|\psi\rangle - \langle\psi|L_zL_y|\psi\rangle$$

$$= i\langle\psi|L_x|\psi\rangle$$

Since L_z is Hermitian we obtain

$$i\langle\psi|L_x|\psi\rangle$$

$$= m\langle\psi|L_y|\psi\rangle - m\langle\psi|L_y|\psi\rangle$$

$$= 0$$

or $\langle\psi|L_x|\psi\rangle = 0$

Similarly $\langle\psi|L_y|\psi\rangle = 0$

2nd formation:

Relative to an arbitrary state axis z' the operator of orbital angular mom. is given by

$$L_{z'} = L_x\cos(xz')$$

$$+ L_y\cos(yz')$$

$$+ L_z\cos(zz')$$

So now in the state $|\psi\rangle$ for which $L_z|\psi\rangle = m|\psi\rangle$ the average value of the angular mom. $L_{z'}$ about an axis z' which makes an angle θ with the z-axis is equal to

$$\langle\psi|L_{z'}|\psi\rangle = \cos(xz')\langle\psi|L_x|\psi\rangle$$

$$+ \cos(yz')\langle\psi|L_y|\psi\rangle$$

$$+ \cos\theta\langle\psi|L_z|\psi\rangle$$

$$= m\cos\theta.$$

This result can be visualized as follows. The angular mom. vector in the state Y_m is evenly "spread out" over a cone with its axis along the z-axis, its slant height equal to $\sqrt{l(l+1)}$, and its height equal to m. The average value of its projection on the xy plane is equal to zero, and its comp along the z'-axis is, after averaging, equal to $m\cos\theta$.

7. The angular mom. of a particle is equal to j, and its z-comp. has its largest possible value. Determine the probability for diff. values of the angular mom. comp. along a direction which makes an angle θ with the z-axis.

Solution

To find the required probability we use a formal method which consists in considering instead of a particle with angular mom. j a system consisting of 2j particles of spin ½. Since in our prob. the angular mom. comp. of the particle is equal to j, all particles in the equivalent system of 2j particles must have a z-comp. of the spin equal to +½. The probability for a spin +½ (or -½) along the z-axis is for each of the particles equal to $\cos^2\theta/2$ [or $\sin^2\theta/2$]. In order that the value of the z-comp. of the total angular mom. of these particles is equal to m, it is necessary that j+m particles have a z-comp. +½ and the remaining j-m particles a z-comp. -½. The required probability $P(m)$ is obtained by multiplying

$$(\cos^2\tfrac{\theta}{2})^{j+m} \quad (\sin^2\tfrac{\theta}{2})^{j-m}$$

by the no. of ways of distributing 2j particles into two such groups, i.e., by $(2j)!/(j+m)!(j-m)!$. Thus,

$$P(m) = \frac{(2j)!}{(j+m)!(j-m)!}(\cos^2\tfrac{\theta}{2})^{j+m} \cdot (\sin^2\tfrac{\theta}{2})^{j-m}$$

$$\sum_{-j}^{j} P(m) = 1.$$

8. A system consists of two particles, one with angular mom. $l_1 = 1$, and the other with angular mom. $l_2 = l$. The total angular mom. j can take on the values $l+1, l, l-1$. Express the eigen functions of the square and z comp. operators J^2 and J_z in terms of the eigen functions of the square and the z comp. of the angular mom. of the separate particles.

Solution

$$|jm\rangle = \sum_{m_1, m_2} |l_1 m_1 l_2 m_2\rangle\langle l_1 m_1 l_2 m_2|jm\rangle$$

$$m = m_1 + m_2$$

$$\Rightarrow |jm\rangle = \sum_{m_1} |l_1 m_1 l_2\, m-m_1\rangle\langle l_1 m_1 l_2\, m-m_1|jm\rangle$$

$$= |1\,{-1}\,l\,m+1\rangle\langle 1\,{-1}\,l\,m+1|jm\rangle$$

$$+ |1\,0\,l\,m\rangle\langle 1\,0\,l\,m|jm\rangle$$

$$+ |1\,1\,l\,m-1\rangle\langle 1\,1\,l\,m-1|jm\rangle$$

Now we evaluate the CG coeffs for diff. values of j & m_1.

From the table for $j_1 = 1$, j_2 arbitrary we find that $(j = l+1)$

$$\langle 1 1\, l\, m-1 | l+1, m\rangle = \left[\frac{(l+m)(l+m+1)}{2(l+1)(2l+1)}\right]^{1/2}$$

$$\langle 1 0\, l\, m | l+1, m\rangle = \left[\frac{(l+m+1)(l-m+1)}{(2l+1)(l+1)}\right]^{1/2}$$

$$\langle 1,-1\, l\, m+1 | l+1, m\rangle = \left[\frac{(l-m+1)(l-m)}{2(2l+1)(l+1)}\right]^{1/2}$$

$j = l$

$$\langle 1 1\, l\, m-1 | l\, m\rangle = -\left[\frac{(l+m)(l-m+1)}{2l(l+1)}\right]^{1/2}$$

$$\langle 1 0\, l\, m | l\, m\rangle = \left[\frac{m}{l(l+1)}\right]^{1/2}$$

$$\langle 1,-1\, l\, m+1 | l\, m\rangle = \left[\frac{(l+m+1)(l-m)}{2l(l+1)}\right]^{1/2}$$

$j = l-1$.

$$\langle 1 1\, l\, m-1 | l-1, m\rangle = \left[\frac{(l-m+1)(l-m)}{2l(2l+1)}\right]^{1/2}$$

$$\langle 1 0\, l\, m | l-1, m\rangle = -\left[\frac{(l+m)(l-m)}{(2l+1)l}\right]^{1/2}$$

$$\langle 1,-1\, l\, m+1 | l-1, m\rangle = \frac{(l+m+1)(l+m)}{2l(2l+1)}$$

Thus we can write the coupled states in terms of the uncoupled states in the form of matrix product as follows.

$$\begin{bmatrix} |l+1, m\rangle \\ |l\, m\rangle \\ |l-1, m\rangle \end{bmatrix} = \begin{bmatrix} \left[\frac{(l+m+1)(l+m)}{2(2l+1)(l+1)}\right]^{1/2} & \left[\frac{(l+m+1)(l-m+1)}{(2l+1)(l+1)}\right]^{1/2} & \left[\frac{(l-m+1)(l-m)}{2(2l+1)(l+1)}\right]^{1/2} \\ -\left[\frac{(l+m)(l-m+1)}{2l(l+1)}\right]^{1/2} & \frac{m}{[l(l+1)]^{1/2}} & \left[\frac{(l+m+1)(l-m)}{2l(l+1)}\right]^{1/2} \\ \left[\frac{(l-m+1)(l-m)}{2l(2l+1)}\right]^{1/2} & -\left[\frac{(l+m)(l-m)}{l(2l+1)}\right]^{1/2} & \left[\frac{(l+m+1)(l+m)}{2l(2l+1)}\right]^{1/2} \end{bmatrix}$$

$$\times \begin{bmatrix} |1 1\, l\, m-1\rangle \\ |1 0\, l\, m\rangle \\ |1,-1, l\, m+1\rangle \end{bmatrix}$$

Since the matrix is orthogonal its inverse is the same as its transpose and therefore each of the functions $|1 1\, l\, m-1\rangle$, $|1 0\, l\, m\rangle$, $|1 -1\, l\, m+1\rangle$ can be expressed as linear combinations of $|l+1, m\rangle$, $|l\, m\rangle$ and $|l-1\, m\rangle$.

9. Express the operator of rotation over a finite angle φ_0 around the direction n in terms of the angular mom. operator (of a system of N particles).

Solution

The required operator (which we shall denote by $\hat{R}_{\varphi_0}$) must transform any arbitrary function of the coordinates of the system $\psi(\bar{r}_1, \bar{r}_2, \bar{r}_3 \ldots \bar{r}_i, \ldots \bar{r}_N)$ into the same function ψ but of coordinates which are rotated over the given angle.

For a rotation over an infinitesimal angle $d\varphi_0$ around the direction $\bar{n}$ ($\vec{n}$ unit vector) the radius vector of the i^{th} particle $\bar{r}_i$ will be increased by $d\bar{r}_i$ where

$$d\bar{r}_i = d\varphi \; \bar{n} \times \bar{r}_i$$

under a rotation over a finite angle $\int d\varphi = \varphi_0$ around the same axis $\bar{n}$ we find for the finite change in $\bar{r}_i$

$$\int d\bar{r}_i = \int d\varphi \, (\bar{n} \times \bar{r}_i)$$

$$= \bar{n} \times \int \bar{r}_i d\varphi \equiv \delta \bar{r}_i \quad \ldots (1)$$

Let us consider the most general operator $\bar{R}$ which is the operator of arbitrary finite displacements of the particles in the system. (These displacements must satisfy the cond. $\operatorname{div} \delta\bar{r} = 0$.) To determine $\bar{R}$ we have

$$R\chi(\bar{r}_1, \ldots \bar{r}_i, \ldots \bar{r}_N)$$

$$= \psi(\bar{r}_1 + \delta\bar{r}_1, \ldots \bar{r}_i + \delta\bar{r}_i, \ldots \bar{r}_N + \delta\bar{r}_N) \quad \ldots (2)$$

Expanding into a Taylor series

$$\psi(\bar{r}_1 + \delta\bar{r}_1, \ldots \bar{r}_i + \delta\bar{r}_i, \ldots \bar{r}_N + \delta\bar{r}_N)$$

$$= \psi(\bar{r}_1, \ldots \bar{r}_i, \ldots \bar{r}_N)$$

$$+ \sum_{i=1}^{N} \frac{\partial\psi}{\partial\bar{r}_i} \cdot \delta\bar{r}_i + \frac{1}{2!} \sum_{i=1}^{N} \frac{\partial^2\psi}{\partial\bar{r}_i^2} (\delta\bar{r}_i)^2 \ldots$$

$$= \left\{ 1 + \sum_{i=1}^{N} \delta\bar{r}_i \cdot \frac{\partial}{\partial\bar{r}_i} + \frac{1}{2!} \sum_{i=1}^{N} \left(\delta\bar{r}_i \cdot \frac{\partial}{\partial\bar{r}_i}\right)^2 + \ldots \right\} \psi(\bar{r}_1, \ldots \bar{r}_i, \ldots \bar{r}_N) \quad \ldots (3)$$

Comparing (2) & (3) we can see that the expression in the ~~square~~ curly brackets is the operator R.

Since the expression is the power series expansion of the exponential function we can write R in the following symbolic form (we introduce the operators of the mom. of the particles $\bar{p}_i = \hbar/i \, (\partial/\partial\bar{r}_i)$):

$$R = \exp\left[\sum_{i=1}^{N} \left(\delta\bar{r}_i \cdot \frac{\partial}{\partial\bar{r}_i}\right)\right]$$

$$= \exp\left[\frac{i}{\hbar} \sum_{i=1}^{N} \delta\bar{r}_i \cdot \hat{p}\right] \quad \ldots 4$$

In the particular case of a parallel translation of all particles over a distance a we have

$$\delta r_1 = \delta r_2 = \ldots \delta r_i = \ldots = \delta r_N = a$$

and eq. 4 will be of the form

$$R_a = T_a = \exp\left[\tfrac{i}{\hbar} (\bar{a} \cdot \hat{p})\right]$$

where $\hat{p} = \sum_{i=1}^{N} \bar{p}_i$ is the operator of the total mom. of the system of particles.

Going over to the case of a rotation over a finite angle φ_0, if we take $\vec{n}$ along the z-axis of cylindrical coordinates (ρ, φ, z) and substitute $\delta\varphi_i = \varphi_0$, $\delta\rho_i = \delta z_i = 0$, we have

$$\delta\bar{r}_i \cdot \frac{\partial}{\partial\bar{r}_i} = \delta\varphi_i \frac{\partial}{\partial\varphi_i} + \delta\rho_i \frac{\partial}{\partial\rho_i} + \delta z_i \frac{\partial}{\partial z_i}$$

$$= \varphi_0 \, \partial/\partial\varphi_i$$

Substituting this in eq. (4) and noting that $\frac{\hbar}{i} \partial/\partial\varphi_i$ is the operator of the z-comp. of the angular mom. of the i^{th} particle we have

$$R_{\varphi_0} = \exp\left(\varphi_0 \sum_i \frac{\partial}{\partial \varphi_i}\right)$$

$$= \exp\left(\tfrac{i}{\hbar}\varphi_0 \hat{M}_z\right)$$

where $\hat{M}_z = \sum_{i=1}^{N} \frac{\hbar}{i}\frac{\partial}{\partial \varphi_i}$ is the operator of the comp. of the angular mom. of the system of particles about the z-axis.

Finally for a rotation about an arbitrary axis (direction $\bar{n}$, $M_z \to M_n = \bar{n}\cdot\bar{M}$) we have

$$R_{\varphi_0,\bar{n}} = \exp\left(\tfrac{i}{\hbar}\varphi_0(\bar{n}\cdot\bar{M})\right)$$

10. Is it possible for a photon to decay spontaneously into (i) two photons (ii) three photons

Solution

(i) A photon has spin 1 directed along its direction of motion. Then the fact that linear mom. should be conserved implies the resulting photons move in opposite direction. But here the total spin is zero which violates the conservation of angular mom (spin) since the original photon has spin 1. So no. (True if the photon was initially at rest)

(ii) The other hand, if the two move in the same direction in such a way that their total mom. be equal to that of the original photon, angular mom. is not conserved $(1 \neq 2)$.

ii. We can bring about the conservation of energy, mom. and angular mom. as follows.

In one simple way let the two photons move in the original direction of the mother photon and the 3rd one in the opposite direction. We can arrange in such a way that the net mom. of the final system be equal to that of the mother photon. Obviously angular mom. ~~spin~~ is conserved by making the relative motion of the final system attain a zero orbital angular mom. So here yes.

11. Consider two spin-1/2 particles interacting through a magnetic dipole-dipole interaction

$$V = A\,\frac{(\hat{\sigma}_1\cdot\hat{\sigma}_2)r^2 - (\hat{\sigma}_1\cdot\bar{r})(\hat{\sigma}_2\cdot\bar{r})}{r^5}$$

If the two spins are at a fixed distance d apart if at $t=0$ one spin is parallel to $\bar{r}$ and the other one antiparallel to $\bar{r}$, calculate the time after which the parall spin is antiparallel and the antiparallel spin parallel.

Solution

The possible state vectors ~~WFs~~ for the two-particle system are

$|11\rangle = |\tfrac{1}{2}\tfrac{1}{2}\rangle$ $\quad |10\rangle = \frac{1}{\sqrt{2}}\left[|\tfrac{1}{2},-\tfrac{1}{2}\rangle + |-\tfrac{1}{2},\tfrac{1}{2}\rangle\right]$

$|1,-1\rangle = |-\tfrac{1}{2},-\tfrac{1}{2}\rangle$

$|00\rangle = \frac{1}{\sqrt{2}}\left[|\tfrac{1}{2},-\tfrac{1}{2}\rangle - |-\tfrac{1}{2},\tfrac{1}{2}\rangle\right]$

The first three functions correspond to the triplet state and the last one to the singlet state.

The dipole-dipole interaction energies corresponding to these states are

$$\langle V \rangle_{11} = V_{11} = V_{1,-1} = -2A/d^3$$

$$V_{10} = 4A/d^3 ; \quad V_{00} = 0$$

The state at $t=0$ is

$$|\tfrac{1}{2},-\tfrac{1}{2}\rangle = \sqrt{2}\,[\,|10\rangle + |00\rangle\,]$$

and to find the state at any later time we solve Schrödingers eq

$$i\hbar \frac{d\psi}{dt} = H\psi = E\psi$$

$$\rightarrow \psi(t) = C\, e^{-\frac{iEt}{\hbar}}$$

In light of the normalization we take $C=1$, and each state vector evolves in time according to the factor $e^{-\frac{iEt}{\hbar}}$

$$\therefore \; |\psi(t)\rangle = \sqrt{2}\,\{\, |10\rangle\, e^{-\frac{i4A\,t}{\hbar d^3}} + |00\rangle \}$$

and after a time $\hbar d^3/4\pi A$, the state vector becomes

$$|\psi(t)\rangle = -\sqrt{2}\,\{\, |10\rangle - |00\rangle \}$$

i.e., now the parallel spin becomes antiparallel and the antiparallel spin becomes parallel.

12. The W.F. of a particle with coordinates x,y,z is $(x^2-y^2)(x^2+y^2+z^2)$. Calculate the probability of the z-comp. of angular mom. having any particular value.

Solution

Passing over to spherical coordinates we write

$$\psi = (\cos^2\varphi - \sin^2\varphi)\,F(r,\theta)$$

$$= \frac{1}{2}\,[\, e^{2i\varphi} + e^{-2i\varphi}\,]\,F(r,\theta)$$

and it is seen that ψ is a mixture of $m=+2$ & $m=-2$ with equal probabilities.

13. Use the result at the back of this page to determine the energy of a charged particle moving in a constant, uniform magnetic field.

Solution

Let us take the z-axis along the direction of the magnetic field, the intensity of which we shall denote by H.

The velocity comps of the particle satisfy the following commutation rules

$$[\hat{v}_x, \hat{v}_y] = \frac{ie\hbar}{m^2c} H, \quad [\hat{v}_y, \hat{v}_z] = 0$$

$$[\hat{v}_z, \hat{v}_x] = 0$$

The energy operator is equal to

$$\hat{H} = \frac{m\hat{v}_x^2}{2} + \frac{m\hat{v}_y^2}{2} + \frac{m\hat{v}_z^2}{2}$$

$\hat{H}$ (since $\hat{v}_z$ commutes with $\hat{v}_x$ & $\hat{v}_y$) can be written as a sum of two commuting operators:

$$\hat{H}_1 = \frac{m\hat{v}_x^2}{2} + \frac{m\hat{v}_y^2}{2}$$

$$\hat{H}_2 = \frac{m\hat{v}_z^2}{2}$$

and its eigenvalues are equal to the sum of the eigenvalues of $\hat{H}_1$

and $\hat{H}_2$. Let us first find the eigenvalues $\hat{H}_1$.

Define $\hat{v}_x = \alpha \hat{Q}$

$\hat{v}_y = \alpha \vec{p}$

Where $\alpha = \sqrt{e\hbar H/m^2c}$. In the variables $\vec{p}, \hat{Q}$ the commutation relation is of the form

$$\hat{P}\hat{Q} - \hat{Q}\vec{p} = -i$$

and the operator $\hat{H}_1$ is given by

$$\hat{H}_1 = \hbar(\sqrt{eH/2mc})^2(\hat{P}^2 + \hat{Q}^2)$$

From the result at the back of pages 9 & 10, we find the eigen values of $\hat{H}_1$

$$E_{1n} = \hbar \frac{eH}{mc}(n + 1/2)$$

$n = 0, 1, 2, \ldots$

The eigen values of $\hat{H}_2$ form a continuous spectrum. The energy for the motion in a magnetic field is thus given by

$$E_{nv_z} = \hbar \frac{eH}{mc}(n + 1/2) + \frac{mv_z^2}{2}$$

14. Determine the energy spectrum of a charged particle (charge -e) moving in a uniform magnetic and a uniform electric field which are at right angles to one another.

Solution

Let $\vec{H} = H\vec{k}$, $\vec{E} = \mathcal{E}\vec{i}$

and $A_y = xH$, $A_x = 0$, $A_z = 0$.

Now,

$$\hat{H} = \frac{\vec{P}_x^2}{2m} + (\hat{P}_y - \frac{exH}{c})^2 + \frac{\vec{P}_z^2}{2m} - e\mathcal{E}x$$

We introduce the following notation

$$\frac{exH}{c} - P_y - \frac{mc\mathcal{E}}{H} = \hat{\pi}$$

$$\Rightarrow \hat{H} = \frac{\vec{P}_x^2}{2m} + \frac{\hat{\pi}^2}{2m} - \frac{\hat{P}_y c\mathcal{E}}{H} + \frac{\hat{P}_z^2}{2m} - \frac{mc^2\mathcal{E}^2}{2H^2}$$

and $[\hat{P}_x, \hat{\pi}] = -i\hbar \frac{eH}{c}$

Thus we find that the eigenvalues of the operator

$$\hat{H}_1 = \frac{P_x^2 + \hat{\pi}^2}{2m}$$

are the same as the energy levels of an oscillator vibrating with twice the Larmor frequency ($\omega_L = eH/2mc$)

$$E_{1n} = \hbar \frac{eH}{mc}(n + 1/2)$$

Since $\hat{P}_y$ & P_z which occur in the last expression of $\hat{H}$ above commute with $\hat{H}_1$ the operator

$$\hat{H}_2 = \frac{\hat{P}_z^2}{2m} - \frac{\hat{P}_y c\mathcal{E}}{H} - \frac{mc^2\mathcal{E}^2}{2H^2}$$

can be put in diagonal form as $\hat{H}_1$. Now, the energy spectrum of the particle is

$$E_{nP_yP_z} = \hbar \frac{eH}{mc}(n + 1/2) + \frac{P_z^2}{2m} - \frac{P_y c\mathcal{E}}{H} - \frac{mc^2\mathcal{E}^2}{2H^2}$$

Comparing this result with the previous prob. shows that the electric field lifts the degeneracy which occurs for the case of a magnetic field only: the energy levels in an electric field depend on three q. nos.

15. What will be the W.F of the particle in prob. 14?

Solution

It will be found that

$\psi_{n p_y p_z}(x,y,z)$

$= \exp\left[\frac{i}{\hbar}(y p_y + z p_z)\right]$

$\cdot \exp\left[-\frac{eH}{2\hbar c}\left(x - \frac{c p_y}{eH} - \frac{mc^2\varepsilon}{eH^2}\right)^2\right]$

$\cdot H_n\left[\sqrt{\frac{eH}{c\hbar}}\left(x - \frac{c p_y}{eH} - \frac{mc\varepsilon}{eH^2}\right)\right]$

16. Determine the energy levels and wave functions of a charged particle in a uniform magnetic field. Use cylindrical coordinates. Write the vector potential in the form $A_\varphi = \frac{1}{2} H \rho$, $A_\rho = A_z = 0$

Solution

The Schrödinger eq. in cylindrical coordinates ρ, φ, z is of the form

$$-\frac{\hbar^2}{2M}\left(\frac{\partial^2\psi}{\partial z^2} + \frac{\partial^2\psi}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial\psi}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\varphi^2}\right) - \frac{ie\hbar}{2Mc} H \frac{\partial\psi}{\partial\varphi} + \frac{e^2H^2}{8Mc^2}\rho^2 \psi = E\psi$$

Let us write

$$\psi(\rho,\varphi,z) = \frac{1}{\sqrt{2\pi}} R(\rho) \cdot e^{ik_z z} \cdot e^{im\varphi}$$

Define, $\gamma = \frac{eH}{2c\hbar}$, $\beta = \frac{2ME}{\hbar^2} - k_z^2$

In the eq. which determines the radial function $R(\rho)$,

$$R'' + \frac{R'}{\rho} + \left(\beta - \gamma^2\rho^2 - 2\gamma m - \frac{m^2}{\rho^2}\right)R = 0$$

With a new independent variable $\xi = \gamma\rho^2$,

$$\xi R'' + R' + \left(-\frac{\xi}{4} + \lambda - \frac{m^2}{4\xi}\right)R = 0 \quad \cdots (1)$$

Where $\lambda = \beta/4\gamma - \frac{m}{2}$

The required function behaves as $e^{-\xi/2}$ as $\xi \to \infty$ and is for small ξ proportional to $\xi^{|m|/2}$. Then for (1) we take

$$R = e^{-\xi/2}\, \xi^{\frac{|m|}{2}}\, \omega(\xi)$$

i.e.,
$$\xi\omega'' + (1 + |m| - \xi)\omega' + \left(\lambda - \frac{|m|+1}{2}\right)\omega = 0$$

the solution of which is a confluent hypergeometric function,

$$\omega = F\left[-\left(\lambda - \frac{|m|+1}{2}\right), |m|+1, \xi\right]$$

In order that the W.F. remains finite it is necessary that the quantity $\lambda - \frac{1}{2}(|m|+1)$ is equal to a non negative integer k. The energy levels are thus determined by the expression

$$E = \hbar\frac{eH}{Mc}\left(n + \frac{|m|}{2} + \frac{m}{2} + \frac{1}{2}\right) + \frac{\hbar^2 k_z^2}{2M}$$

17. Find the current density comps for a particle in a uniform magnetic field in the state characterized by the q. nos (ℓ, n, k_z) (See preceding prob.).

Solution

Solution

In cylindrical coordinates

$J_\rho = 0$

$J_\varphi = \left(\frac{e\hbar m}{M\rho} - \frac{e^2 H \rho}{2Mc} \right) \cdot |\psi_{nmk_z}|^2$

$J_z = \frac{e\hbar k_z}{M} |\psi_{nmk_z}|^2$

18. Find in cylindrical co-ordinates the energy levels of a charged particle in a uniform magnetic field using the semi-classical approximation.

Solution

The eq. for the radial W.F. R goes over to the form

$$u'' + \left[\frac{2M}{\hbar^2} E - k_z^2 - \frac{1}{\rho^2} \left(m + \frac{eH}{2\hbar c} \rho^2 \right)^2 \right] u = 0$$

μ - mass, m - e. no.

Where $u = \sqrt{\rho}\, R$ and where $n^2 - 1/4$ is change to n^2. (this change is analogous to the change $l(l+1) \to (l+\frac{1}{2})^2$ and can be based on the method for solving the radial eq. of a particle in a central force field) the expression

$$V_{eff} = \frac{\hbar^2}{2M\rho^2} \left(m + \frac{eH}{2\hbar c} \rho^2 \right)^2$$

Can be considered to be the effective potential energy for a 1-D motion.

From the quantization cond.

$$\int_{\rho_1}^{\rho_2} \left[\frac{2M}{\hbar^2} E - k_z^2 - 1/\rho^2 \left(m + \frac{eH}{2\hbar c} \rho^2 \right)^2 \right]^{1/2} d\rho = \pi (n + 1/2)$$

We get the energy spectrum

$$E - \frac{\hbar^2 k_z^2}{2\mu} = \frac{e\hbar H}{\mu c} \left(n + \frac{|m|}{2} + \frac{m}{2} + \frac{1}{2} \right)$$

The energy calculated from the min. of $V_{eff}(\rho)$ is equal to

$$E' = \frac{e\hbar H}{\mu c} (n + 1/2)$$

and is the energy of the radial motion, while the energy

$$E'' = \frac{e\hbar H}{2\mu c} (m + |m|)$$

Corresponds to the energy of rotational motion. The transition to the classical circular orbit is realised provided $E' \ll E''$ or $n \ll \frac{1}{2}(m + |m|)$. This cond. is clearly only satisfied for positive values of m and can thus be written in the form $n \ll m$.

19. Determine the classically accessible of the radial motion of a particle in a magnetic field (see preceding prob.)

Solution

If we put the expression under the square root sign equal to zero we get for $m > 0$,

$$\rho_{1,2} = \sqrt{\frac{2\hbar c}{eH}} \left[(n + m + 1/2)^{1/2} \pm (n + 1/2)^{1/2} \right]$$

20. Show that in a variable uniform magnetic field the wave function of a particle with spin can be written as the product of a space function and a spin function.

Solution

The pauli eq. is of the form

$$i\hbar \frac{\partial}{\partial t} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = H_0 \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} - \mu_0 (\vec{\sigma} \cdot \vec{H}) \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

where $\vec{H}_0 = \frac{1}{2\mu}\left(\hat{p} - (e/c)\bar{A}\right)^2 + eV$, and where μ_0 is the magnetic moment of the particle. We shall write the W.F. in the form

$$\begin{pmatrix}\psi_1 \\ \psi_2\end{pmatrix} = \varphi(x,y,z,t)\begin{pmatrix}\chi_1(t) \\ \chi_2(t)\end{pmatrix}$$

φ is a solution of the eq.

$$i\hbar \frac{\partial \varphi}{\partial t} = \vec{H}_0 \varphi$$

For the spin function $\begin{pmatrix}\chi_1 \\ \chi_2\end{pmatrix}$ we get

$$i\hbar \frac{\partial}{\partial t}\begin{pmatrix}\chi_1 \\ \chi_2\end{pmatrix} = -\mu_0(\vec{\sigma}\cdot\vec{H})\begin{pmatrix}\chi_1 \\ \chi_2\end{pmatrix}$$

21. A particle of mass μ and charge e is confined to move on a circle of radius a in the x-y plane, but is otherwise free. A magnetic field H is applied in the z-direction. Find the ground state energy as a function of H.

Solution

If we use for the vector potential the gauge where $A_\varphi = \frac{\rho H}{2}$, $A_\rho = A_z = 0$ (ρ, φ & z are cylindrical polars; compare prob. 16 d), we get the Hamiltonian

$$-\frac{\hbar^2}{2\mu}\left(\frac{\partial^2\psi}{\partial z^2} + \frac{\partial^2\psi}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\psi}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\varphi^2}\right)$$

$$-\frac{ie\hbar}{2\mu c}H\frac{\partial\psi}{\partial\varphi} + \frac{e^2H^2}{8\mu c^2}\rho^2\psi = E\psi$$

which for a particle which is constrained to move along the circle $\rho = a$ reduces to

$$-\frac{\hbar^2}{2\mu a^2}\frac{d^2\psi}{d\varphi^2} - \frac{ie\hbar}{2\mu c}H\frac{d\psi}{d\varphi} + \frac{e^2H^2}{8\mu c^2}a^2\psi = E\psi$$

with solution

$$\psi = \exp(im\varphi), \quad m = 0, \pm1, \pm2, \ldots$$

The corresponding energy eigenvalues are

$$E_m = \frac{1}{2\mu}\left[\frac{m\hbar}{a} + \frac{eaH}{2c}\right]^2.$$

and we see that the ground state energy as function of H is given by the eq.

$$E_{gr\,st} = \frac{1}{2\mu}\left(\frac{m\hbar}{a} + \frac{eaH}{2c}\right)^2$$

where $-\frac{ea^2H}{2\hbar c} - \frac{1}{2} \le m \le -\frac{ea^2H}{2\hbar c} + \frac{1}{2}$

This means that $E_{gr\,st}$ is a periodic function of H with periodicity $2\hbar c/ea^2$ and with discontinuities in slope at $H = (2m+1)\hbar c/ea^2$

22. A spin-½ particle with magnetic moment $\bar{\mu} = g\mu_B\bar{J}$ ($\bar{J}$ is the total angular mom. in units $\hbar$, μ_B the Bohr magneton) is in a magnetic field H in the z-direction. For times $t \le 0$ the spin is in $+z$ direction.

At $t=0$, the magnetic field is instantaneously rotated through $90°$ so that it points in x-direction.

i. Find the W.F. of the particle for all times $t > 0$.

ii. Find the expectation values of J_x, J_y & J_z for $t > 0$.

iii. If the field is rotated slowly to the x-direction, taking a total time T, the expectation value of J_x is approximately equal to $+\frac{1}{2}$ for all $t > T$. Estimate the shortest time T for which this is a correct description.

Solution

The eq. of motion for the two comp. W.F. is for $t > 0$ ($\bar{J} = \frac{1}{2}\bar{\sigma}$)

$$\tfrac{1}{2}g\mu_B\sigma_x H\psi = i\hbar\dot{\psi}$$

or

$$-i\omega_0 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \dot{a} \\ \dot{b} \end{pmatrix}$$

where $\omega_0 = g\mu_B H / 2\hbar$

$$-i\omega_0 \begin{pmatrix} b \\ a \end{pmatrix} = \begin{pmatrix} \dot{a} \\ \dot{b} \end{pmatrix}$$

$$\Rightarrow \dot{a} = -i\omega_0 b$$
$$\dot{b} = -i\omega_0 a$$

whose solution becomes

$$a = \cos \omega_0 t$$
$$b = -i \sin \omega_0 t$$

ii. $\langle \hat{J}_x \rangle = 0$

$\langle \hat{J}_y \rangle = -\frac{1}{2} \sin 2\omega_0 t$

$\langle \hat{J}_z \rangle = \frac{1}{2} \cos 2\omega_0 t$

iii. $T \gtrsim \omega_0^{-1}$

23. Find the energy levels of a system of two spin-½ particles in an external magnetic field H, described by the Hamiltonian

$$\hat{H} = A[1 - (\vec{\sigma}_1 \cdot \vec{\sigma}_2)] + \tfrac{1}{2} g \mu_B H (\hat{\sigma}_{1z} + \hat{\sigma}_{2z})$$

where A is a const, g an effective Landé factor, μ_B the Bohr magneton, and $\vec{\sigma}_1$ & $\vec{\sigma}_2$ vectors whose comps are the Pauli matrices referring to the two particles.

Solution

The singlet energy eigenvalue is $-3A$. The triplet energy eigenvalues are $A + \frac{1}{2} g \mu_B H M_z$ ($M_z = 0, \pm 1$).

24. Positronium is a bound state of an electron and a positron. The effective Hamiltonian for the system in the 1S state in a magnetic field H can be written in the form

$$\hat{H} = \hat{H}_0 + A(\vec{\sigma}_1 \cdot \vec{\sigma}_2) + \mu_B H (\hat{\sigma}_{1z} - \hat{\sigma}_{2z})$$

where the notation is the same as the previous prob., where 1 & 2 refer resp. to the electron and positron, and where $\hat{H}_0$ contains K.E.s and central force potentials.

i. Find A if when H = 0 1^1S and 1^3S states are separated by $2\,10^5$ Mc/s, with the singlet state the lowest.

ii. Discuss the principles which restrict the 1^1S & 1^3S states to decay primarily by two and three quantum emission resp.

iii. Find the energy eigenvalues and eigenfunctions for non-vanishing values of H.

iv. If the lifetime of the 1^1S state is 10^{-10} sec and of the 1^3S is 10^{-7} sec, estimate the value of H which will cause the lifetime of the 1^3S state to be reduced to 10^{-8} sec.

Solution

i. From the results of the preceding prob. for H = 0 we obtain

$$4A/\hbar = 2\,10^5 \text{ Mc/s} \quad \text{or } A = 5\,10^{-17} \text{ erg}$$

ii. Single-quantum decay

is impossible because linear momentum cannot be conserved.

Two-quantum decay means that the two photons must go off into opposite directions to conserve linear mom. As the angular mom of a photon is ħ along its direction of propagation, the total angular mom. of the two photons must be 0 or 2. Hence, the singlet state can decay by two-photon emission, but for the decay of the triplet state we need at least three photons.

iii. When the magnetic field is zero, the spin eigenfunctions are

$\psi|11\rangle = |\tfrac{1}{2}\tfrac{1}{2}\rangle$

$|10\rangle = \frac{1}{\sqrt{2}}\{|\tfrac{1}{2},-\tfrac{1}{2}\rangle + |-\tfrac{1}{2},\tfrac{1}{2}\rangle\}$

$|1,-1\rangle = |-\tfrac{1}{2},-\tfrac{1}{2}\rangle$

$|00\rangle = \frac{1}{\sqrt{2}}\{|\tfrac{1}{2},-\tfrac{1}{2}\rangle - |-\tfrac{1}{2},\tfrac{1}{2}\rangle\}$

(or w.f.s)

$\psi_{11} = \alpha_1\alpha_2$

$\psi_{10} = \frac{1}{\sqrt{2}}(\alpha_1\beta_2 + \beta_1\alpha_2)$

$\psi_{1,-1} = \beta_1\beta_2$

$\psi_{00} = \frac{1}{\sqrt{2}}(\alpha_1\beta_2 - \beta_1\alpha_2)$

where $\alpha = \binom{1}{0}$, $\beta = \binom{0}{1}$

with spin energies

$E_{0,0} = -3A$, $E_{10} = E_{1,\pm1} = A$

When the field is nonvanishing we must solve the eigenvalue prob.

$$\begin{pmatrix} A-\lambda & 0 & 0 & 0 \\ 0 & A-\lambda & 0 & 2\mu_B H \\ 0 & 0 & A-\lambda & 0 \\ 0 & 2\mu_B H & 0 & -3A-\lambda \end{pmatrix} = 0$$

with eigenvalues $\lambda_{1,2} = \lambda_{1,\pm1} = A$

$\lambda_{3,4} = -A \pm 2[A^2 + \mu_B^2 H^2]^{1/2}$, and the eigenfunctions are $\psi_{1,\pm1}$ and

$\psi_3 = \gamma\psi_{10} + [-2+2\sqrt{1+\gamma^2}]\psi_{00}$

$\psi_4 = [2-2\sqrt{1+\gamma^2}]\psi_{10} + \gamma\psi_{00}$ ...(1)

with $\gamma = 2\mu_B H/A$

iv. The presence of a magnetic field mixes the singlet and triplet states (see iii), thus reducing the triplet lifetime.

To find the lifetime τ for a given value of H, we write

$$1/\tau = P_1/\tau_1 + P_3/\tau_3 \quad \text{---}(2)$$

where P_1 & P_3 are probabilities that the system, originally in the triplet state when $H=0$, is in the singlet state or triplet state. As the three states ψ_{10}, $\psi_{1,\pm1}$ are equally probable when $H=0$ and as for $H \neq 0$, the admixture of the singlet state to ψ_{10} is (see 1)

$[-2+2\sqrt{1+\gamma^2}]/\gamma^2$

we find that

$$P_3 = 1 - \frac{4[\sqrt{1+\gamma^2}-1]^2}{3\gamma^2}$$

$$P_1 = 4[-1+\sqrt{1+\gamma^2}]^2/3\gamma^2$$

As $\tau_1 = 10^{-10}$ sec, $\tau_3 = 10^{-7}$ sec, and $A = 5\cdot10^{-17}$ erg, we find for the value of H which reduces

25. Use the inequality

$$\int |\nabla\psi + Z\psi\nabla r|^2 d\tau \geq 0$$

to find the minimum energy of a one-electron atom and the corresponding W.F. Show that for the ground state of the atom the inequality $2\bar{T} \geq |\bar{V}|$ is satisfied.

Solution

$$|\nabla\psi + Z\psi\nabla r|^2$$

$$= (\nabla\psi^* + Z\psi^*\nabla r)(\nabla\psi + Z\psi\nabla r)$$

$$= |\nabla\psi|^2 + \nabla\psi^* \cdot Z\psi\nabla r + Z\psi^*\nabla r\cdot\nabla\psi + Z^2|\psi|^2(\nabla r\cdot\nabla r)$$

$$= |\nabla\psi|^2 + Z^2|\psi|^2 + Z\nabla r\cdot(\psi\nabla\psi^* + \psi^*\nabla\psi) \quad (\nabla r\cdot\nabla r = 1)$$

$$= |\nabla\psi|^2 + Z^2|\psi|^2 + Z\nabla r\cdot\nabla(\psi^*\psi)$$

$$= |\nabla\psi|^2 + Z^2|\psi|^2 + Z\nabla r\cdot\nabla|\psi|^2$$

$$\Rightarrow \int|\nabla\psi|^2 d\tau + Z^2\int|\psi|^2 d\tau + Z\int(\nabla|\psi|^2\cdot\nabla r)d\tau \geq 0$$

Integrating the last term by parts ($u = \nabla r$, $dv = \nabla|\psi|^2 d\tau$)

$$du = \nabla^2 r = 2/r$$

$$v = \int\nabla|\psi|^2 d\tau = \ldots$$

we obtain

$$\int|\nabla\psi|^2 d\tau + Z^2\int|\psi|^2 d\tau + Z\left[\overset{0}{\cancel{\nabla r|\psi|^2}} - \int\frac{2}{r}|\psi|^2 d\tau\right] \geq 0$$

or $$\frac{1}{2}\int|\nabla\psi|^2 d\tau - \int\frac{Z}{r}|\psi|^2 d\tau \geq -\frac{Z^2}{2}\int|\psi|^2 d\tau$$

The L.H.S is the average value of the Hamiltonian operator (in units, $e=1$, $\hbar=1$, $\mu=1$) $\hat{H} = -\frac{\nabla^2}{2} - \frac{Z}{r}$ for the state ψ. The lower limit of the energy, $-Z^2/2$, is reached for the state with W.F ψ_0, which satisfies the first order differential eq.

$$\nabla\psi_0 + Z\psi_0\nabla r = 0$$

from which it follows that

$$\psi_0 \sim e^{-Zr}$$

26. Find the mom. distribution of the hydrogen atom electron in the 1S, 2S & 2p state.

Solution

We evaluate first of all the W.F. in the mom. representation from the general formula

$$\varphi(\vec{p}) = \frac{1}{2(\pi\hbar)^{3/2}}\iiint e^{-\frac{i}{\hbar}(\vec{p}\cdot\vec{r})}\psi(\vec{r})\,d\tau$$

For the 1S state we find

$$\varphi_{1s}(p) = \frac{1}{\pi}\left(\frac{2a}{\hbar}\right)^{3/2}\frac{1}{[(p^2a^2/\hbar^2)+1]^2}$$

Similarly for the 2S state

$$\varphi_{2s}(p) = \frac{1}{2\pi}\left(\frac{2a}{\hbar}\right)^{3/2}\frac{\frac{p^2a^2}{\hbar^2} - \frac{1}{4}}{\left(\frac{p^2a^2}{\hbar^2} + \frac{1}{4}\right)}$$

For the 2p state we find there are three eigenfunctions ($m_z = -1, 0, +1$)

$$\varphi_{2p}^{(0)}(p) = \frac{-i}{\pi}\left(\frac{a}{\hbar}\right)^{3/2}\frac{p_z a}{\hbar[(p^2a^2/\hbar^2 + 1/4)]^3}$$

$$\varphi_{2p}^{(\pm1)}(p) = \frac{-i}{\sqrt{2}\pi}\left(\frac{a}{\hbar}\right)^{3/2}\frac{(p_x \pm ip_y)a}{\hbar[(p^2a^2/\hbar^2 + 1/4)]^3}$$

from these expressions we can find the normalized mom. distribution

$$\omega(\bar{p}) = |\varphi(\bar{p})|^2$$

27. What is the change in the expression for the magnetic moment of the hydrogen atom if the motion of the nucleus is taken into account.

Solution

$$\bar{\mu} = \frac{e}{2mc}\,\bar{l} \quad \text{(due to orbital mot.)}$$

But if the charge fluctuates

$$\bar{\mu} = \frac{e}{2\mu c}\int \psi^* \bar{l}\, \psi\, d\tau$$

In the case of two particles we introduce new variables; the center of mass coordinates (X, Y, Z) and the relative coordinates (x, y, z). The average value of the magnetic moment expressed in the new variables will be

$$\overline{M}_z = \frac{e}{2c}\int \psi^* \Big[\left(\tfrac{1}{m} + \tfrac{1}{M}\right)\left(X\frac{\partial}{\partial y} - Y\frac{\partial}{\partial x}\right)$$

$$+ \frac{1}{m+M}\left(x\frac{\partial}{\partial Y} - y\frac{\partial}{\partial X}\right)$$

$$+ \frac{M-m}{mM}\left(x\,\partial/\partial y - y\,\partial/\partial x\right)\Big]\psi\, d\tau$$

and similar expressions for $\overline{m}_x$ and $\overline{m}_y$.

In a stationary state, the average values of the coordinates x, y, z and of the momenta $(-i\hbar\,\partial/\partial x)$, $(-i\hbar\,\partial/\partial y)$, $(-i\hbar\,\partial/\partial z)$ are zero. Thus the above expression simplifies to

$$\overline{M}_z = -\frac{e}{2mc}\left(1 - \frac{m}{M}\right)$$

$$\cdot \int \psi^* \left[-i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\right]\psi\, d\tau$$

where m is the mass of the electron and M is the mass of the nucleus.

Note: The separation of the hyperfine structure terms for an s electron in the ground state is

$$\Delta E = 0.00844 \times 2.79\, \frac{Z^3}{n^3}\, cm^{-1}$$

for the hydrogen atom ground state $(Z = n = 1)$

$$\Delta E = 0.0235\, cm^{-1}$$

28. Determine the shift in the term levels of a one electron atom for the case of an intermediate field $(e\hbar/(2\mu c))H \approx (\Delta E_{nl})$.

Solution

The energy of an atom in a magnetic field is of the same order of magnitude as the spin-orbit interaction. The spin orbit interaction operator which is equal to $\varphi(r)\,\hat{l}\hat{s}$ (considering relative motion of the nucleus and the electron $\varphi(r) = \frac{\hbar^2}{2\mu^2 c^2}\frac{1}{r}\frac{dV}{dr}(\hat{L}\cdot\hat{S})$) is of the same operator as the operator $(\hat{L}_z + 2\hat{S}_z)$, and their sum

$$V = \varphi(r)(\hat{l}\hat{s}) + \frac{e\hbar}{2\mu c}H(\hat{L}_z + 2\hat{S}_z)$$

can be considered to be a small perturbation. (L^2, L_z, S^2 & S_z are constants of motion in the unperturbed state. Since L^2, S^2, J^2 and J_z are also constants of motion, we characterize the stationary unperturbed states by the q. nos (n, l, j, m_j). The degree of the degeneracy in the case of a coulomb field is equal to $2n^2$, and in the case arbitrary central field of force $2(2l+1)$. It is unnecessary for us to solve a secular eq. of such a high order. We note that

In the perturbed state the square of the orbital angular mom. and the z-comp. of the total angular mom. are integrals of motion. The w.f. of the perturbed prob. can therefore be constructed from the w.f.s $\psi^{(0)}_{nljm_j}$ corresponding to the same value of n, l, m_j or

$$\psi = c_1 \psi^{(0)}_{nl(l-\frac{1}{2})m_j} + c_2 \psi^{(0)}_{nl(l+\frac{1}{2})m_j}$$

$$\text{or } \psi = c_1 \frac{R^{(0)}_{nl}}{\sqrt{2l+1}} \begin{pmatrix} \sqrt{l+m_j+\frac{1}{2}}\; Y_{lm_j-\frac{1}{2}} \\ \sqrt{l-m_j+\frac{1}{2}}\; Y_{lm_j+\frac{1}{2}} \end{pmatrix}$$

$$+ c_2 \frac{R^{(0)}_{nl}}{\sqrt{2l+1}} \begin{pmatrix} \sqrt{l+m_j+\frac{1}{2}}\; Y_{lm_j-\frac{1}{2}} \\ -\sqrt{l+m_j+\frac{1}{2}}\; Y_{lm_j+\frac{1}{2}} \end{pmatrix}$$

The matrix elements of V operator are

$$V^{nl\,l+\frac{1}{2}m_j}_{nl\,l+\frac{1}{2}m_j} = A\frac{l}{2} + H\mu_0 m_j\left(1+\frac{1}{2l+1}\right)$$

$$V^{nl\,l-\frac{1}{2}m_j}_{nl\,l-\frac{1}{2}m_j} = -A\frac{l+1}{2} + H\mu_0 m_j\left(1-\frac{1}{2l+1}\right)$$

$$V^{nl\,l-\frac{1}{2}m_j}_{nl\,l+\frac{1}{2}m_j} = V^{nl\,l+\frac{1}{2}m_j}_{nl\,l-\frac{1}{2}m_j} = \frac{H\mu_0}{2l+1}\left[(l+\tfrac{1}{2})^2 - m_j^2\right]^{1/2}$$

Where $A = \int_0^\infty R_{nl}(r)\,\varphi(r)\,R_{nl}(r)\,r^2\,dr$

and $\mu_0 = \frac{e\hbar}{2\mu c}$

The energy value E follows from the solution of the secular eq.

$$\begin{vmatrix} E^0_{nl} + A\frac{l}{2} + H\mu_0 m_j\left(1+\frac{1}{2l+1}\right) - E & \frac{H\mu_0}{2l+1}\left[(l+\frac{1}{2})^2 - m_j^2\right]^{1/2} \\ \frac{H\mu_0}{2l+1}\left[(l+\frac{1}{2})^2 - m_j^2\right]^{1/2} & E^0_{nl} - A\frac{l+1}{2} + H\mu_0 m_j\left(1-\frac{1}{2l+1}\right) - E \end{vmatrix} = 0$$

Denoting the energies of the one electron atom, taking spin-orbit interaction into account where E_+ refers to the state $j=l+\frac{1}{2}$ and E_- to $j=l-\frac{1}{2}$.

It will be found that

$$E_+ = E^{(0)}_{nl} + A\,l/2$$

$$E_- = E^{(0)}_{nl} - A\,\frac{l+1}{2}$$

The solution of the secular eq. is

$$E = \frac{1}{2}(E_+ + E_-) + H\mu_0 m_j$$

$$\pm\left[\tfrac{1}{4}(E_+ - E_-)^2 + H\mu_0 \frac{m_j}{2l+1}(E_+ - E_-) + \tfrac{1}{4}H^2\mu_0^2\right]^{1/2}$$

Limiting cases:

a) weak field, $\mu_0 H \ll E_+ - E_-$ we find

$$E = E_+ + H\mu_0 m_j \frac{2(l+1)}{2l+1}$$

$$E = E_- - H\mu_0 m_j \frac{2l}{2l+1}$$

The first corresponds to the energy of the nth level of the state $j=l+\frac{1}{2}$ and the second to $j=l-\frac{1}{2}$.

b) strong field, $\mu_0 H \gg E_+ - E_-$. Here we find

$$E = \tfrac{1}{2}(E_+ + E_-) + H\mu_0 m_j$$

$$\pm \tfrac{1}{2} H\mu_0 \pm \frac{m_j}{2l+1}(E_+ - E_-)$$

Let E_c be the energy of the center of gravity of the level, when there is no field present i.e.,

$$E_c = \frac{E_+(l+1) + E_- l}{2l+1}$$

(statistical weights of states E_+ & E_- have ratio $(l+1)/l$) and let ΔE be difference $E_+ - E_-$. With this notation

$$E = E_c + H\mu_0(m_j \pm \tfrac{1}{2}) \pm \frac{\Delta E}{2l+1}(m_j \mp \tfrac{1}{2})$$

The upper sign corresponds to the state with $m_l = m_j - \frac{1}{2}$ and the lower to the state with $m_l = m_j + \frac{1}{2}$.

29. Find the w.f.s of the electron under the conditions of the preceding prob.

Solution

$c_1 = \sqrt{\tfrac{1}{2}(1+\delta)}$ $c_2 = \sqrt{\tfrac{1}{2}(1-\delta)}$

(for the upper state)

$c_1 = \sqrt{\tfrac{1}{2}(1-\delta)}$ $c_2 = -\sqrt{\tfrac{1}{2}(1+\delta)}$

(for the lower state)

where $\delta = \dfrac{\tfrac{1}{2}\Delta E + \dfrac{m_j}{2l+1} H\mu_0}{\sqrt{\tfrac{1}{4}(\Delta E)^2 + \dfrac{m_j}{2l+1}\Delta E\, H\mu_0 + \tfrac{1}{4}H^2\mu_0^2}}$

Limiting cases

a) A vanishingly small magnetic field ($\Delta E \gg H\mu_0$) c $\delta \approx 1$.

$c_1 = 1,\ c_2 = 0$ — upper level

b) strong field ($\Delta E \ll H\mu_0$).

Here $\delta = m_j/(l+\tfrac{1}{2})$ and

$c_1 = \sqrt{\dfrac{l+m_j+\tfrac{1}{2}}{2l+1}}$ $c_2 = \sqrt{\dfrac{l-m_j+\tfrac{1}{2}}{2l+1}}$

(upper level)

$c_1 = \sqrt{\dfrac{l-m_j+\tfrac{1}{2}}{2l+1}}$ $c_2 = -\sqrt{\dfrac{l+m_j+\tfrac{1}{2}}{2l+1}}$

(lower level)

Substituting in the expressions of the w.f.s (prob. 28) we find

$$\psi = R_{nl}^{(0)} \begin{pmatrix} Y_{l m-\frac{1}{2}}(\theta,\varphi) \\ 0 \end{pmatrix}$$

... upper level

$$\psi = R_{nl}^{(0)} \begin{pmatrix} 0 \\ Y_{l m-\frac{1}{2}}(\theta,\varphi) \end{pmatrix}$$

... lower level

30. Determine the splitting of the hydrogen energy levels in a strong magnetic field $e\hbar/2\mu c\, H \gg |E_{nlj} - E_{nlj'}|$. To apply perturbation theory it is necessary to assume that the energy of the atom in the magnetic field is small compared to the difference in energy between diff. multiplets, (i.e.)

$$\frac{e\hbar H}{2\mu c} < |E_{nlj} - E_{nlj'}|$$

Solution

Since the energy in the magnetic field is considerably larger than the spin orbit energy, we can neglect the latter to a first approximation.

In this case L_z & S_z are constants of motion and the energy splitting follows from the eq.

$$E^{(1)} = \frac{e\hbar}{2\mu c} H(m_l + 2m_s)$$

In the 2nd approximation we must take the spin-orbit interaction into account. The multiplet splitting which is added to the splitting in the magnetic field is equal to the average value of the operator

$$\frac{e^2}{2\mu^2 c^2}\cdot\frac{1}{r^3}(\vec{l}\cdot\vec{s})$$

over the state with given values of m_l and m_s. For a given value of one of the comps of the angular mom. the average value of the other two is equal to zero so that

$$(\vec{l}\cdot\vec{s}) = m_l m_s$$

The energy splitting of the level, when spin-orbit interaction is taken into account, is thus

$$E^{(1)} = \frac{e\hbar}{2\mu c} H(m_l + 2m_s) + \frac{e^2\hbar^2}{2\mu^2 c^2}\overline{r^{-3}}\, m_l m_s$$

It can be found that

$$\frac{e^2\hbar^2}{2\mu^2 c^2}\overline{r^{-3}} = \frac{E_{nl\,l+\frac{1}{2}} - E_{nl\,l-\frac{1}{2}}}{l+\frac{1}{2}} = \frac{\Delta E}{l+\frac{1}{2}}$$

We finally get

$$E^{(1)} = \frac{e\hbar}{2Mc} H(m_l + 2m_s) + \frac{\Delta E}{l+\frac{1}{2}} m_l m_s$$

N.B. The magnetic moment of a hydrogen atom in a weak field is

$$M_0 = \frac{e\hbar}{2Mc} \frac{(j+\frac{1}{2})^2}{j(j+1)} m_j \quad \text{(check)}$$

31. Evaluate the shift of the $n=2$ hydrogen atom term in an electric field of intermediate intensity (Stark effect and fine structure splitting of the same order of magnitude).

Solution

In the case under consideration the spin-orbit interaction V_1, the relativistic correction due to the change in mass V_2 and the energy of the electron in the external homogeneous electric field $V_3 = -e\mathcal{E}z$ are all of the same order of magnitude. We shall therefore consider their sum as a small perturbation of the original system. Now the states with well defined values of l, l_z, S & S_z are considered and the matrix elements of V_1, V_2 & V_3 in atomic units are

$$(V_1)^{l'm_l'}_{lm_l} = \begin{cases} \frac{1}{2}\alpha^2 \frac{m_l(m-m_l)}{n^3 l(l+\frac{1}{2})(l+1)} \delta_{ll'} & \text{for } m_l = m_{l'} \\ \frac{1}{2}\alpha^2 \frac{[(l+\frac{1}{2})^2 - m^2]^{\frac{1}{2}}}{n^3 l(l+\frac{1}{2})(l+1)} \delta_{ll'} & \text{for } m_l' = m + \frac{1}{2},\ m_l = m - \frac{1}{2} \text{ or the other way round} \\ 0 & \text{otherwise} \end{cases}$$

$$(V_2)^{l'm_l'}_{lm_l} = -\frac{1}{2}\frac{\alpha^2}{n^3}\left(\frac{1}{l+\frac{1}{2}} - \frac{3}{4n}\right)\delta_{ll'}\delta_{m_l m_l'}$$

$$(V_3)^{l'm_l'}_{lm_l} = -\frac{3n}{2}\left[\frac{(n^2-l^2)(l^2-m_l^2)}{4l^2-1}\right]^{\frac{1}{2}} e\mathcal{E}\,\delta_{l',l-1}\delta_{m_l m_l'}.$$

If $n=2$, the energy of the state with q. nos $l=1$, $m = m_l + m_s = \pm\frac{3}{2}$ $(j=\frac{3}{2})$ will not change in the electric field. The shift of this level due to V_1 and V_2 is equal to $\alpha^2/128$ at. un.

The splitting of the level with q. nos $n=2$, $m=\pm\frac{1}{2}$, can be found from the solution of the secular eq.

$$\begin{vmatrix} -\frac{11}{4}\delta - E^{(1)} & \sqrt{2}\delta & 0 \\ \sqrt{2}\delta & -\frac{7}{4}\delta - E^{(1)} & -3e\mathcal{E} \\ 0 & -3e\mathcal{E} & -\frac{15}{4}\delta - E^{(1)} \end{vmatrix} = 0$$

where $3\delta = \alpha^2/32$ at un is the fine structure splitting of the level $n=2$ when there is no external field. If we introduce in that eq. ϵ which is connected with $E^{(1)}$ by the relation

$$E^{(1)} = \epsilon - \frac{11}{4}\delta$$

[$-\frac{11}{4}\delta$ is the energy of the center of gravity of the three energy levels: $(E_1^{(1)} + E_2^{(1)} + E_3^{(1)})/3 = -\frac{11}{4}\delta$], we get

$$\begin{vmatrix} -\epsilon & \sqrt{2}\delta & 0 \\ \sqrt{2}\delta & \delta-\epsilon & -3e\mathcal{E} \\ 0 & -3e\mathcal{E} & -\delta-\epsilon \end{vmatrix} = 0$$

$$\epsilon^3 - \epsilon[3\delta^2 + 9e^2\mathcal{E}^2] - 2\delta^3 = 0$$

We shall solve this eq. both for weak fields ($e\mathcal{E} \ll \delta$) and for strong fields ($e\mathcal{E} \gg \delta$). In the first case we find

$$\epsilon_1 = -\delta - \sqrt{3}\, e\mathcal{E} - \frac{e^2\mathcal{E}^2}{\delta},$$

$$\epsilon_2 = -\delta + \sqrt{3}\, e\mathcal{E} - \frac{e^2\mathcal{E}^2}{\delta}$$

$$\epsilon_3 = 2\delta + 2\frac{e^2\mathcal{E}^2}{\delta}$$

In the second case the result is

$$\epsilon_1 = -3e\mathcal{E} - \frac{\delta^2}{2e\mathcal{E}} + \frac{1}{9}\frac{\delta^3}{e^2\mathcal{E}^2}$$

$$\epsilon_2 = -\frac{2}{9}\frac{\delta^3}{e^2\mathcal{E}^2}$$

$$\epsilon_3 = 3e\mathcal{E} + \frac{\delta^3}{2e\mathcal{E}} + \frac{1}{9}\frac{\delta^3}{e^2\mathcal{E}^2}$$

32. Evaluate the average value of the n^{th} power of the radius r for the ground state of the hydrogen atom.

Solution

For the ground state the normalised w.F. is

$$\psi_0(r) = \frac{1}{\sqrt{\pi a_B^3}} e^{-r/a_B} \left(a_B = \frac{\hbar^2}{Me^2}\right)$$

$$\Rightarrow \overline{r_n} = \int r^n \psi_0^2(r)\, d\tau$$

$$= \frac{4}{a_B^3} \int_0^\infty r^n e^{-2r/a_B} dr$$

$$= \frac{4}{a_B^3} \left(\frac{a_B}{2}\right)^{n+3} \int_0^\infty t^{n+2} e^{-t} dt$$

where $t = 2r/a_B$ and

$$\overline{r_n} = \frac{\Gamma(n+3)}{2^{n+1}} a_B \quad , n > -3$$

For $n=1$ & $n=2$ resp.,

$$\overline{r} = \frac{3}{2} a_B, \quad \overline{r^2} = 3 a_B^2$$

33. Show that the average value of the dipole moment of a system charged particles which is in a state of well defined parity is equal to zero.

Solution

The average value of the dipole moment of a system of N particles is

$$\langle \bar{\mu} \rangle = \int \cdots \int \psi^*(r_1, \ldots, r_N) \left(\sum_{i=1}^N e_i \bar{r}_i\right) \psi(r_1, \ldots, r_N)\, d\tau_1 \ldots d\tau_N$$

where e_i is the charge of the i^{th} particle, $\sum_{i=1}^N e_i \bar{r}_i = \hat{\mu}$, the dipole moment operator, which in the coordinate representation is simply equivalent to multiplying by $\bar{\mu} = \sum e_i \bar{r}_i$ and the integration is over the configuration space of all particles, while the dipole moment is odd in those coordinates; i.e., the complete expression under the integral sign is thus odd.

Since the integration goes to infinity for each of the coordinates i.e., is over a symmetrical domain, it is clear that $\langle \bar{\mu} \rangle = 0$.

On the other hand, in spherical basis $\bar{\mu}$ has got three projection comps $M = -1, 0, 1 \Leftrightarrow j = l = 1$. It is thus seen $\bar{\mu}$ is changeable under the inversion of the coordinate i.e., has odd parity. Thus it cannot connect states of the same parity or $\langle \bar{\mu} \rangle = 0$ in a state of definite parity.

34. Show that the sum total of the changes produced by an arbitrary magnetic field in the energies of all states of a given value of M_j is equal to

$$\frac{e\hbar}{2Mc} H M_j \sum \left[1 + \frac{j(j+1) - l(l+1) + s(s+1)}{2j(j+1)}\right]$$

The summation is over j, satisfying the conditions

$$|l+s| \le j \le |l-s|, \; j \ge M_j$$

Solution

From the projection theorem for a 1^{st} rank tensor we have

$$\langle jm' | V_\mu | jm \rangle = \frac{\langle jm' | J_\mu (\bar{J} \cdot \bar{V}) | jm \rangle}{j(j+1)}$$

Suppose then an atom is kept in a magnetic field $\bar{\mathcal{H}} = \mathcal{H} \bar{k}$, the interaction energy will be

$$V = -\bar{\mu} \cdot \bar{\mathcal{H}} = -\mu_z \mathcal{H}$$

Now for a single state of a certain j (fixed values).

$$\langle V \rangle = -\mathcal{H} \langle \mu_z \rangle = -\mathcal{H} \langle jm' | \mu_z | jm \rangle$$

$$= -\mathcal{H} \frac{\langle jm' | J_z (\bar{J} \cdot \bar{\mu}) | jm \rangle}{j(j+1)}$$

But, $\bar{\mu} = \mu_0 (\bar{L} + 2\bar{S})$
$\bar{J} = \bar{L} + S$

$$\Rightarrow \bar{J}\cdot\bar{\mu} = \mu_0 (\bar{L}+\bar{S})\cdot(\bar{L}+2\bar{S})$$

$$= \mu_0 (L^2 + 2S^2 + 3\bar{L}\cdot\bar{S})$$

$$= \mu_0 [L^2 + 2S^2 + \tfrac{3}{2}(J^2 - L^2 - S^2)]$$

$$= \frac{\mu_0}{2} (3J^2 - L^2 + S^2)$$

i.e., $\langle V \rangle$

$$= -\mathcal{H}\mu_0 \frac{\langle jm' | J_z (3J^2 - L^2 + S^2) | jm \rangle}{2j(j+1)}$$

$$= -\mathcal{H}\mu_0\, m\hbar^2 \frac{3j(j+1) - l(l+1) + s(s+1)}{2j(j+1)} \delta_{mm'}$$

$$= -\mathcal{H}\mu_0\, m\hbar^2 \left[1 + \frac{j(j+1) - l(l+1) + s(s+1)}{2j(j+1)}\right] \delta_{mm'}$$

$$= \frac{-e\hbar^2\mathcal{H} m}{2\mu c} \left[1 + \frac{j(j+1) - l(l+1) + s(s+1)}{2j(j+1)}\right] \delta_{mm'}$$

Now for all states with diff. values of but the same m (due to the factor $\delta_{mm'}$) we have for the magnitude of the energy ~~[illegible]~~

$$\frac{e\hbar^2\mathcal{H}}{2\mu c} m \left[1 + \frac{j(j+1) - l(l+1) + s(s+1)}{2j(j+1)}\right]$$

Solution to problems taken from a book "Quantum mechanics and Path Integrals."

Chapter 2

2-1. The classical trajectory $\bar{x}$ corresponds to a least action S_{cl} that it satisfies the Lagrangian eq.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\bar{x}}}\right) - \frac{\partial L}{\partial \bar{x}} = 0$$

for a free particle $L = \frac{m}{2}\dot{\bar{x}}^2$

which using Lagranges eq. gives

$$m\ddot{\bar{x}} = 0$$

$$\ddot{\bar{x}} = 0$$

$$\Rightarrow \dot{\bar{x}} = C_1 \text{ (constant)}$$

$$\bar{x}(t) = C_1 t + C_2, \; C_2 \text{- constant}$$

$$\bar{x}(t_a) = x_a = C_1 t_a + C_2$$

$$\bar{x}(t_b) = x_b = C_1 t_b + C_2$$

$$\Rightarrow C_1 = \frac{x_b - x_a}{t_b - t_a}$$

$$C_2 = \frac{t_b x_a - t_a x_b}{t_b - t_a}$$

i.e., $\bar{x}(t) = \frac{x_b - x_a}{t_b - t_a} t + \frac{t_b x_a - t_a x_b}{t_b - t_a}$

$$\dot{\bar{x}} = \frac{x_b - x_a}{t_b - t_a}$$

$$L = \frac{m}{2}\left(\frac{x_b - x_a}{t_b - t_a}\right)^2$$

We could have guessed this result since for a free particle the classical trajectory is a straight line.

$$S_{cl} = \int_{t_a}^{t_b} L(\bar{x}, \dot{\bar{x}}, t)\, dt$$

$$= \frac{m}{2}\left(\frac{x_b - x_a}{t_b - t_a}\right)^2 \int_{t_a}^{t_b} dt$$

$$= \frac{m}{2}\left(\frac{x_b - x_a}{t_b - t_a}\right) \cdot t \Big|_{t_a}^{t_b}$$

or $S_{cl} = \frac{m}{2} \frac{(x_b - x_a)^2}{t_b - t_a}$

2-2. For a harmonic oscillator

$$L = \frac{m}{2}(\dot{\bar{x}}^2 - \omega^2 \bar{x}^2)$$

Corresponding to a classical trajectory.

Applying Lagranges eq.

$$m\ddot{\bar{x}} + m\omega^2 \bar{x} = 0$$

$$\ddot{\bar{x}} + \omega^2 \bar{x} = 0$$

$$\Rightarrow \bar{x}(t) = A \sin\omega t + B\cos\omega t$$

$$\bar{x}(t_a) = x_a = A\sin\omega t_a + B\cos\omega t_a$$

$$\bar{x}(t_b) = x_b = A\sin\omega t_b + B\cos\omega t_b$$

To find A we elliminate B:

$$x_a \cos\omega t_b = A\sin\omega t_a \cos\omega t_b + B\cos\omega t_a \cos\omega t_b$$

$$x_b \cos\omega t_a - x_a \cos\omega t_b = A(\cos\omega t_a \sin\omega t_b - \sin\omega t_a \cos\omega t_b)$$

or $A = \frac{x_b \cos\omega t_a - x_a \cos\omega t_b}{\sin\omega(t_b - t_a)}$

$$B = (x_a - A \sin\omega t_a)/\cos\omega t_a$$

$$= \frac{x_a \sin\omega(t_b - t_a)}{\sin\omega(t_b-t_a)\cdot\cos\omega t_a} - \frac{x_b \cos\omega t_a \sin\omega t_a}{\sin\omega(t_b-t_a)\cos\omega t_a} + \frac{x_a \cos\omega t_b \sin\omega t_a}{\sin\omega(t_b-t_a)\cos\omega t_a}$$

or $$B = \frac{x_a \sin\omega t_b - x_b \sin\omega t_a}{\sin\omega(t_b-t_a)}$$

$$L = \tfrac{1}{2} m \dot{\bar{x}}^2 - \tfrac{1}{2} m\omega^2 \bar{x}^2$$

$$\dot{\bar{x}} = \omega A \cos\omega t - \omega B \sin\omega t$$

i.e.) $$L = \frac{m\omega^2}{2}\{A^2\cos^2\omega t + B^2\sin^2\omega t - 2AB\sin\omega t\cdot\cos\omega t - A^2\sin^2\omega t - B^2\cos^2\omega t - 2AB\sin\omega t\cos\omega t\}$$

$$= \frac{m\omega^2}{2}\left[(A^2-B^2)\cos 2\omega t - 2AB\sin 2\omega t\right]$$

... with A & B given by the above expressions.

$$S_{cl} = \int_{t_a}^{t_b} L(\bar{x},\dot{\bar{x}},t)\,dt$$

$$= \frac{m\omega^2}{2}\left[\frac{(A^2-B^2)}{2\omega}\sin 2\omega t + \frac{AB}{\omega}\cos 2\omega t\right]_{t_a}^{t_b}$$

$$= \frac{m\omega}{2}\left[\frac{(A^2-B^2)}{2}(\sin 2\omega t_b - \sin 2\omega t_a) + AB(\cos 2\omega t_b - \cos 2\omega t_a)\right]$$

$$A^2 = (x_b^2\cos^2\omega t_a + x_a^2\cos^2\omega t_b - 2x_a x_b \sin\omega t_a \cos\omega t_b)\frac{1}{\sin^2\omega(t_b-t_a)}$$

$$B^2 = (x_a^2\sin^2\omega t_b + x_b^2\sin^2\omega t_a - 2x_a x_b\sin\omega t_a\cdot\sin\omega t_b)\frac{1}{\sin^2\omega(t_b-t_a)}$$

$$A\cdot B = (x_a x_b \cos\omega t_a \sin\omega t_b - x_b^2\sin\omega t_a\cos\omega t_a - x_a^2\sin\omega t_b\cos\omega t_a + x_a x_b\sin\omega t_a\cos\omega t_b)\frac{1}{\sin^2\omega(t_a-t_b)}$$

$$A^2-B^2 = (x_a^2\cos 2\omega t_b + x_b^2\cos 2\omega t_a - 2x_a x_b\cos\omega(t_b+t_a))\frac{1}{\sin^2\omega(t_b-t_a)}$$

i.e.) $$S_{cl} = \frac{m\omega}{2\sin^2\omega(t_b-t_a)}$$

$$\Big\{\frac{x_a^2}{2}\cos 2\omega t_b(\sin 2\omega t_b - \sin 2\omega t_a) + \frac{x_b^2}{2}\cos 2\omega t_a(\sin 2\omega t_b - \sin 2\omega t_a) - x_a x_b\cos\omega(t_b+t_a)(\sin 2\omega t_b - \sin 2\omega t_a) + x_a x_b\sin\omega(t_b+t_a)(\cos 2\omega t_b - \cos 2\omega t_a) - \frac{x_a^2}{2}\sin 2\omega t_b(\cos 2\omega t_b - \cos 2\omega t_a) - \frac{x_b^2}{2}\sin 2\omega t_a(\cos 2\omega t_b - \cos 2\omega t_a)\Big\}$$

$$= \frac{m\omega}{2\sin^2\omega(t_b-t_a)}$$

$$\Big\{\tfrac{1}{2}(x_a^2+x_b^2)\sin 2\omega(t_b-t_a) - x_a x_b[\cos\omega(t_b+t_a)\sin 2\omega t_b - \cos\omega(t_b+t_a)\sin 2\omega t_a - \sin\omega(t_b+t_a)\cos 2\omega t_b + \sin\omega(t_b+t_a)\cos 2\omega t_a]\Big\}$$

$$= \frac{m\omega}{2\sin^2\omega(t_b-t_a)}\Big\{\tfrac{1}{2}(x_a^2+x_b^2)\sin 2\omega(t_b-t_a) - 2x_a x_b\sin\omega(t_b-t_a)$$

or $S_{cl} = \frac{m\omega}{2\sin\omega T}\{(x_a^2+x_b^2)\cos\omega T - 2x_a x_b\}$

with $T = t_b - t_a$

2-3. $L = \frac{m}{2}\dot{\bar{x}}^2 - F\bar{x}$

which in Lagrange's eq. gives

$$m\ddot{\bar{x}} + F = 0$$

$$\ddot{\bar{x}} + F/m = 0$$

$$\dot{\bar{x}} + F/m\, t = c_1$$

$$\bar{x} + \frac{F}{2m}t^2 = c_1 t + c_2$$

or $\bar{x}(t) = -\frac{F}{2m}t^2 + c_1 t + c_2$

$$\bar{x}(t_a) = x_a = -\frac{F}{2m}t_a^2 + c_1 t_a + c_2$$

$$\bar{x}(t_b) = x_b = -\frac{F}{2m}t_b^2 + c_1 t_b + c_2$$

Solving for c_1 and c_2

$$c_1 = \frac{x_b - x_a}{t_b - t_a} + \frac{F}{2m}(t_b + t_a)$$

$$c_2 = \frac{t_b x_a - t_a x_b}{t_b - t_a} - \frac{F}{2m}t_b t_a$$

$$\Rightarrow \bar{x}(t) = -\frac{F}{2m}t^2 + \left\{\frac{(x_b - x_a)}{t_b - t_a} + \frac{F}{2m}(t_b + t_a)\right\}t + \frac{t_b x_a - t_a x_b}{t_b - t_a} - \frac{F}{2m}t_b \cdot t_a$$

$$\dot{\bar{x}} = -\frac{F}{m}t + \frac{x_b - x_a}{t_b - t_a} + \frac{F}{2m}(t_b + t_a)$$

$$S_{cl} = \int_{t_a}^{t_b} L(\bar{x}, \dot{\bar{x}}, t)\, dt$$

$$L(\bar{x}, \dot{\bar{x}}, t) = \frac{m}{2}\left(-\frac{F}{m}t + c_1\right)^2 - F\left(-\frac{F}{2m}t^2 + c_1 t + c_2\right)$$

$$= \frac{m}{2}\left(\frac{F^2}{m^2}t^2 - \frac{2F}{m}c_1 t + c_1^2\right) + \frac{F^2}{2m}t^2 - Fc_1 t - Fc_2$$

$$= -2Fc_1 t + \frac{m}{2}c_1^2 - Fc_2$$

i.e., $S_{cl} = \left[-Fc_1 t^2 + \left(\frac{m}{2}c_1^2 - Fc_2\right)t\right]_{t_a}^{t_b}$

$$= -Fc_1(t_b^2 - t_a^2) + \left(\frac{m}{2}c_1^2 - Fc_2\right)(t_b - t_a)$$

$$= -F(x_b - x_a)(t_b + t_a) - \frac{F^2}{2m}(t_b^2 - t_a^2)(t_b + t_a)$$
$$+ \left[\frac{m}{2}\left(\frac{x_b - x_a}{t_b - t_a}\right)^2 + \frac{F^2}{8m}(t_b + t_a)^2 + \frac{F}{2}\frac{x_b - x_a}{t_b - t_a}(t_b + t_a) - \frac{F(t_b x_a - t_a x_b)}{t_b - t_a} + \frac{F^2}{2m}t_b \cdot t_a\right](t_b - t_a)$$

$$= \frac{m}{2}\frac{(x_b - x_a)^2}{t_b - t_a} - F(x_b t_b - x_a t_a) + \frac{F^2}{m}\left(\tfrac{1}{8} - \tfrac{1}{2}\right)(t_b + t_a)^2(t_b - t_a) + \frac{F^2}{2m}(t_b - t_a)t_b t_a + \frac{F}{2}(x_b - x_a)(t_b + t_a)$$

$$= \frac{m}{2}\frac{(x_b - x_a)^2}{t_b - t_a} - \frac{F}{2}(x_b t_b - x_a t_a + x_a t_b - x_b t_a) - \frac{3F^2}{8m}(t_b + t_a)^2(t_b - t_a) + \frac{F^2}{2m}(t_b - t_a)t_b t_a$$

or $S_{cl} = \frac{m}{2}\frac{(x_b - x_a)^2}{t_b - t_a} - \frac{F}{2}(x_b - x_a)(t_b + t_a) - \frac{F^2}{8m}(t_b - t_a)\left[3(t_b^2 + t_a^2) + 2t_b t_a\right]$

2-4. Let us consider the variation of the classical action.

$$S(\bar{x}+\delta x) = \int_{t_a}^{t_b} L(\bar{x}+\delta x, \dot{\bar{x}}+\delta\dot{x}, t)\,dt$$

To the first order in δx

$$S(\bar{x}+\delta x) = \int_{t_a}^{t_b} \left\{ L(\bar{x},\dot{\bar{x}},t) + \frac{\partial L}{\partial \bar{x}}\delta x + \frac{\partial L}{\partial \dot{\bar{x}}}\delta \dot{x} \right\} dt$$

$$= S(\bar{x}) + \int_{t_a}^{t_b} \left(\frac{\partial L}{\partial \bar{x}}\delta x + \frac{\partial L}{\partial \dot{\bar{x}}}\delta\dot{x} \right) dt$$

$$\Rightarrow \delta S_{cl} = \int_{t_a}^{t_b} \left(\frac{\partial L}{\partial \bar{x}}\delta x + \frac{\partial L}{\partial \dot{\bar{x}}}\delta\dot{x} \right) dt$$

$$= \int_{t_a}^{t_b} \left[\frac{\partial L}{\partial \bar{x}}\delta x + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\bar{x}}}\delta x\right) - \delta x \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\bar{x}}}\right) \right] \cdot dt$$

$$= \frac{\partial L}{\partial \dot{\bar{x}}}\delta x \Big|_{t_a}^{t_b} - \int_{t_a}^{t_b} \delta x \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\bar{x}}}\right) - \frac{\partial L}{\partial \bar{x}} \right] dt$$

For the classical trajectory

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\bar{x}}}\right) - \frac{\partial L}{\partial \bar{x}} = 0$$

i.e., $\delta S_{cl} = + \frac{\partial L}{\partial \dot{\bar{x}}}\delta x \Big|_{t_a}^{t_b}$

Suppose the end point (x_b, t_b) is fixed,

$$\Rightarrow \delta S_{cl} = - \frac{\partial L}{\partial \dot{\bar{x}}}\Big|_{x=x_a} \delta x_a$$

$$\frac{\delta S_{cl}}{\delta x_a} = - \frac{\partial L}{\partial \dot{\bar{x}}}\Big|_{x=x_a}$$

or $$\left(\frac{\partial L}{\partial \dot{\bar{x}}}\right)_{x=x_a} = - \frac{\partial S_{cl}}{\partial x_a}$$

which is the momentum at an end point.

2-5. $$S_{cl} = \int_{t_a}^{t_b} L(\bar{x},\dot{\bar{x}},t)\,dt$$

$$\Rightarrow \delta S_{cl} = \int_{t_a}^{t_b} \delta L\, dt$$

$$\delta L = \frac{\partial L}{\partial \bar{x}}\delta x + \frac{\partial L}{\partial \dot{\bar{x}}}\delta\dot{x} + \frac{\partial L}{\partial t}\delta t$$

$$\delta S_{cl} = \int_{t_a}^{t_b} \left\{ \frac{\partial L}{\partial \bar{x}}\delta x + \frac{\partial L}{\partial \dot{\bar{x}}}\delta\dot{x} + \frac{\partial L}{\partial t}\delta t \right\} dt$$

$$= \delta x \frac{\partial L}{\partial \dot{\bar{x}}}\Big|_{t_a}^{t_b} - \int_{t_a}^{t_b} \left\{ \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\bar{x}}}\right) - \frac{\partial L}{\partial \bar{x}} \right]\delta x - \frac{\partial L}{\partial t}\delta t \right\} dt$$

The middle term vanishes since it satisfies the classical trajectory, i.e.,

$$\delta S_{cl} = \delta x \frac{\partial L}{\partial \dot{\bar{x}}}\Big|_{t_a}^{t_b} + \int_{t_a}^{t_b} \frac{\partial L}{\partial t}\delta t\, dt$$

Since x_a is fixed

$$\delta S_{cl} = \delta x_b \left(\frac{\partial L}{\partial \dot{\bar{x}}}\right)_{x_b} + \int_{t_a}^{t_b} dL\, \delta t$$

Chapter 3.

3-1. The probability that a free particle arrives at the point b is proportional to the absolute square of the kernel $K(b,a)$ and for the free particle

$$P(b)\,dx = \frac{m}{2\pi\hbar(t_b-t_a)}\,dx$$

— This is a relative probability since the integral over the complete range of x diverges. This means that the free particle is described by a plane wave which spreads out in space and time.

— Now as stated above we may assume that the amplitude of the free particle $\sim e^{ikx}$ which gives a constant relative probability. Classically this would mean that, at the point of departure, a, the particle is likely to have all momenta.

$$p = \frac{m}{2\pi\hbar}\frac{dx}{t_b-t_a}$$

$$= \frac{m\,dv}{2\pi\hbar}$$

$$= \frac{dp}{2\pi\hbar}$$

Which is the relative probability that the momentum of the particle lies in the range dp.

3-2. The free particle kernel is given by

$$K(b,a) = \left[\frac{2\pi i\hbar(t_b-t_a)}{m}\right]^{-1/2} e^{\frac{im}{2\hbar}\frac{(x_b-x_a)^2}{t_b-t_a}}$$

$$\Rightarrow \frac{\partial K}{\partial t_b} = -\frac{1}{2}\left(\frac{2\pi i\hbar}{m}\right)^{-1/2}(t_b-t_a)^{-3/2}\, e^{\frac{im}{2\hbar}\frac{(x_b-x_a)^2}{t_b-t_a}}$$

$$-\left[\frac{2\pi i\hbar(t_b-t_a)}{m}\right]^{-1/2}\frac{im}{2\hbar}\left(\frac{x_b-x_a}{t_b-t_a}\right)^2 e^{\frac{im}{2\hbar}\frac{(x_b-x_a)^2}{t_b-t_a}}$$

$$= -\left[\frac{2\pi i\hbar(t_b-t_a)}{m}\right]^{-1/2}\left[\frac{1}{2(t_b-t_a)} + \frac{im}{2\hbar}\left(\frac{x_b-x_a}{t_b-t_a}\right)^2\right] e^{\frac{im}{2\hbar}\frac{(x_b-x_a)^2}{t_b-t_a}} \quad \ldots (1)$$

$$\frac{\partial^2 K}{\partial x_b^2} = \frac{\partial}{\partial x_b}\left\{\frac{im}{\hbar}\left(\frac{x_b-x_a}{t_b-t_a}\right)\left[\frac{2\pi i\hbar(t_b-t_a)}{m}\right]^{-1/2} e^{\frac{im}{2\hbar}\frac{(x_b-x_a)^2}{t_b-t_a}}\right\}$$

$$= \frac{im}{\hbar(t_b-t_a)}\left[\frac{2\pi i\hbar(t_b-t_a)}{m}\right]^{-1/2} e^{\frac{im}{2\hbar}\frac{(x_b-x_a)^2}{t_b-t_a}}$$

$$-\frac{m^2}{\hbar^2}\left(\frac{x_b-x_a}{t_b-t_a}\right)^2\left[\frac{2\pi i\hbar(t_b-t_a)}{m}\right]^{-1/2} e^{\frac{im}{2\hbar}\frac{(x_b-x_a)^2}{t_b-t_a}}$$

$$= \left[\frac{2\pi i\hbar(t_b-t_a)}{m}\right]^{-1/2}\left[\frac{im}{\hbar(t_b-t_a)} - \frac{m^2}{\hbar^2}\left(\frac{x_b-x_a}{t_b-t_a}\right)^2\right] e^{\frac{im}{2\hbar}\frac{(x_b-x_a)^2}{t_b-t_a}} \quad \ldots (2)$$

from (1), $-\frac{\hbar}{i}\frac{\partial K}{\partial t_b}$

$$= \left[\frac{2\pi i\hbar(t_b-t_a)}{m}\right]^{-1/2}\left[\frac{m}{2}\left(\frac{x_b-x_a}{t_b-t_a}\right)^2 - \frac{i\hbar}{2(t_b-t_a)}\right] e^{\frac{im}{2\hbar}\frac{(x_b-x_a)^2}{t_b-t_a}}$$

from (2),

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2 K}{\partial x_b^2}\right)$$

$$= \left[\frac{2\pi i\hbar(t_b-t_a)}{m}\right]^{-1/2}$$

$$\cdot \left[\frac{m}{2}\left(\frac{x_b-x_a}{t_b-t_a}\right)^2 - \frac{i\hbar}{2(t_b-t_a)}\right] \cdot e^{\frac{im}{2\hbar}\frac{(x_b-x_a)^2}{t_b-t_a}}$$

i.e., $-\frac{\hbar}{i}\frac{\partial K}{\partial t_b} = -\frac{\hbar^2}{2m}\frac{\partial^2 K}{\partial x_b^2}$

3-3.

3-4. Assume that the point of departure is $(x',0)$ and the point of arrival is (x,t).

$$K(x,t;x',0) = \left(\frac{2\pi i\hbar t}{m}\right)^{-1/2} e^{\frac{im}{2\hbar t}(x-x')^2}$$

$$\psi(x',0) = c\, e^{\frac{ipx'}{\hbar}}$$

$$\Rightarrow \psi(x,t) = \int_{-\infty}^{\infty} \left(\frac{2\pi i\hbar t}{m}\right)^{-1/2} e^{\frac{im}{2\hbar t}(x-x')^2} \cdot c\, e^{\frac{ipx'}{\hbar}}\, dx'$$

$$= c\left(\frac{2\pi i\hbar t}{m}\right)^{-1/2} \int_{-\infty}^{\infty} e^{\frac{im}{2\hbar t}(x-x')^2 + \frac{ipx'}{\hbar}}\, dx'$$

$$\frac{im}{2\hbar t}(x-x')^2 + \frac{ipx'}{\hbar}$$

$$= \frac{im}{2\hbar t}\left(x^2 + x'^2 - 2xx' + \frac{2tpx'}{m}\right)$$

$$= \frac{im}{2\hbar t}\left\{x^2 + x'^2 - 2\left(x - \frac{tp}{m}\right)x'\right\}$$

$$= \frac{im}{2\hbar t}\left\{x^2 + \left[x' - \left(x - \frac{tp}{m}\right)\right]^2 - \left(x - \frac{tp}{m}\right)^2\right\}$$

$$= \frac{im}{2\hbar t}\left\{x^2 + \left[x' - \left(x - \frac{tp}{m}\right)\right]^2 - x^2 - \frac{t^2p^2}{m^2} + \frac{2tpx}{m}\right\}$$

$$= -\frac{m}{2i\hbar t}\left[x' - \left(x - \frac{tp}{m}\right)\right]^2 - \frac{ip^2}{2m\hbar}t + \frac{ipx}{\hbar}$$

i.e., $$\psi(x,t) = c\left(\frac{2\pi i\hbar t}{m}\right)^{-1/2} e^{\frac{ipx}{\hbar}} e^{-\frac{ip^2}{2m\hbar}t} \int_{-\infty}^{\infty} e^{-\frac{m}{2i\hbar t}\left[x' - \left(x - \frac{tp}{m}\right)\right]^2} dx'$$

$$= c\left(\frac{2\pi i\hbar t}{m}\right)^{-1/2} e^{\frac{ipx}{\hbar}} e^{-\frac{ip^2}{2m\hbar}t} \int_{-\infty}^{\infty} e^{-\frac{m}{2i\hbar t}u^2}\, du$$

$$= c\left(\frac{2\pi i\hbar t}{m}\right)^{-1/2} e^{\frac{ipx}{\hbar}} e^{-\frac{ip^2}{2m\hbar}t} \cdot \sqrt{\frac{\pi}{m/2i\hbar t}}$$

or $$\psi(x,t) = c\, e^{\frac{ipx}{\hbar}} e^{-\frac{ip^2}{2m\hbar}t}$$

3-5.

$$\psi(x,t) = \int_{-\infty}^{\infty} K(x,t;x',t')\,\psi(x',t')\,dx'$$

Where x',t' is initial point and x,t is final point.

$$\Rightarrow -\frac{\hbar}{i}\frac{\partial\psi}{\partial t} = -\frac{\hbar}{i}\int_{-\infty}^{\infty}\frac{\partial K}{\partial t}\psi(x',t')\,dx'$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} = -\frac{\hbar^2}{2m}\int_{-\infty}^{\infty}\frac{\partial^2 K}{\partial x^2}\psi(x',t')\,dx'$$

From 3-2, $-\hbar/i\,\frac{\partial K}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 K}{\partial x^2}$

i.e., $-\hbar/i\,\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}$, Q.E.D.

3-6. We have described every possible path between any two points (x_a,t_a) & (x_b,t_b) in terms of the classical path $\bar{x}$ as

$$x(t) = \bar{x}(t) + y(t)$$

where $y(t)$ characterizes the displacement of $x(t)$ from $\bar{x}(t)$ and $y(t_a) = y(t_b) = 0$.

So for a Lagrangian of the form

$$L = a(t)\dot{x}^2 + b(t)x\dot{x} + c(t)x^2 + d(t)x + e(t)\dot{x} + f(t)$$

the action will become

$$S[x(t)] = S[\bar{x}(t) + y(t)] = \int_{t_a}^{t_b} \{a(t)(\dot{\bar{x}}^2 + 2\dot{\bar{x}}\dot{y} + \dot{y}^2)\}dt$$

If all terms independent of y, are collected the resulting integral is $S[\bar{x}(t)] = S_{cl}$. If all terms which contain y & $\dot{y}$ as a linear factor are collected, the resulting integral vanishes. Those terms which are 2nd order in y, $\dot{y}$ can be picked out and we may write

$$S[x(t)] = S_{cl}[b,a] + \int_{t_a}^{t_b} \{a(t)\dot{y}^2 + b(t)y\dot{y} + c(t)y^2\} dt.$$

Since $dx_i = dy_i$, $Dx(t) = Dy(t)$ and the integral over paths becomes

$$K(b,a) = e^{\frac{i}{\hbar}S[b,a]} \cdot \int_0^0 \exp\left[\frac{i}{\hbar}\int_{t_a}^{t_b}\{a(t)\dot{y}^2 + b(t)y\dot{y} + c(t)y^2\}dt\right] Dy(t)$$

Since every path y(t) start from and return to the point y=0, the integral over paths can be a function only of times at the end points. Or,

$$K(b,a) = e^{\frac{i}{\hbar}S_{cl}[b,a]} F(t_a, t_b)$$

Now from prob. 2.1, for a free particle

$$S_{cl} = \frac{1}{2}\frac{m(x_b - x_a)^2}{t_b - t_a}$$

Whence in line with the above arguement

$$K(b,a) = F(t_b, t_a)\, e^{\frac{im}{2\hbar}\frac{(x_b - x_a)^2}{t_b - t_a}} \quad \dots (*)$$

As it is seen $F(t_b, t_a)$ is a gaussian integral and in section 3.1 we have obtained (if we imagine y is substituted in place of x) that

$$K(b,a) \to F(t_b, t_a) = \left(\frac{2\pi i\hbar n\epsilon}{m}\right)^{-1/2} e^{\frac{im}{n2\hbar\epsilon}(y_n - y_0)^2}$$

But $y_n = y(t_b) = 0$, $y_0 = y(t_a) = 0$

i.e., $F(t_b, t_a) = \left[\frac{2\pi i\hbar \epsilon n}{m}\right]^{-1/2}$

Now $t_b = t_a + n\epsilon$ whereby

$$F(t_b, t_a) = \left[\frac{2\pi i\hbar(t_b - t_a)}{m}\right]^{-1/2} \equiv F(t_b - t_a)$$

N.B. $F(t_b, t_a)$ is the integral of a term of the 2nd order y or

$$F(t_b, t_a) = \int_0^0 \exp\left[\frac{i}{\hbar}\int_{t_a}^{t_b}\frac{m}{2}\dot{y}^2 dt\right] Dy(t)$$

or solving (*) $F(t_b, t_a) = K(b,a) f(t_b - t_a) \equiv F(t_b - t_a)$

3-7. from 3-6, $F(t_b, t_a) \equiv F(t_b - t_a)$.

$$K(b,a) = F(t_b - t_a)\, e^{\frac{im}{2\hbar}\frac{(x_b - x_a)^2}{t_b - t_a}} \quad \dots (1)$$

$$K(b,c) = F(t_b - t_c)\, e^{\frac{im}{2\hbar}\frac{(x_b - x_c)^2}{t_b - t_c}}$$

$$K(c,a) = F(t_c - t_a)\, e^{\frac{im}{2\hbar}\frac{(x_c - x_a)^2}{t_c - t_a}}$$

Also, $K(b,a) = \int_{x_c} K(b,c)K(c,a)\,dx_c$

$$= F(t_b - t_c)F(t_c - t_a)\int_{x_c} e^{\frac{im}{2\hbar}\left\{\frac{(x_b - x_c)^2}{t_b - t_c} + \frac{(x_c - x_a)^2}{t_c - t_a}\right\}} dx_c$$

Let $I = \int_{x_c} e^{\frac{im}{2\hbar}\left\{\frac{(x_b-x_c)^2}{t_b-t_c}+\frac{(x_c-x_a)^2}{t_c-t_a}\right\}} dx_c$

$$\frac{(x_b-x_c)^2}{t_b-t_c}+\frac{(x_c-x_a)^2}{t_c-t_a}$$

$$= \frac{x_b^2-2x_bx_c+x_c^2}{t_b-t_c}+\frac{(x_c^2-2x_ax_c+x_a^2)}{t_c-t_a}$$

$$= \frac{\{x_b^2(t_c-t_a)+x_a^2(t_b-t_c)\}}{(t_b-t_c)(t_c-t_a)} + \frac{x_c^2(t_b-t_c+t_c-t_a)}{(t_b-t_c)(t_c-t_a)} - 2\frac{\{x_b(t_c-t_a)+x_a(t_b-t_c)\}x_c}{(t_b-t_c)(t_c-t_a)}$$

$$= \frac{x_b^2(t_c-t_a)+x_a^2(t_b-t_c)}{(t_b-t_c)(t_c-t_a)} + \frac{(t_b-t_a)\,x_c^2}{(t_b-t_c)(t_c-t_a)} - 2\frac{\{x_b(t_c-t_a)+x_a(t_c-t_a)\}x_c}{(t_b-t_c)(t_c-t_a)}$$

$$= \alpha_b x_b^2 + \alpha_a x_a^2 + \gamma x_c^2 + \beta x_c$$

$$\Rightarrow I = e^{\frac{im}{2\hbar}(\alpha_b x_b^2+\alpha_a x_a^2)} \int_{-\infty}^{\infty} e^{\frac{im}{2\hbar}(\gamma x_c^2+\beta x_c)}\, dx_c$$

$$= e^{\frac{im}{2\hbar}(\alpha_b x_b^2+\alpha_a x_a^2)} \cdot \left[\frac{2\pi i\hbar}{m\gamma}\right]^{1/2} e^{-\left(\frac{im}{2\hbar}\beta\right)^2 / 4\left(\frac{im}{2\hbar}\right)\gamma}$$

$$= e^{\frac{im}{2\hbar}(\alpha_b x_b^2+\alpha_a x_a^2)} \cdot \left[\frac{2\pi i\hbar}{m}\frac{(t_b-t_c)(t_c-t_a)}{(t_b-t_a)}\right]^{1/2} e^{(\ )}$$

$$-\left(\frac{im}{2\hbar}\beta\right)^2 \Big/ 4\left(\frac{im}{2\hbar}\gamma\right)$$

$$= -\frac{im}{2\hbar}\left\{\frac{4\{x_b(t_c-t_a)+x_a(t_b-t_c)\}^2}{(t_b-t_c)^2(t_c-t_a)^2 \cdot \frac{4(t_b-t_a)}{(t_b-t_c)(t_c-t_a)}}\right\}$$

$$= -\frac{im}{2\hbar}\left\{\frac{x_b^2(t_c-t_a)^2+x_a^2(t_b-t_c)^2+2x_ax_b(t_b-t_c)(t_c-t_a)}{(t_b-t_c)(t_c-t_a)(t_b-t_a)}\right\}$$

$$= -\frac{im}{2\hbar}\left\{\frac{(t_c-t_a)x_b^2}{(t_b-t_c)(t_b-t_a)}+\frac{(t_b-t_c)x_a^2}{(t_c-t_a)(t_b-t_a)}+\frac{2x_ax_b}{t_b-t_a}\right\}$$

$$\Rightarrow I = e^{\frac{im}{2\hbar}\left\{\frac{x_b^2}{t_b-t_c}+\frac{x_a^2}{t_c-t_a}-\frac{(t_c-t_a)x_b^2}{(t_b-t_c)(t_b-t_a)}-\frac{(t_b-t_c)x_a^2}{(t_c-t_a)(t_b-t_a)}-\frac{2x_ax_b}{t_b-t_a}\right\}} \times \left[\frac{2\pi i\hbar(t_b-t_c)(t_c-t_a)}{m(t_b-t_a)}\right]^{1/2}$$

$$= e^{\frac{im}{2\hbar}\left\{\frac{x_b^2+x_a^2-2x_ax_b}{m(t_b-t_a)}\right\}} \cdot \left[\frac{2\pi i\hbar(t_b-t_c)(t_c-t_a)}{m(t_b-t_a)}\right]$$

$$= e^{\frac{im}{2\hbar}\frac{(x_b-x_a)^2}{t_b-t_a}} \left[\frac{2\pi i\hbar(t_b-t_c)(t_c-t_a)}{m(t_b-t_a)}\right]^{1/2}$$

or $K(b,a) = F(t_b-t_c)F(t_c-t_a) \cdot \left[\frac{2\pi i\hbar(t_b-t_c)(t_c-t_a)}{t_b-t_a}\right]^{1/2} e^{\frac{im}{2\hbar}\frac{(x_b-x_a)^2}{t_b-t_a}}$ ...... (9)

Comparing (1) & (2)

$$F(t_b-t_a)=\left[\frac{2\pi i\hbar\,(t_b-t_c)(t_c-t_a)}{m(t_b-t_a)}\right]^{1/2}$$

$$\cdot F(t_b-t_c)\,F(t_c-t_a)$$

Or if $t=t_b-t_c$, $s=t_c-t_a$,

$$F(t+s)=\left[\frac{2\pi i\hbar\, ts}{m(t+s)}\right]^{1/2} F(t)\cdot F(s) \quad \dots (*)$$

But $F(t)=\sqrt{\dfrac{m}{2\pi i\hbar t}}$

$$F(s)=\sqrt{\frac{m}{2\pi i\hbar s}}$$

$$\Rightarrow F(t+s)=\sqrt{\frac{m}{2\pi i\hbar(t+s)}}$$

If $F(y)=\sqrt{\dfrac{m}{2\pi i\hbar y}}\,f(y)$, then in accordance with (*)

$$f(t+s)=f(t)\,f(s)$$

i.e., $f(y)$ must be of the form e^{at} where a may be complex no., $a=\alpha+i\beta$. However note that the special choice of A in 2-21 implies $f(\epsilon)=1$ to first order in ϵ. In the above case $a=0$.

3-8. $x=\bar{x}+y$

For a harmonic oscillator

$$L=\frac{m}{2}\dot{x}^2-\frac{m\omega^2}{2}x^2$$

$$=\frac{m}{2}(\dot{\bar{x}}+\dot{y})^2-\frac{m\omega^2}{2}(\bar{x}+y)^2$$

$$=\frac{m}{2}\dot{\bar{x}}^2-\frac{m\omega^2}{2}\bar{x}^2$$

$$+m(\dot{\bar{x}}\dot{y}-\omega^2\bar{x}y)$$

$$+\frac{m}{2}\dot{y}^2-\frac{m\omega^2}{2}y^2$$

$$S=\int_{t_a}^{t_b}L\,dt$$

$$=S_{cl}+m\int_{t_a}^{t_b}(\dot{\bar{x}}\dot{y}-\omega^2\bar{x}y)\,dt$$

$$+\int_{t_a}^{t_b}\left(\frac{m}{2}\dot{y}^2-\frac{m\omega^2}{2}y^2\right)dt$$

Middle term:

$$I=m\int_{t_a}^{t_b}(\dot{\bar{x}}\dot{y}-\omega^2\bar{x}y)\,dt$$

$u=\dot{\bar{x}}$, $\dfrac{du}{dt}=\ddot{\bar{x}}$

$dv=\dot{y}\,dt$, $v=y$

$$\Rightarrow I=m\dot{\bar{x}}\,y\Big|_{t_a}^{t_b}-m\int_{t_a}^{t_b}(\ddot{\bar{x}}+\omega^2\bar{x})\,y\,dt$$

$$=0 \qquad\qquad (\ddot{\bar{x}}+\omega^2\bar{x}=0)$$

$$S=S_{cl}+\int_{t_a}^{t_b}\left(\frac{m}{2}\dot{y}^2-\frac{m\omega^2}{2}y^2\right)dt$$

$$\Rightarrow K(b,a)=e^{\frac{i S_{cl}}{\hbar}}\int_0^0\left(\exp\left\{\frac{i}{\hbar}\int_{t_a}^{t_b}\left(\frac{m}{2}\dot{y}^2-\frac{m\omega^2}{2}y^2\right)dt\right\}\right)\mathcal{D}x(t)$$

$$=e^{\frac{iS_{cl}}{\hbar}}F(t_b,t_a)$$

From prob. 2-2,

$$S_{cl}=\frac{m\omega}{2\sin\omega T}\left\{(x_a^2+x_b^2)\cos\omega T-2x_ax_b\right\}$$

where $T=t_b-t_a$,

Also, $F(t_b,t_a)=\left(\dfrac{m\omega}{2\pi i\hbar\sin\omega T}\right)^{1/2}$

(see at the back.)

$$\therefore K(b,a)=\left(\frac{m\omega}{2\pi i\hbar\sin\omega T}\right)^{1/2}$$

$$\cdot\exp\left\{\frac{im\omega}{2\hbar\sin\omega T}\left[(x_a^2+x_b^2)\cos\omega T-2x_ax_b\right]\right\}$$

3-9.

$$L = \frac{m}{2}\dot{x}^2 + fx$$

$$x = \bar{x} + y$$

$$\Rightarrow L = \frac{m}{2}(\dot{\bar{x}} + \dot{y})^2 + f(\bar{x} + y)$$

$$= \frac{m}{2}\dot{\bar{x}}^2 + f\bar{x} + m\dot{\bar{x}}\dot{y} + \frac{m}{2}\dot{y}^2 + fy$$

$$S = S_{cl} + \int_{t_a}^{t_b} (m\dot{\bar{x}}\dot{y} + fy)\,dt + \int_{t_a}^{t_b} \frac{m}{2}\dot{y}^2\,dt$$

$$I = \int_{t_a}^{t_b} (m\dot{\bar{x}}\dot{y} + fy)\,dt$$

$$u = \dot{\bar{x}}, \quad \frac{du}{dt} = \ddot{\bar{x}}$$

$$dv = \dot{y}\,dt, \quad v = y$$

$$I = \dot{\bar{x}}y\Big|_{t_a}^{t_b} - \int_{t_a}^{t_b} (m\ddot{\bar{x}} - f)y\,dt$$

$$= 0$$

$$\Rightarrow S = S_{cl} + \int_{t_a}^{t_b} \frac{m}{2}\dot{y}^2\,dt$$

$$K(b,a) = e^{\frac{i}{\hbar}S_{cl}} \underbrace{\int_0^0 \left(\exp\left\{\frac{i}{\hbar}\int_{t_a}^{t_b} \frac{m}{2}\dot{y}^2\,dt\right\}\right) Dy(t)}_{F(t_b,t_a)}$$

$$F(t_b,t_a) = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{\frac{im}{2\hbar\epsilon}\sum_{i=1}^{n}(y_{i+1}-y_i)^2}\,dy_1\cdots dy_{n-1}$$

Integrating term by term up to the $(n-1)^{th}$ term,

$$F(t_b,t_a) = \sqrt{\frac{m}{2\pi i\hbar(t_b-t_a)}}$$

Solving Lagrange's eq for $\bar{x}$ and finding the corresponding action we have

$$S_{cl} = \frac{m}{2}\frac{(x_b-x_a)^2}{t_b-t_a} + \frac{f}{2}(t_b-t_a)(x_a+x_b) - f^2/24\,(t_b-t_a)^3$$

With $T = t_b - t_a$ we obtain

$$\therefore K(b,a) = \sqrt{\frac{m}{2\pi i\hbar T}} \exp \frac{i}{\hbar}\left\{\frac{m(x_b-x_a)^2}{2T} + \frac{fT}{2}(x_a+x_b) - f^2T^3/24\right\}$$

3-10.

$$L = \frac{m}{2}(\dot{x}^2+\dot{y}^2+\dot{z}^2) + \frac{eB}{2c}(x\dot{y} - y\dot{x})$$

Let
$$x = \bar{x} + x^*$$
$$y = \bar{y} + y^*$$
$$z = \bar{z} + z^*$$

$$\Rightarrow L = \frac{m}{2}(\dot{\bar{x}}^2 + \dot{\bar{y}}^2 + \dot{\bar{z}}^2) + \frac{eB}{2c}(\bar{x}\dot{\bar{y}} - \bar{y}\dot{\bar{x}})$$
$$+ m(\dot{\bar{x}}\dot{x}^* + \dot{\bar{y}}\dot{y}^* + \dot{\bar{z}}\dot{z}^*)$$
$$+ \frac{eB}{2c}(\bar{x}\dot{y}^* + x^*\dot{\bar{y}} - \bar{y}\dot{x}^* - y^*\dot{\bar{x}})$$
$$+ \frac{m}{2}(\dot{x}^{*2} + \dot{y}^{*2} + \dot{z}^{*2})$$
$$+ \frac{eB}{2c}(x^*\dot{y}^* - y^*\dot{x}^*)$$

$$S = \int L\,dt$$
$$= S_{cl} + \int_{t_a}^{t_b}\left\{\frac{m}{2}(\dot{x}^{*2} + \dot{y}^{*2} + \dot{z}^{*2}) + \frac{eB}{2c}(x^*\dot{y}^* - y^*\dot{x}^*)\right\}dt$$

Since the middle term vanishes upon integration.

$$\Rightarrow K(b,a) = e^{\frac{i}{\hbar}S_{cl}} \int_0^0 \left(\exp\left\{\frac{i}{\hbar}\int_{t_a}^{t_b}\left[\frac{m}{2}(\dot{x}^{*2} + \dot{y}^{*2} + \dot{z}^{*2}) + \frac{eB}{2c}(x^*\dot{y}^* - y^*\dot{x}^*)\right]dt\right\}\right) Dx^*Dy^*Dz^*$$

$$L(\bar{x},\bar{y},\bar{z}) = \frac{m}{2}(\dot{\bar{x}}^2 + \dot{\bar{y}}^2 + \dot{\bar{z}}^2) + \frac{eB}{2c}(\bar{x}\dot{\bar{y}} - \bar{y}\dot{\bar{x}})$$

$$\frac{\partial L}{\partial \dot{\bar{x}}} = m\dot{\bar{x}} - \frac{eB}{2c}\bar{y}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\bar{x}}}\right) = m\ddot{\bar{x}} - \frac{eB}{2c}\dot{\bar{y}}$$

$$\frac{\partial L}{\partial \bar{x}} = \frac{eB}{2c}\dot{\bar{y}}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\bar{x}}}\right) - \frac{\partial L}{\partial \bar{x}} = 0$$

$$\Rightarrow m\ddot{\bar{x}} - \frac{eB}{c}\dot{\bar{y}} = 0$$

Similarly,

$$m\ddot{\bar{y}} + \frac{eB}{c}\dot{\bar{x}} = 0$$

Now,
$$m(\ddot{\bar{x}} + i\ddot{\bar{y}}) - \frac{eB}{c}(\dot{\bar{y}} - i\dot{\bar{x}}) = 0$$

$$m(\ddot{\bar{x}} + i\ddot{\bar{y}}) + i\frac{eB}{c}(\dot{\bar{x}} + i\dot{\bar{y}}) = 0$$

Let $Z = \bar{x} + i\bar{y}$

$$\Rightarrow m\ddot{z} + i\frac{eB}{c}\dot{z} = 0$$

$$\ddot{z} + i\frac{eB}{mc}\dot{z} = 0$$

$$\ddot{z} + i\omega\dot{z} = 0, \quad \omega = eB/mc$$

Let $w = \dot{z}$

$$\Rightarrow \dot{w} + i\omega w = 0$$

$$\frac{dw}{w} = -i\omega\, dt$$

$$\ln w = -i\omega t + \text{const.}$$

$$w = c_1 e^{-i\omega t}$$

$$\dot{z} = c_2 e^{-i\omega t}$$

$$Z(t) = \tilde{c}_1 e^{-i\omega t} + c_2$$

Take $\tilde{c}_1 = A e^{-i\alpha}$

$$c_2 = a + ib$$

i.e., $Z(t) = A e^{-i(\alpha+\omega)t} + a + ib$

$$\bar{x} + i\bar{y} = A e^{-i(\alpha+\omega)t} + a + ib$$

$$\Leftrightarrow \bar{x} = A\cos(\alpha+\omega)t + a$$

$$\bar{y} = -A\sin(\alpha+\omega)t + b$$

In particular we can choose the phase $\alpha = 0$.

$$\bar{x} = A\cos\omega t + a$$

$$\bar{y} = -A\sin(\alpha+\omega)t + b$$

Now,

$$x_a = A\cos\omega t_a + a$$
$$x_b = A\cos\omega t_b + a$$
$$y_a = -A\sin\omega t_a + b \qquad (*)$$
$$y_b = -A\sin\omega t_b + b$$

$$\Rightarrow (x_b - x_a)^2 + (y_b - y_a)^2$$

$$= -2A^2\cos\omega(t_b - t_a)$$

$$\text{or } A^2 = +\frac{(x_b - x_a)^2 + (y_b - y_a)^2}{2 - 2\cos\omega(t_b - t_a)}$$

Also,
$$m\ddot{z} = 0$$
$$\dot{z} = c_1$$
$$z = c_1 t + c_2$$

$$z_a = c_1 t_a + c_2$$
$$z_b = c_1 t_b + c_2$$

$$\Rightarrow c_1 = \frac{z_b - z_a}{t_b - t_a}$$

$$c_2 = \frac{z_a t_b - z_b t_a}{t_b - t_a}$$

$$\text{or } z = \frac{z_b - z_a}{t_b - t_a}t + \frac{z_a t_b - z_b t_a}{t_b - t_a}$$

$$- \quad T = \frac{m}{2}(\dot{x}^2+\dot{y}^2+\dot{z}^2)$$

$$= \frac{m}{2}\left\{\omega^2A^2\sin^2\omega t + \omega^2A^2\cos^2\omega t + \left(\frac{z_b-z_a}{t_b-t_a}\right)^2\right\}$$

$$= \frac{m}{2}\left\{\omega^2A^2 + \left(\frac{z_b-z_a}{t_b-t_a}\right)^2\right\}$$

$$\Rightarrow \int_{t_a}^{t_b} T\,dt$$

$$= \frac{m}{2}\left\{\omega^2A^2(t_b-t_a) + \frac{(z_b-z_a)^2}{t_b-t_a}\right\}$$

(with A as given above.)

$$- \quad (\bar{x}\dot{\bar{y}} - \bar{y}\dot{\bar{x}})$$

$$= (-\omega A^2\cos^2\omega t - a\omega A\cos\omega t - \omega A^2\sin^2\omega t + b\omega A\sin\omega t)$$

$$= -\omega A^2 + \omega A(b\sin\omega t - a\cos\omega t)$$

We solve $(*)$ for a & b:

$$a = \frac{x_a+x_b - A(\cos\omega t_b + \cos\omega t_a)}{2} \qquad (**)$$

$$b = \frac{y_a+y_b + A(\sin\omega t_b + \sin\omega t_a)}{2}$$

$$\int_{t_a}^{t_b} (\bar{x}\dot{\bar{y}} - \bar{y}\dot{\bar{x}})dt$$

$$= -\omega A^2(t_b-t_a) + \omega A\int_{t_a}^{t_b}(b\sin\omega t - a\cos\omega t)dt$$

$$= -\omega A^2(t_b-t_a) - A\left\{b\cos\omega t - a\sin\omega t\right\}_{t_a}^{t_b}$$

$$= -\omega A^2(t_b-t_a) - A\left\{b(\cos\omega t_b - \cos\omega t_a) + a(\sin\omega t_b - \sin\omega t_a)\right\}$$

Again from $(*)$

$$A(\cos\omega t_b - \cos\omega t_a) = (x_b-x_a)$$

$$-A(\sin\omega t_b - \sin\omega t_a) = (y_b-y_a)$$

$$\Rightarrow \int_{t_a}^{t_b}(\bar{x}\dot{\bar{y}} - \bar{y}\dot{\bar{x}})dt$$

$$= -\omega A^2(t_b-t_a) + a(y_b-y_a) - b(x_b-x_a)$$

$$= -\omega A^2(t_b-t_a) + \frac{(x_a+x_b)(y_b-y_a)}{2} - \frac{A(y_b-y_a)}{2}(\cos\omega t_b+\cos\omega t_a) - \frac{(y_a+y_b)(x_b-x_a)}{2} - \frac{A(x_b-x_a)}{2}(\sin\omega t_b + \sin\omega t_a)$$

$$= -\omega A^2(t_b-t_a) + (x_ay_b - x_by_a) + \frac{A^2}{2}\left\{(\sin\omega t_b - \sin\omega t_a)(\cos\omega t_b+\cos\omega t_a) - (\cos\omega t_b - \cos\omega t_a)(\sin\omega t_b + \sin\omega t_a)\right\}$$

$$= -\omega A^2(t_b-t_a) + (x_ay_b - x_by_a) + \frac{A^2}{2}\left\{\cdot 2\sin\omega(t_b-t_a)\right\}$$

$$= -\omega A^2(t_b-t_a) + (x_ay_b - x_by_a) + A^2\sin\omega(t_b-t_a)$$

$$\text{(i.e.)} \quad S_{cl} = \int_{t_a}^{t_b} L\,dt$$

$$= \frac{m}{2}\left\{\omega^2A^2(t_b-t_a) + \frac{(z_b-z_a)^2}{t_b-t_a}\right\} - \frac{eB}{mc}\omega A^2(t_b-t_a) + \frac{eB}{mc}(x_ay_b - x_by_a) + eB/mc\, A^2\sin\omega(t_b-t_a)\}$$

$$S_{cl} = \frac{m}{2}\Big\{ \omega^2 A^2 (t_b - t_a) + \frac{(z_b - z_a)^2}{t_b - t_a} - \omega^2 A^2 (t_b - t_a) + \omega (x_a y_b - x_b y_a) + \omega A^2 \sin \omega (t_b - t_a) \Big\}$$

$$= \frac{m}{2}\Big\{ \frac{(z_b - z_a)^2}{t_b - t_a} + \omega (x_a y_b - x_b y_a) + \omega A^2 \sin \omega (t_b - t_a) \Big\}$$

$$= \frac{m}{2}\Big\{ \frac{(z_b - z_a)^2}{t_b - t_a} + \omega (x_a y_b - y_b y_a) + \omega \frac{\left[(x_b - x_a)^2 + (y_b - y_a)^2 \right] . \sin \omega (t_b - t_a)}{2(1 - \cos \omega (t_b - t_a))} \Big\}$$

$$= \frac{m}{2}\Big\{ \frac{(z_b - z_a)^2}{T} + \omega (x_a y_b - y_b y_a) + \frac{\omega}{2}\left[(x_b - x_a)^2 + (y_b - y_a)^2 \right] . \times \frac{2 \sin \frac{\omega T}{2} \cos \frac{\omega T}{2}}{2 \sin^2 \frac{\omega T}{2}} \Big\}$$

or
$$S_{cl} = \frac{m}{2}\Big\{ \frac{(z_b - z_a)^2}{T} + \frac{\omega}{2} \cot \frac{\omega T}{2} \left[(x_b - x_a)^2 + (y_b - y_a)^2 \right] + \omega (x_a y_b - x_b y_a) \Big\}$$

$\Rightarrow$ $K(b, a)$

$$= F(T) \exp \Big\{ \frac{im}{2\hbar} \Big(\frac{(z_b - z_a)^2}{T} + \frac{\omega}{2} \cot \frac{\omega T}{2} \left[(x_b - x_a)^2 + (y_b - y_a)^2 \right] + \omega (x_a y_b - x_b y_a) \Big) \Big\}$$

Evaluation of $F(T)$:

3-11

$$L = \frac{m}{2}(\dot{x}^2 - \omega^2 x^2) + f(t)\cdot x$$

Assuming $x(t)$ is displaced by an amount $y(t)$ from the classical trajectory $\bar{x}(t)$ we have

$$x(t) = \bar{x}(t) + y(t)$$

$$\Rightarrow \quad L = \frac{m}{2}(\dot{\bar{x}}^2 - \omega^2 \bar{x}^2) + f(t)\bar{x}$$
$$+ m(\dot{\bar{x}}\dot{y} - \omega^2 \bar{x} y) + f(t) y$$
$$+ \frac{m}{2}(\dot{y}^2 - \omega^2 y^2)$$

The integral of the first term wrt t from t_a to t_b gives S_{cl}, the 2nd term vanishes and the last term as in prob. 3-8 (when exponentiated and path integrated) gives

$$F(T) = \sqrt{\frac{m\omega}{2\pi i\hbar \sin\omega T}}$$

$$\therefore K(b,a) = \sqrt{\frac{m\omega}{2\pi i\hbar \sin\omega T}}\, e^{\frac{i}{\hbar}S_{cl}}$$

Let us now find S_{cl}.

$$S_{cl} = \int_{t_a}^{t_b} \left[\frac{m}{2}(\dot{\bar{x}}^2 - \omega^2 \bar{x}^2) + f(t)\bar{x}\right] dt$$

$\bar{x}$ is a solution of Lagrange's eq.

$$\frac{\partial L}{\partial \dot{\bar{x}}} = m\dot{\bar{x}}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\bar{x}}}\right) = m\ddot{\bar{x}}$$

$$\frac{\partial L}{\partial \bar{x}} = -m\omega^2 \bar{x} + f(t)$$

$$\Rightarrow \ddot{\bar{x}} + \omega^2 \bar{x} = \frac{f(t)}{m}$$

The solution to this eq. the sum of the soln. to the homogeneous DE and the particular solution i.e.,

$$\bar{x} = \bar{x}_H + \bar{x}_p$$

From prob. 2-1

$$\bar{x}_H(t) = A \sin\omega t + B\cos\omega t$$

With $A = \dfrac{x_b \cos\omega t_a - x_a \cos\omega t_b}{\sin T}$

$$B = \frac{x_a \sin\omega t_b - x_b \sin\omega t_a}{\sin\omega T}$$

$(T = t_b - t_a)$

$$\Rightarrow \bar{x}_H(t) = \frac{x_b \sin\omega(t-t_a) + x_a \sin\omega(t_b - t)}{\sin\omega T}$$

By using the variational method

$$\bar{x}_p(t) = \frac{1}{m\omega \sin\omega T}\left\{\sin\omega(t-t_a)\int_{t}^{t_b} f(t)\sin\omega(t_b-t)\,dt - \sin\omega(t_b-t)\int_{t_a}^{t} f(t)\sin\omega(t-t_a)\,dt\right\}$$

$$\equiv \frac{1}{m\omega}\int_{t_a}^{t} f(t')\sin\omega(t-t')\,dt'$$

Now, $L = \frac{m}{2}(\dot{\bar{x}}_H^2 - \omega^2 \bar{x}_H^2) + f(t)\bar{x}_H$
$$+ m(\dot{\bar{x}}_H \dot{\bar{x}}_p - \omega^2 \bar{x}_H \bar{x}_p)$$
$$+ \frac{m}{2}(\dot{\bar{x}}_p^2 - \omega^2 \bar{x}_p^2) + f(t)\bar{x}_p$$

Integration of the first term gives the action obtained in prob. 2-1 and the additional term

$$\frac{1}{\sin\omega T}\left\{x_a \int_{t_a}^{t_b} f(t)\sin\omega(t_b - t)\,dt + x_b\int_{t_a}^{t_b} f(t)\sin\omega(t-t_a)\,dt\right\}$$

...... (*)

consider the last two terms

$$m\,(\dot{\bar{x}}_H\dot{\bar{x}}_p-\omega^2\bar{x}_H\bar{x}_p)$$
$$+\frac{m}{2}(\dot{\bar{x}}_p^2-\omega^2\bar{x}_p^2)+f(t)\bar{x}_p$$

$$=\frac{m}{2}\{2\dot{\bar{x}}_H\dot{\bar{x}}_p-2\omega^2\bar{x}_H\bar{x}_p+\dot{\bar{x}}_p^2-\omega^2\bar{x}_p^2\}+f(t)\bar{x}_p$$

$$=\frac{m}{2}\{\dot{\bar{x}}_p(\dot{\bar{x}}_p+2\dot{\bar{x}}_H)-\omega^2\bar{x}_p(\bar{x}_p+2\bar{x}_H)\}+f(t)\bar{x}_p$$

$$I=\frac{m}{2}\int_{t_a}^{t_b}\{\tfrac{m}{2}[\dot{\bar{x}}_p(\dot{\bar{x}}_p+2\dot{\bar{x}}_H)-\omega^2\bar{x}_p(\bar{x}_p+2\bar{x}_H)]+f(t)\bar{x}_p\}dt$$

integration by parts:

Let $u=\dot{\bar{x}}_p+2\dot{\bar{x}}_H$

$$\frac{du}{dt}=\ddot{\bar{x}}_p+2\ddot{\bar{x}}_H$$

$dv=\dot{\bar{x}}_p\,dt,\; v=\bar{x}_p$

$$\Rightarrow I=\frac{m}{2}\int_{t_a}^{t_b}\frac{d}{dt}\{\bar{x}_p(\dot{\bar{x}}_p+2\dot{\bar{x}}_H)\}dt$$
$$-\frac{m}{2}\int_{t_a}^{t_b}\{\bar{x}_p[\ddot{\bar{x}}_p+2\ddot{\bar{x}}_H]+\omega^2\bar{x}_p(\bar{x}_p+2\bar{x}_H)-\frac{2f(t)}{m}\bar{x}_p\}dt$$

$$=\frac{m}{2}\int_{t_a}^{t_b}\frac{d}{dt}\{\bar{x}_p(\dot{\bar{x}}_p+2\dot{\bar{x}}_H)\}dt$$
$$-\frac{m}{2}\int_{t_a}^{t_b}\{\bar{x}_p(\ddot{\bar{x}}_p+\omega^2\bar{x}_p-\frac{f(t)}{m})+2\bar{x}_p(\ddot{\bar{x}}_H+\omega^2\bar{x}_H)-\frac{f(t)}{m}\bar{x}_p\}dt$$

For a periodic motion since the velocities at the end points are the same this term is zero.

Note:

$$\bar{x}=\bar{x}_H+\bar{x}_p$$
$$\ddot{\bar{x}}+\omega^2\bar{x}=\frac{f(t)}{m}$$
$$\Rightarrow\ \ddot{\bar{x}}_H+\omega^2\bar{x}_H+\ddot{\bar{x}}_p+\omega^2\bar{x}_p=\frac{f(t)}{m}$$

The first term here is zero since it is a solution to the homogeneous D.E. i.e.,

$$I=\frac{m}{2}\int_{t_a}^{t_b}\frac{d}{dt}\{\bar{x}_p(\dot{\bar{x}}_p+2\dot{\bar{x}}_H)\}dt+\frac{1}{2}\int_{t_a}^{t_b}f(t)\bar{x}_p\,dt$$

$$=\frac{m}{2}\{\bar{x}_p(\dot{\bar{x}}_p+2\dot{\bar{x}}_H)\}\Big|_{t_a}^{t_b}+\tfrac{1}{2}\int_{t_a}^{t_b}f(t)\bar{x}_p\,dt$$

$\dot{\bar{x}}_H$ & $\dot{\bar{x}}_p$ are velocities and all the assuming that the particle starts from rest at (x_a,t_a) and comes to rest again at (x_b,t_b) we obtain

$$I=\frac{1}{2}\int_{t_a}^{t_b}f(t)\bar{x}_p\,dt$$

$$=\frac{1}{2m\omega\sin\omega T}\Big\{\int_{t_a}^{t_b}\int_{t_a}^{t}f(t)f(s)\sin\omega(t-t_a)\cdot\sin\omega(t_b-s)\,ds\,dt$$
$$-\int_{t_a}^{t_b}\int_{t_a}^{t}f(t)f(s)\sin\omega(t_b-t)\sin\omega(s-t_a)\,ds\,dt\Big\}$$

In the first integral replace t by s & s by t to get

$$I=\frac{-1}{m\omega\sin\omega T}\int_{t_a}^{t_b}\int_{t_a}^{t}f(t)f(s)\sin\omega(t_b-t)\cdot\sin\omega(s-t_a)\,ds\,dt\quad\ldots(**)$$

So now collecting the results from prob. 2-1, (*) & (* *) we obtain

$$S_{cl} = \frac{m\omega}{2\sin\omega T}\Big\{\cos\omega T(x_a^2+x_b^2)-2x_ax_b + \frac{2x_a}{m\omega}\int_{t_a}^{t_b} f(t)\sin\omega(t_b-t)dt + \frac{2x_b}{m\omega}\int_{t_a}^{t_b} f(t)\sin\omega(t-t_a)dt - \frac{2}{m^2\omega^2}\int_{t_a}^{t_b}\int_{t_a}^{t} f(t)f(s)\sin\omega(t_b-t)\cdot\sin\omega(s-t_a)\,ds\,dt\Big\}$$

3-12. Initial w.f.

$$\psi(x',0)=\exp\left[-\frac{m\omega}{2\hbar}(x'-a)^2\right]$$

The kernel to get to the particle a point (x,T) from $(x',0)$ is (prob. 3-8)

$$K=\sqrt{\frac{m}{2\pi i\hbar\sin\omega T}}\cdot\exp\left\{\frac{im\omega}{2\hbar\sin\omega T}\left[(x^2+x'^2)\cos\omega T - 2xx'\right]\right\}$$

The w.f. of the particle is thus

$$\psi(x,T)=\int_{-\infty}^{\infty} K(x,T;x',0)\psi(x',0)dx'$$

$$K\cdot\psi=\sqrt{\frac{m\omega}{2\pi i\hbar\sin\omega T}}\cdot\exp\left\{i\alpha(x^2+x'^2)\beta-2\alpha xx' - \gamma(x'-a)^2\right\}$$

Where $\alpha=\frac{m\omega}{2\hbar\sin\omega T}$, $\beta=\cos\omega T$ and $\gamma=\frac{m\omega}{2\hbar}$

$$i\alpha(x^2+x'^2)\beta-2\alpha xx' - \gamma(x'-a)^2$$
$$= i\alpha(x^2+x'^2)\beta-2\alpha xx' - \gamma x'^2-\gamma a^2+2\gamma ax'$$
$$= i\alpha\beta x^2 + (i\alpha\beta-\gamma)x'^2 + 2(\gamma a-\alpha x)x' - \gamma a^2$$
$$= i\alpha\beta x^2 - \frac{(\gamma a-\alpha x)^2}{i\alpha\beta-\gamma} - \gamma a^2 - (\gamma-i\alpha\beta)\left[x'+\frac{\gamma a-\alpha x}{i\alpha\beta-\gamma}\right]^2$$

(i.e.) $\psi(x,T)$

$$=\sqrt{\frac{m\omega}{2\pi i\hbar\sin\omega T}}\exp\left\{i\alpha\beta x^2-\frac{(\gamma a-\alpha x)^2}{i\alpha\beta-\gamma}-\gamma a^2\right\}\int_{-\infty}^{\infty}\exp\left\{-(\gamma-i\alpha\beta)\left[x'+\frac{\gamma a-\alpha x}{i\alpha\beta-\gamma}\right]^2\right\}dx'$$

$$=\sqrt{\frac{m\omega}{2\pi i\hbar\sin\omega T}}\cdot\sqrt{\frac{\pi}{\gamma-i\alpha\beta}}\cdot\exp\left\{i\alpha\beta x^2-\frac{(\gamma a-\alpha x)^2}{i\alpha\beta-\gamma}-\gamma a^2\right\}$$

$$i\alpha\beta x^2-\frac{(\gamma a-\alpha x)^2}{i\alpha\beta-\gamma}-\gamma a^2$$
$$= -\frac{(\alpha^2\beta^2+i\alpha\beta\gamma+\alpha^2)x^2}{i\alpha\beta-\gamma} - \frac{i\alpha\beta\gamma a^2}{i\alpha\beta-\gamma} + \frac{2\alpha\gamma ax}{i\alpha\beta-\gamma}$$

$$\alpha^2(\beta^2+1)+i\alpha\beta\gamma$$
$$=\frac{m^2\omega^2}{4\hbar^2\sin^2\omega T}\left(\cos^2\omega T+1+i\cos\omega T\sin\omega T\right)$$
$$=\frac{m^2\omega^2\cos\omega T}{4\hbar^2\sin^2\omega T}\left(\frac{1}{\cos\omega T}+e^{i\omega T}\right)$$

$$i\alpha\beta\gamma = \frac{i\, m^2\omega^2 \cos\omega T}{4\hbar^2 \sin\omega T}$$

$$2\gamma a\alpha = \frac{m^2\omega^2 a}{2\hbar^2 \sin\omega T}$$

$$i\alpha\beta - \gamma = \frac{i m\omega}{2\hbar \sin\omega T}\, e^{i\omega T}$$

i.e.,

	Coeff.
x^2	$-\frac{i m\omega}{2\hbar \sin\omega T}\left(e^{-i\omega T} + \cos\omega T\right)$
a^2	$-\frac{m\omega}{2\hbar}\cos\omega T\, e^{-i\omega T}$
	$= -\frac{m\omega}{4\hbar}\left(1 + e^{-2i\omega T}\right)$
$2ax$	$-\frac{i m\omega}{2\hbar}\, e^{-i\omega T}$

Also, $\sqrt{\frac{m\omega}{2\pi i\hbar \sin\omega T}} \cdot \sqrt{\frac{\pi}{-i\alpha\beta + \gamma}} = e^{-\frac{i\omega T}{2}}$

$\therefore\ \psi(x, T)$

$$= \exp\left\{-\frac{i\omega T}{2} - i\frac{m\omega}{2\hbar}\left[x^2\,\frac{e^{-i\omega T} + \cos\omega T}{\sin\omega T} - 2ax\, e^{-i\omega T} - \frac{1}{2}a^2\left(1 + e^{-2i\omega T}\right)\right]\right\}$$

Chapter 4

4-1. By analogy with the 1-D case the w.f. of a single particle moving in 3-D is given by

$$\psi(x_2, y_2, z_2; t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K(x_2, y_2, z_2, t_2; x_1, y_1, z_1, t_1)\, \psi(x_1, y_1, z_1, t_1)\, dx_1 dy_1 dz_1$$

If $t_2 = t_1 + \epsilon \equiv t+\epsilon$; $x_2, y_2, z_2 \to x, y, z$; $x_1, y_1, z_1 \to x_0, y_0, z_0$

$$\psi(x, y, z, t+\epsilon) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{A^3}\left[\exp\left\{\frac{i}{\hbar}\int_t^{t+\epsilon} L(x,y,z,\dot{x},\dot{y},\dot{z},t)dt\right\}\right] \psi(x_0,y_0,z_0,t)\, dx_0 dy_0 dz_0$$

$$\cong \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{A^3}\left[\exp\left\{\frac{i\epsilon}{\hbar} L\left(\frac{x-x_0}{\epsilon}, \frac{y-y_0}{\epsilon}, \frac{z-z_0}{\epsilon}, \frac{x+x_0}{2}, \frac{y+y_0}{2}, \frac{z+z_0}{2}\right)\right\}\right] \psi(x_0,y_0,z_0,t)\, dx_0 dy_0 dz_0$$

(since ϵ is very small.)

For $V \equiv V(x,y,z,t)$, K.E. $= \frac{m}{2}(\dot{x}^2+\dot{y}^2+\dot{z}^2)$

$$\psi(x,y,z,t+\epsilon) = \int_{-\infty}^{\infty} \frac{1}{A^3}\left[\exp\left\{\frac{im}{2\hbar}\left[\frac{(x-x_0)^2}{\epsilon} + \frac{(y-y_0)^2}{\epsilon} + \frac{(z-z_0)^2}{\epsilon}\right] - \frac{i\epsilon}{\hbar} V\left(\frac{x+x_0}{2}, \frac{y+y_0}{2}, \frac{z+z_0}{2}, t\right)\right\}\right] \psi(x_0,y_0,z_0,t)\, dx_0 dy_0 dz_0$$

If the exponential of the K.E. is to contribute more we must have x_0, y_0, z_0 close to x, y, z i.e.,

$$x_0 = x+\eta$$

$$y_0 = y+\zeta$$

$$z_0 = z+\xi$$

Since (x, y, z) is a fixed point

$$\psi(x,y,z,t+\epsilon) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{A^3} e^{\frac{im}{2\hbar\epsilon}(\eta^2+\zeta^2+\xi^2)} \cdot e^{-\frac{i\epsilon}{\hbar} V\left(\frac{x+\eta}{2}, \frac{y+\zeta}{2}, \frac{z+\xi}{2}, t\right)} \cdot \psi(x+\eta, y+\zeta, z+\xi)\, d\eta\, d\zeta\, d\xi$$

Since the error in η is higher order we replace $\epsilon V\left(\frac{x+\eta}{2}, \ldots\right)$ by $V(x,y,z,t)$.

For the first factor to contribute more η, ζ, ξ should be of order $\sqrt{\frac{\hbar\epsilon}{m}}$ so now the expand the L.H.S to order ϵ and the R.H.S to order ϵ $\xi\eta^2$ i.e.,

$$\psi(x,y,z,t) + \epsilon\frac{\partial\psi}{\partial t} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{A^3} e^{\frac{im}{2\hbar\epsilon}(\eta^2+\zeta^2+\xi^2)} \left[1 - \frac{i\epsilon}{\hbar} V(x,y,z,t)\right] \cdot \left[\psi(x,y,z) + \eta\frac{\partial\psi}{\partial x} + \zeta\frac{\partial\psi}{\partial y} + \xi\frac{\partial\psi}{\partial z} + \frac{1}{2}\eta^2\frac{\partial^2\psi}{\partial x^2} + \frac{1}{2}\zeta^2\frac{\partial^2\psi}{\partial y^2} + \frac{1}{2}\xi^2\frac{\partial^2\psi}{\partial z^2}\right] d\eta\, d\zeta\, d\xi$$

Now we can integrate over each variable one by one and for each variable $\eta, \zeta, \xi \equiv \gamma$ we obtain

$$\frac{1}{A}\int_{-\infty}^{\infty} e^{\frac{im\gamma^2}{2\hbar\epsilon}} d\gamma = \frac{1}{A}\left(\frac{2\pi i\hbar\epsilon}{m}\right)^{1/2}$$

Since ψ on the right is multiplied by $1/A^3$ and the integrals over η, ζ, ξ and in order that in the limit $\epsilon \to 0$ both sides agree we must have

$$\frac{1}{A^3}\left(\frac{2\pi i\hbar\epsilon}{m}\right)^{3/2} = 1$$

$$\text{or } A = \left(\frac{2\pi i\hbar\epsilon}{m}\right)^{1/2}$$

Also, $$\frac{1}{A}\int_{-\infty}^{\infty} e^{\frac{im\gamma^2}{2\hbar\epsilon}} \gamma\, d\gamma = 0$$

And $$\frac{1}{A}\int_{-\infty}^{\infty} e^{\frac{im\gamma}{2\hbar\epsilon}} \gamma^2 d\gamma = \frac{i\hbar\epsilon}{m}$$

i.e., as an example

$$\frac{1}{A}\int_{-\infty}^{\infty} e^{\frac{im\eta^2}{2\hbar\epsilon}} \eta^2 d\eta \cdot \frac{1}{A}\int_{-\infty}^{\infty} e^{\frac{im\zeta^2}{2\hbar\epsilon}} d\zeta \cdot \frac{1}{A}\int_{-\infty}^{\infty} e^{\frac{im\xi^2}{2\hbar\epsilon}} d\xi = \frac{i\hbar\epsilon}{m}$$

So now by evaluating all the necessary integrals we get

$$\psi + \epsilon\frac{\partial\psi}{\partial t} = \psi - \frac{i\epsilon}{\hbar}V\psi + \frac{i\hbar\epsilon}{2m}\frac{\partial^2\psi}{\partial x^2} + \frac{i\hbar\epsilon}{2m}\frac{\partial^2\psi}{\partial y^2} + \frac{i\hbar\epsilon}{2m}\frac{\partial^2\psi}{\partial z^2}$$

$$\frac{\partial\psi}{\partial t} = -\frac{i}{\hbar}V\psi + \frac{i\hbar}{2m}\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}\right)$$

or

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}\right) + V\psi$$

$$\Rightarrow -\frac{\hbar}{i}\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi$$

which is the 3-D Schrödinger eq.

4-2. The Classical Hamiltonian of such a particle is

$$\mathcal{H} = \frac{p^2}{2m} + e\Phi$$

$$= \frac{1}{2m}\left(P - \frac{eA}{c}\right)^2 + e\Phi$$

To find the corresponding Q. mechanical operator we use the substitution rule i.e.,

$$H = \frac{1}{2m}\left(-i\hbar\nabla - \frac{eA}{c}\right)^2 + e\Phi$$

$$\therefore -\frac{\hbar}{i}\frac{\partial\psi}{\partial t} = \frac{1}{2m}\left(\frac{\hbar}{i}\nabla - \frac{eA}{c}\right)^2\psi + e\Phi\psi$$

which is Schrödinger's eq.

4-3.

$$-\frac{\hbar}{i}\frac{\partial\psi}{\partial t} = H\psi$$

Taking the complex conjugate

$$\frac{\hbar}{i}\frac{\partial\psi^*}{\partial t} = (H\psi)^* \quad \text{Q.E.D.}$$

4-4.

$$\frac{\partial^2}{\partial x^2}x(\psi)$$

$$= \frac{\partial^2}{\partial x^2}(x\psi)$$

$$= \frac{\partial}{\partial x}\left(\psi + x\frac{\partial\psi}{\partial x}\right)$$

$$= 2\frac{\partial\psi}{\partial x} + x\frac{\partial^2\psi}{\partial x^2}$$

$$= \left(2\frac{\partial}{\partial x} + x\frac{\partial^2}{\partial x^2}\right)\psi$$

i.e., $\frac{\partial^2}{\partial x^2}x = 2\frac{\partial}{\partial x} + x\frac{\partial^2}{\partial x^2}$

4-5.

$$K(2,1) = \int_{-\infty}^{\infty} K(2,3)\,K(3,1)\,dx_3$$

$$\Rightarrow \frac{\hbar}{i}\frac{\partial K(2,1)}{\partial t_1} = \int_{-\infty}^{\infty} K(2,3)\frac{\hbar}{i}\frac{\partial K(3,1)}{\partial t_1}dx_3$$

$$= \int_{-\infty}^{\infty} K(2,3)\,H_1^*\,K(3,1)\,dx_3$$

$$= H_1^*\int_{-\infty}^{\infty} K(2,3)\,K(3,1)\,dx_3$$

$$= H_1^*\,K(2,1)$$

i.e., $\frac{\hbar}{i}\frac{\partial K(2,1)}{\partial t_1} - H_1^*K(2,1) = 0$

4-6.

$$\psi(x_2,t_2) = \int_{-\infty}^{\infty} K(x_2,t_2;x_1,t_1)\,\psi(x_1,t_1)\,dx_1$$

If $t_2 \to t_1 + 0$, then

$$\psi(x_2,t_2) \to \psi(x_1,t_1)$$

$$= \int_{-\infty}^{\infty} K(x_2\to x_1, t_2\to t_1; x_1,t_1)\,\psi(x_1,t_1)\,dx_1$$

$$\Rightarrow \int_{-\infty}^{\infty} K(x_2\to x_1, t_2\to t_1; x_1,t_1)\,dx_1$$

$$= \int_{-\epsilon}^{+\epsilon} K(x_2\to x_1, t_2\to t_1)\,dx_1 = 1$$

i.e., $K(x_2,t_2) \equiv K(2,1) \to \delta(x_2 - x_1)$ as $t_2 \to t_1$

4-7.

$$\int K^*(2,1')\, K(2,1)\, dx_2 = \delta(x_1'-x_1)$$

$$\Rightarrow \int K^*(2,1')\, K(2,1)\, K(1,3)\, dx_2\, dx_1$$

$$= \int K^*(2,1')\, K(2,1)\, K^*(3,1)\, dx_2\, dx_1$$

$$= \int \delta(x_1'-x_1)\, K^*(3,1)\, dx_1$$

(... if $t_1 < t_3$)

$$= K^*(3,1)$$

4-8.

$$H\Phi = E\Phi$$

If f & g are functions which falls of to zero at infinity and for H Hermitian

$$\int_{-\infty}^{\infty} (Hg)^* f\, dx = \int_{-\infty}^{\infty} g^* (Hf)\, dx$$

But here $g = f = \Phi$

i.e., $\int_{-\infty}^{\infty} (H\Phi)^* \Phi\, dx = \int_{-\infty}^{\infty} \Phi^* (H\Phi)\, dx$

Since $H\Phi = E\Phi$,

$$\int_{-\infty}^{\infty} (E\Phi)^* \Phi\, dx = \int_{-\infty}^{\infty} \Phi^* (E\Phi)\, dx$$

$$\int_{-\infty}^{\infty} \Phi^* E^* \Phi\, dx = \int_{-\infty}^{\infty} \Phi^* E \Phi\, dx$$

$$E^* \int_{-\infty}^{\infty} |\Phi|^2 dx = E \int_{-\infty}^{\infty} |\Phi|^2 dx$$

$$\therefore E^* = E$$

4-9. Again for H Hermitian

$$\int_{-\infty}^{\infty} (H\Phi_1)^* \Phi_2\, dx = \int_{-\infty}^{\infty} \Phi_1^* (H\Phi_2)\, dx$$

$$\int_{-\infty}^{\infty} (E_1\Phi_1)^* \Phi_2\, dx = \int_{-\infty}^{\infty} \Phi_1^* E_2 \Phi_2\, dx$$

$$E_1^* \int_{-\infty}^{\infty} \Phi_1^* \Phi_2\, dx = E_2 \int_{-\infty}^{\infty} \Phi_1^* \Phi_2\, dx$$

$$E_1 \int_{-\infty}^{\infty} \Phi_1^* \Phi_2\, dx = E_2 \int_{-\infty}^{\infty} \Phi_1^* \Phi_2\, dx$$

... prob. 4-8.

$$\Rightarrow (E_2 - E_1) \int_{-\infty}^{\infty} \Phi_1^* \Phi_2\, dx = 0$$

Since $E_2 \neq E_1$,

$$\int_{-\infty}^{\infty} \Phi_1^* \Phi_2 = 0$$

Had we started with $g = \Phi_2$ & $f = \Phi_1$ we would get

$$\int_{-\infty}^{\infty} \Phi_1 \Phi_2^*\, dx = 0.$$

$\therefore$ In general,

$$\int_{-\infty}^{\infty} \Phi_1^* \Phi_2\, dx = \int_{-\infty}^{\infty} \Phi_1 \Phi_2^*\, dx = 0.$$

4-10.

For a prob. whose eigen-value prob. we solve,

$$K(x_2, t_2; x_1, t_1)$$

$$= \sum_{n=1}^{\infty} \Phi_n(x_2)\, \Phi_n^*(x_1)\, e^{-\frac{i}{\hbar} E_n (t_2 - t_1)} \quad \text{for } t_2 > t_1$$

$$= 0 \quad \text{for } t_2 < t_1.$$

The Schrödinger eq. for the kernel is

$$-\frac{\hbar}{i} \frac{\partial K(2,1)}{\partial t_2} = -\frac{\hbar^2}{2m} \frac{\partial^2 K(2,1)}{\partial x_2^2} + V(2) K(2,1)$$

which for the above case gives

$$\left(-\frac{\hbar}{i}\right)\left(-\frac{i}{\hbar}\right) \sum_{n=1}^{\infty} E_n \Phi_n(x_2)\, \Phi_n^*(x_1)\, e^{-\frac{i}{\hbar} E_n (t_2 - t_1)}$$

$$= \sum_{n=1}^{\infty} \left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x_2^2} + V(2)\right] \Phi_n(x_2)\, \Phi_n^*(x_1)\, e^{-\frac{i}{\hbar} E_n (t_2 - t_1)}$$

or $\sum_{n=1}^{\infty} E_n \Phi_n(x_2) \Phi_n^*(x_1)\, e^{-\frac{i}{\hbar} E_n (t_2 - t_1)}$

$$= \sum_{n=1}^{\infty} E_n\, \Phi_n(x_2)\, \Phi_n^*(x_1)\, e^{-\frac{i}{\hbar} E_n (t_2 - t_1)}$$

i.e., Schrödinger's eq. is satisfied.

4-11. For a free particle in 3-D, the Hamiltonian is

$$H = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)$$

$$= -\frac{\hbar^2}{2m}\nabla^2$$

So that for $\Phi \equiv \Phi_p = e^{\frac{i}{\hbar}\bar{p}\cdot\bar{r}}$,

$$H\Phi_p = -\frac{\hbar^2}{2m}\nabla^2\left(e^{i/\hbar\, \bar{p}\cdot\bar{r}}\right)$$

$$= -\frac{\hbar^2}{2m}\left[\frac{i}{\hbar}\cdot\frac{i}{\hbar}\ \bar{p}\cdot\bar{p}\right]e^{\frac{i}{\hbar}(\bar{p}\bar{r})}$$

$$= -\frac{\hbar^2}{2m}\cdot\left(\frac{-1}{\hbar^2}\right)p^2\ e^{\frac{i}{\hbar}\bar{p}\cdot\bar{r}}$$

$$= \frac{p^2}{2m}\ e^{\frac{i}{\hbar}\bar{p}\cdot\bar{r}}$$

$$= \frac{p^2}{2m}\Phi_p$$

$$= E_p\,\Phi_p$$

Thus $E_p = p^2/2m$ is the energy.

$$\int^{r} \Phi_p^{*}\,\Phi_{p'}\,d^3\bar{r}$$

$$= \int^{r} e^{\frac{i}{\hbar}(\bar{p}'-\bar{p})\cdot\bar{r}}\,d^3\bar{r}$$

$$= \int e^{\frac{i}{\hbar}(P_x'-P_x)x}\,dx \int e^{\frac{i}{\hbar}(P_y'-P_y)y}\,dy \int e^{\frac{i}{\hbar}(P_z'-P_z)}\,dz$$

(... for fixed P_x, P_x' etc.)

$$= (2\pi)\,\delta(P_x'-P_x)\cdot(2\pi)\,\delta(P_y'-P_y)\,(2\pi)\,\delta(P_z'-P_z)$$

$$= (2\pi)^3\,\delta^3(\bar{p}'-\bar{p})$$

$$= 0 \text{ if } \bar{p}'\neq\bar{p}.$$

N.B. for the free particle the expression in Prob. 4-10, of the kernel gives

$$K_0(\bar{r}_2,t_2;\bar{r}_1,t_1)$$

$$= \sum_{p} e^{+\frac{i}{\hbar}\bar{p}\cdot(\bar{r}_2-\bar{r}_1)}\ e^{-\frac{ip^2}{2m}(t_2-t_1)}$$

4-12. In prob. 4-11, since the $\bar{p}$s are distributed over a continuum, the sum over the indices $\bar{p}$ is equivalent to an integral over the values of $\bar{p}$,

$$\sum_{p}(\ ) \to \int^{p}(\ )\,\frac{d^3\bar{p}}{(2\pi\hbar)^3}$$

So now for a free particle,

$$K_0(\bar{r}_2,t_2;\bar{r}_1,t_1)$$

$$= \int^{p} e^{\frac{i}{\hbar}\bar{p}\cdot(\bar{r}_2-\bar{r}_1)}\ e^{-\frac{i}{\hbar}\frac{p^2}{2m}(t_2-t_1)}\ \frac{d^3\bar{p}}{(2\pi\hbar)^3}$$

$$= \frac{1}{(2\pi\hbar)^3}\int^{p} e^{-\frac{i}{\hbar}\left\{\frac{p^2}{2m}(t_2-t_1)-\bar{p}(\bar{r}_2-\bar{r}_1)\right\}}\,d^3\bar{p}$$

$$= \frac{1}{(2\pi\hbar)^3}\int^{p} e^{-\frac{i(t_2-t_1)}{2m\hbar}\left\{p^2-\frac{2m(\bar{r}_2-\bar{r}_1)}{t_2-t_1}\bar{p}\right\}}\,d^3\bar{p}$$

$$= \frac{1}{(2\pi\hbar)^3}\,e^{\frac{im}{2\hbar}\frac{(\bar{r}_2-\bar{r}_1)^2}{t_2-t_1}}\int^{p} e^{-\frac{i(t_2-t_1)}{2m\hbar}\left[\bar{p}-\frac{m(\bar{r}_2-\bar{r}_1)}{t_2-t_1}\right]^2}\,d^3\bar{p}$$

If $|\bar{p}|$ changes from $-\infty$ to $+\infty$,

$$K_0(\bar{r}_2,t_2;\bar{r}_1,t_1)$$

$$= \frac{1}{(2\pi\hbar)^3}\,e^{\frac{im}{2\hbar}\frac{(\bar{r}_2-\bar{r}_1)^2}{t_2-t_1}}\sqrt{\frac{\pi}{i(t_2-t_1)/2m\hbar}}^{\,3}$$

$$= \sqrt{\frac{m}{i2\pi\hbar(t_2-t_1)}}^{\,3}\ e^{\frac{im}{2\hbar}\frac{(\bar{r}_2-\bar{r}_1)^2}{t_2-t_1}}$$

4-13

5-2. From eq. 4-59 for a prob. whose eigenval eq. we solve,

$$K(x_2,t_2;x_1,t_1) = \sum_{n=1}^{\infty} \Phi_n(x_2)\Phi_n^*(x_1)\, e^{-\frac{i}{\hbar}E_n(t_2-t_1)}$$

for $t_2 > t_1$.

If we only transform the time (not spatial variables) we have

$$K(x_2,E_2;x_1,E_1) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i/\hbar E_2 t_2} K(x_2,t_2;x_1,t_1)\, e^{-i/\hbar E_1 t_1}\, dt_1 dt_2$$

$$= \sum_{n=1}^{\infty} \Phi_n(x_2)\Phi_n^*(x_1) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i/\hbar E_n(t_2-t_1)}\, e^{i/\hbar E_2 t_2}\, e^{-i/\hbar E_1 t_1}\, dt_1 dt_2$$

$$= \sum_{n=1}^{\infty} \Phi_n(x_2)\Phi_n^*(x_1) \int_{-\infty}^{\infty} e^{\frac{i}{\hbar}(E_2-E_n)t_2}\, dt_2 \int_{-\infty}^{\infty} e^{-i/\hbar(E_1-E_n)t_1}\, dt_1$$

Let $t_2 = t_1 + \tau$ i.e.,

$$K(x_2,E_2;x_1,E_1) = \sum_{n=1}^{\infty} \Phi_n(x_2)\Phi_n^*(x_1) \int_{-\infty}^{\infty} e^{\frac{i}{\hbar}(E_2-E_1)t_1}\, dt_1 \int_{-\infty}^{\infty} e^{+i/\hbar(E_2-E_n)\tau}\, d\tau$$

$$= 2\pi\hbar\,\delta(E_2-E_1) \sum_{n=1}^{\infty} \Phi_n(x_2)\Phi_n^*(x_1) \int_{-\infty}^{\infty} e^{+\frac{i}{\hbar}(E_2-E_n)\tau}\, d\tau$$

Since E_2-E_n is real our integral does not converge and hence we apply the following technique. Assume E_2-E_n is replaced by ~~####~~ $E_2-E_n+i\epsilon$ where $\epsilon\to 0$.

$$\Rightarrow \int_{-\infty}^{\infty} e^{\frac{i}{\hbar}(E_2-E_n)\tau}\, d\tau = \lim_{\epsilon\to 0} \frac{i}{E_2-E_n+i\epsilon}$$

$$\therefore K(x_2,E_2;x_1,E_1) = 2\pi i\hbar\,\delta(E_2-E_1) \sum_n \frac{\Phi_n(x_2)\Phi_n^*(x_1)}{E_2-E_n+i\epsilon}$$

5-3. $\int_{-\infty}^{\infty} f^*(x) f(x)\, dx = 1$

But those particles having the property G arrive at the point ξ with a probability of certainty and they are described by a w.f. $g(x)$. Now if $f(x)$ is the state which has the highest probability of having the property G, then $f(x) = g(x)$ i.e., $f(x)$ cannot be a diff. function since we are considering one property.

5-4.

$$\psi(x_2,t_2) = \int_{-\infty}^{\infty} K(x_2,t_2;x_1,t_1)\,\psi(x_1)\, dx_1$$

However if the system at time t_2 goes to a state $\chi(x_2)$, $\psi(x_2,t_2)$ may be assumed to be the initial w.f. of the system in this process. Thus the amplitude to end up in this state is

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \chi(x_2)\, K(x_2,t_2;x_1,t_1)\,\psi(x_1)\, dx_1 dx_2$$

for which the probability that the system is found in this state is given by the square of the absolute value of the integral.

5-5.

$$F_{abc\ldots} = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \chi^*_{abc\ldots}(x,y,z,\ldots)\, f(x,y,z,\ldots)\, dx\,dy\,dz\ldots$$

$$f(x,y,z) = \sum_{a'}\sum_{b'}\sum_{c'}\cdots F'_{a'b'c'\ldots}\cdot \chi_{a'b'c'\ldots}(x,y,z,\ldots)$$

$$\Rightarrow F_{abc\ldots}$$

$$= \sum_{a'}\sum_{b'}\sum_{c'}\cdots F'_{a'b'c'\ldots} \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \chi^*_{abc\ldots}(x,y,z,\ldots)\,\chi_{a'b'c'\ldots}(x,y,z,\ldots)\, dx\cdot dy\cdot dz$$

$$= \sum_{a'}\sum_{b'}\sum_{c'}\cdots F'_{a'b'c'\ldots}\cdot \delta(a-a')\,\delta(b-b')\,\delta(c-c')$$

$$= F'_{abc\ldots}$$

5-6. If we take each direction separately

$$\chi_{p_x}(x) = e^{\frac{i}{\hbar}p_x x}$$

$$\chi_{p_y}(y) = e^{\frac{i}{\hbar}p_y\cdot y}$$

$$\chi_{p_z}(z) = e^{\frac{i}{\hbar}p_z\cdot z}$$

$$\Rightarrow \chi_{p_x,p_y,p_z}(x,y,z) = e^{\frac{i}{\hbar}(p_x\cdot x + p_y\cdot y + p_z\cdot z)}$$

$$\text{or } \chi_{p_x,p_y,p_z}(\bar{r}) = e^{\frac{i}{\hbar}\bar{p}\cdot\bar{r}}$$

The result obtained in sec 5-1 is the rule for the transformation between coordinate and mom. representations i.e.,

$$\Psi(\bar{r},t) = \int^{p} e^{\frac{i}{\hbar}(\bar{p}\cdot\bar{r})}\,\Phi(p)\,\frac{d^3p}{(2\pi\hbar)^3}$$

$$\Phi(\bar{p}) = \int^{r} e^{-\frac{i}{\hbar}\bar{p}\cdot\bar{r}}\,\Psi(\bar{r})\,d^3\bar{r}$$

But according to the result of 5-2,

$$\Phi(\bar{p}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \chi^*_{p_x,p_y,p_z}(x,y,z)\cdot f(x,y,z)\,dx\,dy\,dz$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i/\hbar\,\bar{p}\cdot\bar{r}}\,\Psi(\bar{r})\,d^3\bar{r}$$

$$= \int^{r} e^{-i/\hbar\,\bar{p}\cdot\bar{r}}\,\Psi(\bar{r})\,d^3\bar{r}$$

which verifies the result of sec. 5-1.

5-7.

– Transformation between A,B,C,... and coordinate

$$F_{abc\ldots} = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\chi^*_{abc\ldots}(x,y,z,\ldots)\, f(x,y,z,\ldots)\,dx\,dy\,dz$$

And the inverse transformation gives

$$f(x,y,z,\ldots) = \sum_a\sum_b\sum_c\cdots F_{abc\ldots}\,\chi_{abc\ldots}(x,y,z,\ldots)$$

– Transformation between A,B,C,... and mom.

$$F_{abc\ldots} = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\chi^*_{abc\ldots}(p_x,p_y,p_z,\ldots)\, f(p_x,p_y,p_z,\ldots)\,\frac{dp_x\,dp_y\,dp_z}{(2\pi\hbar)^3}$$

Inverse transformation

$$f(p_x,p_y,p_z,\ldots) = \sum_a\sum_b\sum_c\cdots F_{abc\ldots}\,\chi_{abc\ldots}(p_x,p_y,p_z,\ldots)$$

$$\Rightarrow \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \chi^*_{abc\ldots}(x,y,z,\ldots)\, f(x,y,z,\ldots)\,dx\,dy\,dz$$

$$= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \chi^*_{abc\ldots}(P_x,P_y,P_z,\ldots)\, f(P_x,P_y,P_z,\ldots)\, \frac{d^3\bar{p}}{(2\pi\hbar)^3}$$

But from eq. 5-7.,

$$f(x,y,z,\ldots) = \int^P \chi(P_x,P_y,P_z,\ldots\; x,y,z,\ldots)\, f(P_x,P_y,P_z,\ldots)\, \frac{d^3p}{(2\pi\hbar)^3}$$

i.e.,
$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \int^P \chi^*_{abc\ldots}(x,y,z,\ldots)\, \chi_{P_x,P_y,P_z\ldots}(x,y,z)\, f(P_x,P_y,P_z,\ldots) \cdot \frac{d^3\bar{p}}{(2\pi\hbar)^3}\, dx\,dy\,dz$$

$$= \int^P \chi^*_{abc\ldots}(P_x,P_y,P_z,\ldots)\, f(P_x,P_y,P_z,\ldots)\, \frac{d^3p}{(2\pi\hbar)^3}$$

i.e.,
$$\chi^*_{abc\ldots}(P_x,P_y,P_z,\ldots) = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \chi_{abc\ldots}(x,y,z,\ldots)\, \chi_{P_x,P_y,P_z,\ldots}(x,y,z,\ldots)\, dx\,dy\,dz$$

which is the transformation function between the A, B, C, ... and mom. representation.

5-8.

$$I = \int_{-\infty}^{\infty} g^*(x)\, \alpha\, f(x)\,dx$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g^*(x)\, G_A(x,x')\, f(x')\,dx'\,dx$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g^*(x)\, G^*_A(x,x')\, f(x')\,dx'\,dx$$

since $G^*_A = G_A$

$$I = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [G_A(x,x')\, g(x)]^*\, f(x')\,dx'\,dx$$

$$= \int_{-\infty}^{\infty} [\alpha\, g(x)]^*\, f(x)\,dx$$

i.e.,
$$\int_{-\infty}^{\infty} g^*(x)\, \alpha\, f(x)\,dx = \int_{-\infty}^{\infty} [\alpha\, g(x)]^*\, f(x)\,dx$$

5-9. For 1-D

$$G_A(x,x') = \sum_a\sum_b\sum_c\cdots a\, \chi_{abc\ldots}(x)\, \chi^*_{abc\ldots}(x')$$

For $A = P_x$ & in 3-D (considering continuous distribution of mom.)

$$G_{P_x}(x,x')$$

$$= \int_{-\infty}^{\infty} P_x\, e^{\frac{i}{\hbar}\bar{p}\cdot(\bar{r}-\bar{r}')}\, \frac{dP_x\, dP_y\, dP_z}{(2\pi\hbar)^3}$$

$$= \int_{-\infty}^{\infty} P_x\, e^{\frac{i}{\hbar}P_x(x-x')}\, \frac{dP_x}{2\pi\hbar} \int_{-\infty}^{\infty} e^{\frac{i}{\hbar}P_y(y-y')}\, \frac{dP_y}{2\pi\hbar} \int_{-\infty}^{\infty} e^{\frac{i}{\hbar}P_z(z-z')}\, \frac{dP_z}{2\pi\hbar}$$

$$= \frac{\hbar}{i}\frac{d}{dx}\int_{-\infty}^{\infty} e^{\frac{i}{\hbar}P_x(x-x')}\, \frac{dP_x}{2\pi\hbar} \cdot \int_{-\infty}^{\infty} e^{\frac{i}{\hbar}P_y(y-y')}\, \frac{dP_y}{2\pi\hbar} \cdot \int_{-\infty}^{\infty} e^{\frac{i}{\hbar}P_z(z-z')}\, \frac{dP_z}{2\pi\hbar}$$

$G_{P_x}(x,x')$

$= \frac{\hbar}{i}\,\delta'(x-x')\,\delta(y-y')\,\delta(z-z')$

where $\delta' = \frac{d}{dx}\,\delta(x-x')$

$$\hat{P}_x = \int_{-\infty}^{\infty} G_{P_x}(x,x')\,dx'dy'dz'$$

$$\Rightarrow \hat{P}_x f(x) = \frac{\hbar}{i}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta'(x-x')\,\delta(y-y')\,\delta(z-z')\,f(x)\,dx'dy'dz'$$

$$= \frac{\hbar}{i}\int_{-\infty}^{\infty}\delta'(x-x')\,f(x)\,dx'$$

$$= \frac{\hbar}{i}\int_{-\infty}^{\infty}\delta(x-x')\,\frac{df(x)}{dx}\,dx'$$

$$= \frac{\hbar}{i}\,\frac{df(x)}{dx}$$

$$\therefore \hat{P}_x = \frac{\hbar}{i}\,\frac{d}{dx}$$

Also, as defined,

$$\langle \hat{P}_x \rangle = \int_{-\infty}^{\infty} f^*(x)\,\hat{P}_x\,f(x)\,dx$$

$$= \int_{-\infty}^{\infty} f^*(x)\,\frac{\hbar}{i}\,\frac{\partial f}{\partial x}\,dx$$

5-10.

$$\langle x \rangle = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f^*(x)\,G_x(x,x')\,f(x')\,dx'dy'dz'\,dxdydz$$

$$= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f^*(x)\,x\,\delta(x-x')\,\delta'(y-y')\,\delta'(z-z')\,f(x')\,dx'dy'dz'$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big(f^*(x)\,x\,\delta(x-x')\,f(x')\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta'(y-y')\,\delta'(z-z')\,dydz\Big)\,dx'dx\,dy'dz'$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f^*(x)\,x\,\delta(x-x')\,f(x')\,\delta(y-y')\,\delta(z-z')\,dx'dx\,dy\,dz'$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f^*(x)\,x\,\delta(x-x')\,f(x')\,dx'dy$$

$$= \int_{-\infty}^{\infty} f^*(x)\,x\,f(x)\,dx \quad \text{, Q.E.D.}$$

5-11. $G\,\chi_{abc\ldots}(x')$

$$= \int_{-\infty}^{\infty} G_A(x,x')\,\chi_{abc\ldots}(x')\,dx'$$

$$= \sum_{a'}\sum_{b'}\sum_{c'}\cdots\int_{-\infty}^{\infty} a\,\chi_{a'b'c'\ldots}(x)\,\chi^*_{a'b'c'\ldots}(x')\,\chi_{abc\ldots}(x')\,dx'$$

$$= \sum_{a'}\sum_{b'}\sum_{c'}\cdots a\,\chi_{a'b'c'\ldots}(x)\int_{-\infty}^{\infty}\chi^*_{a'b'c'\ldots}(x')\,\chi_{abc\ldots}(x')\,dx'$$

$$= \sum_{a'}\sum_{b'}\sum_{c'}\cdots a\,\chi_{a'b'c'\ldots}(x)\,\delta(a-a')\,\delta(b-b')\,\delta(c-c')\ldots$$

$$= a\,\chi_{abc\ldots}(x)$$

5-12. If x & P_x are simultaneously measurable, they must satisfy the commutation relation

$$[x, P_x] = (xP_x - P_x x) = 0$$

But $(xP_x - P_x x)f$

$$= \frac{\hbar}{i}\Big(x\frac{\partial}{\partial x} - \frac{\partial}{\partial x}x\Big)f(x)$$

$$= \frac{\hbar}{i}\Big(x\frac{\partial}{\partial x} - \frac{\partial}{\partial x}(x f(x))\Big)$$

$$= -\frac{\hbar}{i}\,f(x)$$

i.e., $[x, P_x] = -\hbar/i$ and hence x & P_x cannot be measured simultaneously.

5-13. $\Phi_n(x)$ is the solution of the Schrödinger eq. in which our system has a well determined energy E_n. Thus $\Phi_n(x)$ is the transformation function from the x- rep. to the energy rep.

Chapter 6

6-1. Let $V(x,t) = U(x,t) + v(x,t)$ where v is small in comparison with U.

$$K_{U+v}(b,a) = \int_a^b \left(\exp\left\{\frac{i}{\hbar}\int_{t_a}^{t_b}\left[\frac{m}{2}\dot{x}^2 - U - v\right]dt\right\}\right) \mathcal{D}x(t)$$

$$= \int_a^b \left(\exp\left\{\frac{i}{\hbar}\int_{t_a}^{t_b}\left[\frac{m}{2}\dot{x}^2 - U(x,t)\right]dt\right\} \cdot \exp\left\{-\frac{i}{\hbar}\int_{t_a}^{t_b} v(x,t)\,dt\right\}\right) \mathcal{D}x(t).$$

Expanding the 2nd factor into Taylor's series,

$$\exp\left(-\frac{i}{\hbar}\int_{t_a}^{t_b} v(x,t)\,dt\right) = 1 - \frac{i}{\hbar}\int_{t_a}^{t_b} v(x,t)\,dt + \frac{1}{2!}\left(\frac{i}{\hbar}\right)^2\left(\int_{t_a}^{t_b} v(x,t)\,dt\right)^2 + \ldots$$

$$\Rightarrow K_{U+v} = K_U + K^{(1)}(b,a) + K^{(2)}(b,a) + \ldots \qquad \ldots(*)$$

(assuming that the kernel for U can be worked out.) Where

$$K_U(b,a) = \int_a^b \left(\exp\left\{\frac{i}{\hbar}\int_{t_a}^{t_b}\left[\frac{m}{2}\dot{x}^2 - U(x,t)\right]dt\right\}\right)\mathcal{D}x(t)$$

$$K^{(1)}(b,a) = -\frac{i}{\hbar}\int_a^b \left(\exp\left\{\frac{i}{\hbar}\int_{t_a}^{t_b}\left[\frac{m}{2}\dot{x}^2 - U(x,t)\right]dt\right\} \cdot \int_{t_a}^{t_b} v[x(s),s]\,ds\right)\mathcal{D}x(t)$$

We may also write $K^{(1)}(b,a) = \int_{t_a}^{t_b} F(s)\,ds$

$$K^{(2)}(b,a) = -\frac{1}{2\hbar^2}\int_a^b \left(\exp\left\{\frac{i}{\hbar}\int_{t_a}^{t_b}\left[\frac{m}{2}\dot{x}^2 - U(x,t)\right]dt\right\} \cdot \int_{t_a}^{t_b} v[x(s),s]\,ds \cdot \int_{t_a}^{t_b} v[x(s'),s']\,ds'\right)\mathcal{D}x(t)$$

etc. for the higher terms.

If we analyze $K^{(1)}(b,a)$ we see that the particle is scattered by a potential v at the particular time s. Thus before and after the time s the particle moves in the potential U without being scattered by the additional potential v. Now assuming each path going from a to b passes through the point x_c at the time $s = t_c$ (where particle encounters the potential v) the sum over all paths becomes

$$F(t_c) = \int_{-\infty}^{\infty} K_U(b,c)\, V(x_c,t_c)\, K_U(c,a)\, dx_c$$

where by

$$K^{(1)}(b,a) = -\frac{i}{\hbar}\int_{t_a}^{t_b}\int_{-\infty}^{\infty} K_U(b,c)\, V(x_c,t_c)\, K_U(c,a)\, dx_c\, dt_c \qquad \ldots(**)$$

Similarly,

$$K^{(2)}(b,a) = \left(-\frac{i}{\hbar}\right)^2 \iint K_U(b,c)\, v(c)\, K_U(c,d)\, V(d)\, K_U(d,a)\, d\tau_c\, d\tau_d$$

$$(d\tau = dx\,dt) \qquad \ldots(***)$$

Also, $K(b,a)=0, \quad t_b < t_a \cdots (****)$

i.e., the motion of the particle is described by (*), (**), (***) & (****) and V is treated as a perturbation to U. K_U is the amplitude for the motion in of the system in the unperturbed potential.

6-2. Two particles interacting with a potential $V(x,y)$ (otherwise they are free; x coordinate of one of the particles and y of the other particle.)

$K(b,a)$

$$= \int_a^b \int_a^b \left(\exp\left\{\frac{i}{\hbar}\int_{t_a}^{t_b} \frac{m}{2}\dot{x}^2 dt + \frac{i}{\hbar}\int_{t_a}^{t_b} \frac{M}{2}\dot{y}^2 dt + \frac{i}{\hbar}\int_{t_a}^{t_b} V(x,y,t)dt\right\}\right) Dx(t)\, Dy(t)$$

Assume $V(x,y)$ is small in comparison with the K.E.s of the two particles i.e.,

$$\exp\left(\frac{i}{\hbar}\int_{t_a}^{t_b} V(x,y,t)\,dt\right) = 1 + \frac{i}{\hbar}\int_{t_a}^{t_b} V(x,y,t)\,dt + \frac{1}{2!}\left(\frac{i}{\hbar}\right)^2\left[\int_{t_a}^{t_b} V(x,y,t)\,dt\right]^2 + \cdots$$

$$\Rightarrow K_V(x_b,y_b,t_b;x_a,y_a,t_a) = K_0(x_b,y_b,t_b;x_a,y_a,t_a) + K^{(1)}(x_b,y_b,t_b;x_a,y_a,t_a) + K^{(2)}(x_b,y_b,t_b;x_a,y_a,t_a) + \cdots$$

$K_0(x_b,y_b,t_b;x_a,y_a,t_a)$

$$= \int_a^b \int_a^b \exp\left\{\frac{i}{\hbar}\int_{t_a}^{t_b} \frac{m}{2}\dot{x}^2 dt + \frac{i}{\hbar}\int_{t_a}^{t_b} \frac{M}{2}\dot{y}^2 dt\right\} Dx(t)\, Dy(t)$$

$$= \int_a^b \left(\exp\left(\frac{i}{\hbar} S_x[x(t)]\right)\right) Dx(t) \cdot \int_a^b \left(\exp\left(\frac{i}{\hbar} S_y[y(t)]\right)\right) Dy(t)$$

$$= K_0(x_b,t_b;x_a,t_a) \cdot K_0(y_b,t_b;y_a,t_a)$$

$K^{(1)}(x_b,y_b,t_b;x_a,y_a,t_a)$

$$= \frac{i}{\hbar}\int_a^b \int_a^b \left(\exp\left\{\frac{i}{\hbar}\int_{t_a}^{t_b} \frac{m}{2}\dot{x}^2 dt + \frac{i}{\hbar}\int_{t_a}^{t_b} \frac{M}{2}\dot{y}^2 dt\right\} \cdot \int_{t_a}^{t_b} V[x(s),y(s),s]\,ds\right) Dx(t)\, Dy(t)$$

$K^{(2)}(x_b,y_b,t_b;x_a,y_a,t_a)$

$$= \frac{1}{2!}\left(\frac{i}{\hbar}\right)^2 \int_a^b \int_a^b \left(\exp\left\{\frac{i}{\hbar}\int_{t_a}^{t_b} \frac{m}{2}\dot{x}^2 dt + \frac{i}{\hbar}\int_{t_a}^{t_b} \frac{M}{2}\dot{y}^2 dt\right\} \cdot \int_{t_a}^{t_b} V[x(s),y(s),s]\,ds \cdot \int_{t_a}^{t_b} V[x(s'),y(s'),s']\,ds'\right) Dx(t)\, Dy(t)$$

– $K_0(x_b,y_b,t_b;x_a,y_a,t_a)$ tells us that both particles move without any interaction as two free particles.

– $K^{(1)}(x_b,y_b,t_b;x_a,y_a,t_a)$ indicates that at the particular time s the two particles interact with a potential $V[x(s),y(s),s]$

As in prob. 6-1 we divide each path into two parts i.e., at the time $s = t_c$ all paths pass through the points $x(t_c) = x_c$ & $y(t_c) = y_c$ for each particle resp. Before and after t_c the particles move as free particles.

$$K^{(1)}(b,a) = \frac{i}{\hbar}\int_{t_a}^{t_b} F(s)\,ds$$

where
$$F(s) = \int_a^b\int_a^b \Big(\exp\Big\{\frac{i}{\hbar}\int_{t_a}^{t_b}\frac{m}{2}\dot{x}^2 dt + \frac{i}{\hbar}\int_{t_a}^{t_b}\frac{M}{2}\dot{y}^2 dt\Big\} \cdot V[x(s), y(s)]\Big) \mathcal{D}x(t)\,\mathcal{D}y(t)$$

In line with the above reasoning and using the rules for the combination of amplitudes for events occurring in succession in time we have

$$F(t_c) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K_0(x_b, t_b; x_c, t_c) \cdot K_0(y_b, t_b; y_c, t_c) \cdot V(x_c, y_c, t_c) \cdot K_0(x_c, t_c; x_a, t_a) \cdot K_0(y_c, t_c; y_a, t_a)\, dx_c\, dy_c$$

i.e., $K^{(1)}(x_b, y_b, t_b; x_a, y_a, t_a)$

$$= \frac{i}{\hbar}\int_{t_a}^{t_b}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K_0(x_b, t_b; x_c, t_c) K_0(y_b, t_b; y_c, t_c)\, V(x_c, y_c, t_c) \cdot K_0(x_c, t_c; x_a, t_a) K_0(y_c, t_c; y_a, t_a)\, dx_c\, dy_c\, dt_c$$

Similarly,

$$K^{(2)}(x_b, y_b, t_b; x_a, y_a, t_a)$$
$$= \left(\frac{i}{\hbar}\right)^2 \iint K_0(x_b, t_b; x_c, t_c) K_0(y_b, t_b; y_c, t_c) \cdot V(x_c, y_c, t_c) \cdot K_0(x_c, t_c; x_d, t_d) K_0(y_c, t_c; y_d, t_d)\, V(x_d, y_d, t_d)\, K_0(x_d, t_d; x_a, t_a) K_0(y_d, t_d; y_a, t_a)\, dx_d\, dy_d\, dt_d\, dx_c\, dy_c\, dt_c$$

which can be interpreted as follows.

~~At the times~~ Reading from right to left, before t_d the two particles move without interaction as free particles then interact at t_d. Next they once again move as free particles and interact at t_c at the point (x_c, y_c). Beyond this point they moves as a free particles upto the final point (x_b, y_b). A similar interpretation is given for higher terms.

6-3. For a free particle Schrödinger's eq. for the kernel reduces to

$$-\frac{\hbar}{i}\frac{\partial K_0(b,a)}{\partial t_b} + \frac{\hbar^2}{2m}\frac{\partial^2 K_0(b,a)}{\partial x_b^2} = i\hbar\,\delta(x_b - x_a)\,\delta(t_b - t_a)$$

But for a prticle moving in a potential V

$$K_V(b,a) = K_0(b,a) - \frac{i}{\hbar}\int K_0(b,c)\,V(c)\,K_V(c,a)\,dx_c\,dt_c$$

i.e., $K_0(b,a)$

$$= K_V(b,a) + \frac{i}{\hbar}\int_{t_a}^{t_b}\int_{-\infty}^{\infty} K_0(b,c)V(c)K_V(c,a)\,dx_c\,dt_c$$

$$\Rightarrow i\hbar\,\delta(x_b-x_a)\,\delta(t_b-t_a)$$

$$= -\frac{\hbar}{i}\frac{\partial K_V(b,a)}{\partial t_b} + \frac{\hbar^2}{2m}\frac{\partial^2 K_V(b,a)}{\partial x_b^2}$$

$$+ \frac{i}{\hbar}\int_{t_a}^{t_b}\int_{-\infty}^{\infty}\left(-\frac{\hbar}{i}\frac{\partial K_0(b,c)}{\partial t_b} + \frac{\hbar^2}{2m}\frac{\partial^2 K_0(b,c)}{\partial x_b^2}\right)V(c)K_V(c,a)\,dx_c\,dt_c$$

$$= -\frac{\hbar}{i}\frac{\partial K_V(b,a)}{\partial t_b} + \frac{\hbar^2}{2m}\frac{\partial^2 K_V(b,a)}{\partial x_b^2}$$

$$+ \frac{i}{\hbar}\int_{t_a}^{t_b}\int_{-\infty}^{\infty}\left[i\hbar\,\delta(x_b-x_c)\,\delta(t_b-t_c)\right]V(c)K_V(c,a)\,dx_c\,dt_c$$

Assume t_b is large ($\to\infty$) and t_a is small ($\to -\infty$) so that

$$i\hbar\,\delta(x_b-x_a)\,\delta(t_b-t_a)$$

$$= -\frac{\hbar}{i}\frac{\partial K_V(b,a)}{\partial t_b} + \frac{\hbar^2}{2m}\frac{\partial^2 K_V(b,a)}{\partial x_b^2} + \frac{i}{\hbar}\cdot i\hbar\,V(b)K_V(b,a)$$

$$= -\frac{\hbar}{i}\frac{\partial K_V(b,a)}{\partial t_b} + \frac{\hbar^2}{2m}\frac{\partial^2 K_V(b,a)}{\partial x_b^2} - V(b)K_V(b,a)$$

or

$$-\frac{\hbar}{i}\frac{\partial K_V(b,a)}{\partial t_b} + \frac{\hbar^2}{2m}\frac{\partial^2 K_V(b,a)}{\partial x_b^2} - V(b)K_V(b,a) = i\hbar\,\delta(x_b-x_a)\,\delta(t_b-t_a)$$

6-4.

$$K_V(b,a) = K_0(b,a) - \frac{i}{\hbar}\int K_0(b,c)V(c)K_0(c,a)\,d\tau_c$$

$$-\frac{1}{\hbar^2}\int K_0(b,c)V(c)K_0(c,d)V(d)K_0(d,a)\,d\tau_c\,d\tau_d + \cdots$$

$$\psi(b) = \int K_V(b,a)f(a)\,dx_a$$

Where $f(a)$ is the w.f. at the time $t = t_a$. Thus

$$\psi(b) = \int K_0(b,a)f(a)\,dx_a$$

$$-\frac{i}{\hbar}\iint K_0(b,c)V(c)K_0(c,a)\,d\tau_c\,f(a)\,dx_a$$

$$-\frac{1}{\hbar^2}\iiint K_0(b,c)V(c)K_0(c,d)V(d)K_0(d,a)\,d\tau_c\,d\tau_d\,f(a)\,dx_a$$

$$+ \cdots \qquad \cdots(*)$$

$$= \int K_0(b,a)f(a)\,dx_a$$

$$-\frac{i}{\hbar}\int K_0(b,c)V(c)\Big[\int K_0(c,a)f(a)\,dx_a$$

$$-\frac{i}{\hbar}\iint K_0(c,d)V(d)\Big[\int K_0(d,a)f(a)\,dx_a\Big]\,d\tau_d + \cdots\Big]\,d\tau_c$$

The expression in the brackets has the same form as eq. (*) i.e.,

$$\psi(b) = \int K_0(b,a)f(a)\,dx_a - \frac{i}{\hbar}\int K_0(b,c)V(c)\psi(c)\,d\tau_c$$

With $\phi(b) = \int K_0(b,a)f(a)\,dx_a$,

$$\psi(b) = \phi(b) - \frac{i}{\hbar}\int K_0(b,c)V(c)\psi(c)\,d\tau_c$$

This is an integral eq. which is equivalent to the Schrödinger eq.

$$-\frac{\hbar}{i}\frac{\partial\psi}{\partial t} + \frac{\hbar^2}{2m}\nabla^2\psi + V(\psi) = 0$$

prob. 6-4 continuation.

$$\psi(b) = \varphi(b) - \tfrac{i}{\hbar}\int K_0(b,c)\,V(c)\,\psi(c)\,d\tau_c$$

$$= \int K_0(b,a)\,f(a)\,dx_a$$

$$- \tfrac{i}{\hbar}\int_{t_1}^{t_2}\int_{-\infty}^{\infty} K_0(b,c)\,V(c)\,\psi(c)\,dx_c\,dt_c$$

or
$$\psi(b) = \int_{x_a}\int_{-\infty}^{\infty} K_0(b,c)\,K_0(c,a)\,f(a)\,dx_a\,dx_c$$

$$- \tfrac{i}{\hbar}\int_{t_1}^{t_2}\int_{-\infty}^{\infty} K_0(b,c)\,V(c)\,\psi(c)\,dx_c\,dt_c$$

Where we have used the rules for events occurring in succession in time. Further assume that t_1 & t_2 are only a small time interval ϵ apart i.e.,

$$\psi(b) = \int_{-\infty}^{\infty} K_0(b,c)\,\varphi(c)\,dx_c$$

$$- \frac{i\epsilon}{\hbar}\int_{-\infty}^{\infty} K_0(b,c)\,V(c)\,\psi(c)\,dx_c$$

According to our assumption i.e., $t_2 - t_1 = \epsilon$ (infinitesimal) we can say that as the particle moves from the initial point (x_a, t_a) to the final point (x_b, t_b) it would encounter only a single scattering actually which is the one that occurs at (x_c, t_c). This means in the 2nd integrand we can replace $\psi(c)$, the w.f. of a particle that has been scattered a no. of times by $\varphi(c)$ which is the w.f. of free propagation of our particle. This is just equivalent to taking the 1st order in the perturbation expansion of the particle w.f. of the particle that has scattered more than once i.e.,

$$\psi(b) = \int_{-\infty}^{\infty} K_0(b,c)\,\varphi(c)\,dx_c$$

$$- \frac{i\epsilon}{\hbar}\int_{-\infty}^{\infty} K_0(b,c)\,V(c)\,\varphi(c)\,dx_3$$

$$\equiv \int_{-\infty}^{\infty} K_0(b,c)\,\varphi(c)\left[1 - \frac{i\epsilon}{\hbar}V(c)\right]dx_c$$

Now $\psi(b) = \psi(x_2, t_2)$

$\varphi(c) = \varphi(x_3, t_3)$ and $t_3 < t_2$.

Note that t_3 is the time between t_2 & t_1 and since t_2 & t_1 are a small distance apart so are t_2 & t_3.

Now let $t_2 = t_3 + \epsilon = t + \epsilon$

$x_3 = x_2 + \eta \equiv x + \eta$

Where η could be negative.

$$\Rightarrow \psi(x, t+\epsilon)$$

$$= \int_{-\infty}^{\infty} K_0(x, t+\epsilon;\, x+\eta, t)\,\varphi(x+\eta, t)\left[1 - \frac{i\epsilon}{\hbar}V(x+\eta, t)\right]d\eta$$

$$= \int_{-\infty}^{\infty} \sqrt{\frac{2\pi i\hbar\epsilon}{m}}\; e^{\frac{im}{2\hbar\epsilon}\eta^2}\,\varphi(x+\eta,t)\left[1 - \frac{i\epsilon}{\hbar}V(x+\eta, t)\right]$$

Most of the contribution to the integral comes from η of order $\sqrt{\epsilon\hbar/m}$ and hence we work to first order in ϵ & 2nd order in η. Now ~~since~~ $\epsilon V(x+\eta, t)$ is ~~of higher order that~~ can also be replaced by $\epsilon V(x,t)$ to first order in ϵ.

Now we deduce the 1-D Schrödinger eq. from the integral eq.

6-5.

6-6.

$$v(\bar{q}) = \int e^{\frac{i}{\hbar}\bar{q}\cdot\bar{r}} V(r)\, d^3r$$

where $\bar{q} = \bar{p}_a - \bar{p}_b$

spherical potential

considering of suppose we choose $\bar{q}$ along the z-axis

$$v(q) = \int_0^\infty \int_0^\pi \int_0^{2\pi} V(r) e^{\frac{i}{\hbar} qr\cos\theta} r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$= -2\pi \int_0^\infty \int_0^\pi V(r) e^{\frac{i}{\hbar} qr\cos\theta} r^2 d(\cos\theta)\, dr$$

$$= -2\pi \int_0^\infty V(r) \left[\frac{\hbar}{iqr} e^{\frac{i}{\hbar} qr\cos\theta} \right]_0^\pi r^2 dr$$

$$= -\frac{2\pi\hbar}{iq} \int_0^\infty V(r) \left(e^{-\frac{iqr}{\hbar}} - e^{\frac{iqr}{\hbar}} \right) r\, dr$$

$$= \frac{4\pi\hbar}{q} \int_0^\infty V(r) \frac{\left(e^{\frac{iqr}{\hbar}} - e^{-\frac{iqr}{\hbar}} \right)}{2i} r\, dr$$

$$\text{or } v(q) = \frac{4\pi\hbar}{q} \int_0^\infty r\, V(r) \sin\frac{qr}{\hbar}\, dr$$

for the coulomb potential

$$V(r) = \frac{Ze^2}{r}$$

$$v(q) = \frac{4\pi\hbar}{q} \int_0^\infty r \cdot \frac{Ze^2}{r} \sin\frac{qr}{\hbar}\, dr$$

$$= \frac{4\pi\hbar Ze^2}{q} \int_0^\infty \sin\frac{qr}{\hbar}\, dr$$

Since the integral is oscillatory for the upper limit, we introduce a factor $e^{-\epsilon r}$ (and take the limit $\epsilon \to 0$) so that the integral converges i.e.)

$$\lim_{\epsilon\to 0} v(q,\epsilon) \equiv v(q)$$

$$= \lim_{\epsilon\to 0} \frac{4\pi\hbar Ze^2}{q} \int_0^\infty e^{-\epsilon r} \sin\frac{qr}{\hbar}\, dr$$

$$v(q) = \lim_{\epsilon\to 0} \frac{2\pi\hbar Z e^2}{iq} \int_0^\infty \left(e^{-\epsilon r + \frac{iqr}{\hbar}} - e^{-(\epsilon r + \frac{iqr}{\hbar})} \right) dr$$

where the center of coordinate system is chosen. O is the location of the atom.

$$= \lim_{\epsilon\to 0} \frac{2\pi\hbar Z e^2}{iq} \int_0^\infty \left(e^{(\frac{iq}{\hbar}-\epsilon) r} - e^{-(\frac{iq}{\hbar}+\epsilon) r} \right) dr$$

$$= \lim_{\epsilon\to 0} \frac{2\pi\hbar Z e^2}{iq} \cdot \left[\frac{e^{(\frac{iq}{\hbar}-\epsilon) r}}{\frac{iq}{\hbar}-\epsilon} + \frac{e^{-(\frac{iq}{\hbar}+\epsilon) r}}{\frac{iq}{\hbar}-\epsilon} \right]_0^\infty$$

$$= \lim_{\epsilon\to 0} \frac{2\pi\hbar Z e^2}{iq} \left[\frac{e^{-(\epsilon-\frac{iq}{\hbar}) r}}{\frac{iq}{\hbar}-\epsilon} + \frac{e^{-(\frac{iq}{\hbar}+\epsilon) r}}{\frac{iq}{\hbar}-\epsilon} \right]_0^\infty$$

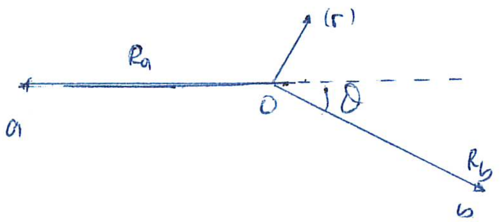
$$= \lim_{\epsilon\to 0} \frac{2\pi\hbar Z e^2}{iq} \cdot \left[-\frac{2}{\frac{iq}{\hbar}-\epsilon} \right]$$

$$= \frac{4\pi\hbar^2 Z e^2}{q^2}$$

$$\frac{d\sigma}{d\Omega} = \left(\frac{m}{2\pi\hbar^2}\right)^2 |v(q)|^2$$

$$= \frac{m^2}{4\pi^2\hbar^4} \cdot \frac{16\pi^2\hbar^4 Z^2 e^4}{q^4}$$

$$= \frac{4m^2 Z^2 e^4}{q^4}$$



$$\bar{q} = \bar{P}_a - \bar{P}_b$$

$$= mu(\bar{i}_a - \bar{i}_b)$$

$\bar{i}_a, \bar{i}_b$ unit vectors in the directions of $-\bar{R}_a$ & $\bar{R}_b$ resp.

$$q^4 = m^4u^4(i_a^2 + i_b^2 - 2i_a \cdot i_b)^2$$

$$= m^4u^4(2 - 2\cos\theta)^2$$

$$= 4m^4u^4(1-\cos\theta)^2$$

$$= 16m^4u^4\left(\frac{1-\cos\theta}{2}\right)^2$$

$$= 16m^4u^4 \sin^4\theta/2$$

$$\Rightarrow \frac{d\sigma}{d\Omega} = \frac{4m^2Z^2e^4}{16m^4u^4\sin^4\theta/2}$$

$$= \frac{Z^2e^4}{16\left(\frac{mu^2}{2}\right)^2\sin^4\theta/2}$$

Which is the Rutherford's scattering cross section. That is the first order Born approximation gives the exact value for the probability of scattering in a Coulomb potential. The contribution of the higher order terms goes to the phase of the amplitude whose square is independent of the phase.

6-7.

$$\nabla^2 V(\bar{r}) = -4\pi e^2 \delta(\bar{r})$$

Fourier transforming $V(\bar{r})$

$$V(\bar{r}) = \frac{1}{2\pi\hbar}\int e^{-\frac{i\bar{q}\cdot\bar{r}}{\hbar}} v(\bar{q})\, d^3\bar{q}$$

$$\Rightarrow \nabla^2 V(\bar r) = -\frac{q^2}{\hbar^2}\frac{1}{2\pi\hbar}\int e^{-\frac{i\bar q\cdot\bar r}{\hbar}} v(\bar q)\, d^3\bar q$$

$$= -\frac{q^2}{\hbar^2} V(\bar r)$$

$$= -4\pi e^2 \rho(\bar r)$$

i.e.) $V(\bar r) = \frac{4\pi e^2\hbar^2}{q^2}\rho(\bar r)$

$$\therefore v(\bar q) = \int e^{\frac{i\bar q\cdot\bar r}{\hbar}} V(\bar r)\, d^3\bar r$$

$$= \frac{4\pi e^2\hbar^2}{q^2}\int e^{\frac{i\bar q\cdot\bar r}{\hbar}}\rho(\bar r)\, d^3\bar r$$

For an atom the charge density is singular at the nucleus so that it can be represented as a δ function of r of strength Z where Z is the charge on the nucleus. Here

$$v(\bar q) = \frac{4\pi e^2\hbar^2}{q^2}\left[Z - \int \rho_e(\bar r)\, e^{\frac{i\bar q\cdot\bar r}{\hbar}} d^3\bar r\right]$$

Where ρ_e is the density of atomic electrons.

6-8.

$$V(r) = \frac{Ze^2}{r} e^{-r/a}$$

$$\Rightarrow v(q) = \frac{4\pi\hbar}{q}\int_0^\infty r V(r) \sin\frac{qr}{\hbar}\, dr$$

$$= \frac{2\pi\hbar Ze^2}{iq}\int_0^\infty e^{-\frac{r}{a}}\left(e^{\frac{iqr}{\hbar}} - e^{-\frac{iqr}{\hbar}}\right) dr$$

$$= \frac{2\pi\hbar Ze^2}{iq}\int_0^\infty \left(e^{-(1/a - \frac{iq}{\hbar})r} - e^{-(\frac{1}{a}+\frac{iq}{\hbar})r}\right) dr$$

$$= \frac{2\pi\hbar Ze^2}{iq}\left[\frac{e^{-(\frac{1}{a}-\frac{iq}{\hbar})r}}{-(\frac{1}{a}-\frac{iq}{\hbar})} - \frac{e^{-(\frac{1}{a}+\frac{iq}{\hbar})r}}{(1/a + \frac{iq}{\hbar})}\right]_0^\infty$$

In the upper limit we get zero so that

$$v(q) = \frac{2\pi\hbar Ze^2}{iq}\left[\frac{1}{1/a - \frac{iq}{\hbar}} - \frac{1}{1/a + \frac{iq}{\hbar}}\right]$$

$$= \frac{2\pi\hbar Ze^2}{iq}\times\frac{2iq/\hbar}{1/a^2 + q^2/\hbar^2}$$

$$\text{or}\quad v(q) = \frac{4\pi Ze^2\hbar^2}{q^2 + \hbar^2/a^2}$$

$$\frac{d\sigma}{d\Omega} = \left(\frac{m}{2\pi\hbar^2}\right)^2\left(\frac{16\pi^2 Z^2 e^4\hbar^4}{(q^2+\hbar^2/a^2)^2}\right)$$

$$= \frac{4m^2 Z^2 e^4}{(q^2 + \hbar^2/a^2)^2}$$

But $q^2 = 4m^2u^2\sin^2\theta/2$ from prob. 6-6.

$$\text{i.e.)}\quad \frac{d\sigma}{d\Omega} = \frac{Z^2e^4}{\left(\frac{m^2u^2}{2}\left[4\sin^2\frac{\theta}{2} + \frac{\hbar^2}{m^2u^2a^2}\right]\right)^2}$$

$$= \frac{Z^2e^4}{\left\{\frac{p^2}{2m}\left(4\sin^2\frac{\theta}{2} + \frac{\hbar^2}{p^2a^2}\right)\right\}^2}$$

$$\sigma = \int d\sigma$$

$$= \int_0^\pi \frac{Z^2e^4\sin\theta\, d\theta}{\left\{\frac{p^2}{2m}\left(4\sin^2\frac{\theta}{2} + \frac{\hbar^2}{p^2a^2}\right)\right\}^2}\int_0^{2\pi} d\phi$$

$$= \frac{2\pi\cdot Z^2e^4}{(p^2/2m)^2}\int_0^\pi \frac{\sin\theta\, d\theta}{\left(4\sin^2\frac{\theta}{2} + \frac{\hbar^2}{p^2a^2}\right)^2}$$

$$I = \int_0^\pi \frac{\sin\theta\, d\theta}{(4\sin^2\theta/2 + c^2)^2}, \quad c^2 = \frac{\hbar^2}{p^2a^2}$$

$$= \int_0^\pi \frac{2\sin\theta/2\,\cos\theta/2\, d\theta}{(4\sin^2\theta/2 + c^2)^2}$$

$$I = \int_0^{\pi} \frac{4 \sin\theta/2 \, d \sin\theta/2}{(4 \sin^2\theta/2 + c^2)^2}$$

$$= \int_0^1 \frac{4x \, dx}{(4x^2+c^2)^2}$$

$$= \frac{1}{2}\int_0^1 \frac{d(4x^2+c^2)}{(4x^2+c^2)^2}$$

$$= -\frac{1}{2(4x^2+c^2)}\Big|_0^1$$

$$= \frac{1}{2}\left[\frac{1}{c^2} - \frac{1}{4+c^2}\right]$$

$$= \frac{2}{c^2(4+c^2)}$$

$$= \frac{2}{4\frac{\hbar^2}{p^2a^2} + \frac{\hbar^4}{p^4a^4}}$$

$$= \frac{p^2a^2/2\hbar^2}{1+\hbar^2/(2pa)^2}$$

$$\Rightarrow \sigma = \frac{2\pi Z^2 e^4 \cdot p^2a^2/2\hbar^2}{(p^2/2m)^2 (1+\hbar^2/(2pa)^2)}$$

$$= \frac{4\pi a^2 \, Z^2e^4/u^2\hbar^2}{1+\hbar^2/(2pa)^2}$$

$$\text{or } \sigma = \frac{\pi(2aZe^2/u\hbar)^2}{1+\hbar^2/(2pa)^2}$$

6-9. Here the incoming electrons strongly interact with the atom i.e., the effective target area on which the electrons fall (cross section) is ~~much larger than they get deflected by large angles~~ must be small just the core of the charge distribution which is thus small.

$$R = R_0 A^{1/3} = 1.2 \times 10^{-13} A^{1/3} \text{ cm}$$

$$V \equiv \text{Volume} = \tfrac{4}{3}\pi R^3$$

$$\rho_e = \frac{Ze}{\frac{4}{3}\pi R^3} \equiv \text{constant for uniform distribution of the charge.}$$

i.e., $$R = \left(\frac{4\pi \rho_e}{3Ze}\right)^{1/3}$$

In order to produce an appreciable effect on the scattering the mom. of the incoming electrons should be large. Also the large scattering angles would be observed since the mom. transfer is large.

6-10. If only each atom is located at the center of a coordinate, say,

$$K_A^{(1)}(5, \bar{r}) = f_A(\bar{\xi}) = \int_{t_r}^{t_s}\int_{-\infty}^{\infty} K_0(s,c) V(c) K_0(c,r) \, d^3\bar{r}_c \, dt_c$$

$$K_B^{(1)}(s, \bar{r}) = f_B(\bar{\xi}) = \int_{t_r}^{t_s}\int_{-\infty}^{\infty} K_0(s,c') V(c') K_0(c',r) \, d^3\bar{r}_{c'} \, dt_{c'}$$

Each has an expression exactly similar to

$$K^{(1)}(b,a) = -\frac{i}{\hbar}\left(\frac{m}{2\pi i\hbar}\right)^{5/2} \frac{u}{T^{1/2} R_a R_b} \left[\exp\left(\frac{imu^2T}{2\hbar}\right) \int \exp\left(\frac{i\bar{q}\cdot\bar{r}}{\hbar}\right) V(r) \, d^3\bar{r}\right]$$

where $\bar{r}$ is the vector from the atom (of the centre) to the scattered particle. Now if the particle gets scattered when the two atoms are at $\bar{a}$ & $\bar{b}$ its vector from the center will be resp.

$\bar{R} = \bar{a} + \bar{r}$

$\bar{R}' = \bar{b} + \bar{r}'$

Thus in $\exp \frac{i\bar{\xi}\cdot\bar{R}}{\hbar}$ or $\exp \frac{i\bar{\xi}\cdot\bar{R}'}{\hbar}$ we have a constant factor $\exp \frac{i\bar{\xi}\cdot\bar{a}}{\hbar}$ or $\exp \frac{i\bar{\xi}\cdot\bar{b}}{\hbar}$ resp so that in the presence of the two atoms the kernel for the scattered particle becomes (using Born's first approximation)

$$K^{(1)} = e^{\frac{i\bar{\xi}\cdot\bar{a}}{\hbar}} f_A(\bar{\xi}) + e^{\frac{i\bar{\xi}\cdot\bar{b}}{\hbar}} f_B(\bar{\xi})$$

probability of scattering at the particular value of $\bar{\xi}$ is

$$|K^{(1)}|^2 = \left(e^{\frac{i\bar{\xi}\cdot\bar{a}}{\hbar}} f_A + e^{\frac{i\bar{\xi}\cdot\bar{b}}{\hbar}} f_B\right) \cdot \left(e^{\frac{-i\bar{\xi}\cdot\bar{a}}{\hbar}} f_A + e^{\frac{-i\bar{\xi}\cdot\bar{b}}{\hbar}} f_B\right)$$

(f_A, f_B are real when integrated)

$$= f_A^2 + f_A f_B e^{\frac{i\bar{\xi}}{\hbar}\cdot(\bar{a}-\bar{b})} + f_A f_B e^{-\frac{i\bar{\xi}}{\hbar}\cdot(\bar{a}-\bar{b})} + f_B^2$$

or $$|K^{(1)}|^2 = f_A^2 + f_B^2 + 2 f_A f_B \cos\left(\frac{\bar{\xi}}{\hbar}\cdot\bar{d}\right)$$

where $\bar{d} = \bar{a} - \bar{b}$.

6-11. If the molecules are distributed randomly we average over the position of the molecules so as to get

$$\overline{|K^{(1)}|^2} \sim \frac{1}{\bar{d}} \int_0^{\bar{d}} (f_A^2 + f_B^2 + 2 f_A f_B \cos(\tfrac{\bar{\xi}}{\hbar}\cdot\bar{d}'))\, d\bar{d}'$$

Actually the relative position of the atoms in a molecule is not zero; it is the sum of the radii if the could touch one another. But since this dist is small we take it to be zero

$$\therefore \overline{|K^{(1)}|^2} \sim \frac{1}{\bar{d}}(f_A^2 + f_B^2)\,\bar{d} + \frac{1}{\frac{\bar{\xi}}{\hbar}\cdot\bar{d}} \sin\left(\frac{\bar{\xi}}{\hbar}\cdot\bar{d}\right)$$

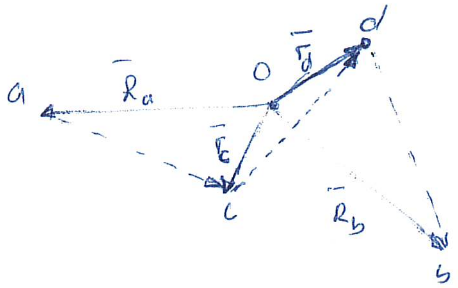
$$= f_A^2 + f_B^2 + \frac{\sin(\frac{\bar{\xi}}{\hbar}\cdot\bar{d})}{(\bar{\xi}/\hbar\cdot\bar{d})}$$

For a polyatomic molecule we can proceed in a similar fashion. (?)

6-12. If $V(\bar{r})$ is independent of time, the time integral of the 2nd order term gives

$$K^{(2)}(b,a) = \left(\frac{m}{2\pi i\hbar^2}\right)^2 \left(\frac{m}{2\pi i\hbar T}\right)^{3/2} \int^{\bar{r}_c} \int^{\bar{r}_d} \frac{r_{cd} + r_{ac} + r_{bd}}{r_{cd}\, r_{ac}\, r_{bd}} \left[\exp\left(\frac{im}{2\hbar T}\right)(r_{cd} + r_{ac} + r_{bd})^2\right] V(\bar{r}_c) V(\bar{r}_d)\, d^3\bar{r}_c\, d\bar{r}_d^3$$

where we have used the arrangement shown below.



If $V(\bar{r})$ becomes negligibly small at distances which are short compared to R_a & R_b, the cross-section can be shown to be given by $|f|^2$ where f, the scattering amplitude including the first order term is

$$f = \frac{m}{2\pi\hbar^2}\int^{r} e^{-\frac{i\bar{p}_b\cdot\bar{r}}{\hbar}} V(\bar{r})\, e^{\frac{i\bar{p}_a\cdot\bar{r}}{\hbar}} d^3\bar{r}$$

$$+\left(\frac{m}{2\pi\hbar^2}\right)^2\int^{r_c}\int^{r_d} e^{-\frac{i}{\hbar}\bar{p}_b\cdot\bar{r}_d} V(\bar{r}_d)\left(\frac{1}{r_{cd}}\right) \cdot e^{\frac{i}{\hbar}p r_{cd}} V(\bar{r}_c)\, e^{\frac{i}{\hbar}\bar{p}_a\cdot\bar{r}_c} d^3\bar{r}_c\, d^3\bar{r}_d$$

+ higher order terms

where $\bar{p}_a$ & $\bar{p}_b$ are the momenta of the electrons traveling in the directions of $-\bar{R}_a$ & $\bar{R}_b$ resp. p is the magnitude of the mom. which is approximately unchanged for elastic scattering.

6-13. W.F.

$$\psi(b) = \int K_0(b,a)\psi(a)\, d^3\bar{r}_a$$

$$-\frac{i}{\hbar}\iiint K_0(b,c)\, V(c)\, K_0(c,a)\, d^3\bar{r}_c\, dt_c\, \psi(a)\, d^3\bar{r}_a$$

$$+ \ldots$$

Let $\psi(a) = e^{\frac{i\bar{p}_a\cdot\bar{r}_a}{\hbar}} e^{\frac{-iE_a t_a}{\hbar}}$

where the incoming electrons are assumed to have a plane w.f. with mom p_a and energy $E_a = p_a^2/2m$.

1st term

$$\phi(b) = \int K_0(b,a)\psi(a)\, d^3\bar{r}_a$$

$$= \left(\frac{m}{2\pi i\hbar t_b}\right)^{3/2}\int e^{\frac{im(\bar{r}_b-\bar{r}_a)^2}{2\hbar t_b}} \cdot e^{\frac{i\bar{p}_a\cdot\bar{r}_a}{\hbar}} d^3\bar{r}_a \quad \ldots(*)$$

$(t_a \equiv 0)$

$$\frac{m}{2t_b}(\bar{r}_b-\bar{r}_a)^2 + \bar{p}_a\cdot\bar{r}_a$$

$$\equiv \frac{m}{2t_b}\left[\bar{r}_a - \left(\bar{r}_b - \frac{t_b}{m}\bar{p}_a\right)\right]^2 + \bar{p}_a\cdot\bar{r}_b - \frac{p_a^2}{2m}t_b$$

which upon integration in (*) gives

$$\phi(b) = e^{\frac{i}{\hbar}\bar{p}_a\cdot\bar{r}_b}\, e^{-\frac{i}{\hbar}E_a t_b}$$

Similarly,

$$\int K_0(c,a)\psi(a)\, d^3\bar{r}_a = e^{\frac{i}{\hbar}\bar{p}_a\cdot\bar{r}_c}\, e^{-\frac{i}{\hbar}E_a t_c}$$

(i.e.) $\psi(\bar{r}_b, t_b)$

$$= e^{\frac{i}{\hbar}\bar{p}_a\cdot\bar{r}_b} \cdot e^{-\frac{i}{\hbar}E_a t_b}$$

$$-\frac{i}{\hbar}\int_0^{t_b}\int^{\bar{r}_c} K_0(\bar{r}_b, t_b; \bar{r}_c, t_c) \cdot V(\bar{r}_c, t_c)\, e^{\frac{i}{\hbar}\bar{p}_a\cdot\bar{r}_c} \cdot e^{-\frac{i}{\hbar}E_a t_c} d^3\bar{r}_c\, dt_c$$

Assume $V(\bar{r}_c, t_c) \equiv V(\bar{r}_c)$, independent of time

$$I = \int_0^{t_b} K_0(\bar r_b, t_b; \bar r_c, t_c)\, e^{-i\frac{E_a}{\hbar}t_c}\, dt_c$$

$$= \int_0^{t_b} \left(\frac{m}{2\pi i\hbar(t_b-t_c)}\right)^3 e^{\frac{im}{2\hbar}\frac{(\bar r_b-\bar r_c)^2}{t_b-t_c}}\, e^{-i\frac{E_a}{\hbar}t_c}\, dt_c$$

$$= -e^{-i\frac{E_a}{\hbar}t_b}\left(\frac{m}{2\pi i\hbar}\right)^{3/2} \int_0^{t_b} \frac{1}{(t_b-t_c)^{3/2}}\, e^{\frac{im}{2\hbar}\frac{(\bar r_b-\bar r_c)^2}{t_b-t_c}} \cdot e^{i\frac{E_a}{\hbar}(t_b-t)} \cdot d(t_b-t_c)$$

$$= -e^{-i\frac{E_a}{\hbar}t_b}\left(\frac{m}{2\pi i\hbar}\right)^{3/2} \int_{t_b}^{0} \frac{1}{t^{3/2}} \cdot e^{\frac{im}{2\hbar}\frac{(\bar r_b-\bar r_c)^2}{t}} \cdot e^{i\frac{E_a}{\hbar}t}\, dt$$

Let $u = t^{-1/2}$, $du = -\frac{t^{-3/2}}{2}dt$

i.e., $$I = 2e^{-i\frac{E_a}{\hbar}t_b}\left(\frac{m}{2\pi i\hbar}\right)^{3/2} \int_{1/\sqrt{t_b}}^{\infty} e^{\frac{im}{2\hbar}(\bar r_b-\bar r_c)^2 u^2} \cdot e^{\frac{iE_a}{\hbar u^2}}\, du$$

$1/\sqrt{t_b} \to 0$ as $t_b \to \infty$.

$\frac{1}{u^2} = t_b - t_c$ and for a free particle moving between the points c & b,

$$v = \frac{r_b - r_c}{t_b - t_c} = \frac{r_{bc}}{t_b - t_c}$$

$$\Rightarrow t_b - t_c = \frac{r_{bc}}{v}$$

$$\frac{E_a}{\hbar u^2} = \frac{p_a^2}{2m\hbar}\cdot\frac{r_{bc}}{v}$$

$$= \frac{1}{\hbar}\frac{p_a^2}{2mv} r_{bc}$$

$$= \frac{1}{\hbar}\frac{p_a^2}{p} r_{bc}$$

$$= \frac{p\, r_{bc}}{\hbar}$$

$$I = 2\left(\frac{m}{2\pi i\hbar}\right)^{3/2} e^{-i\frac{E_a}{\hbar}t_b} \cdot e^{\frac{ipr_{bc}}{\hbar}} \int_0^{\infty} e^{\frac{im}{2\hbar}(\bar r_b-\bar r_c)^2 u^2}\, du$$

$$= \left(\frac{m}{2\pi i\hbar}\right)^{3/2} e^{-i\frac{E_a}{\hbar}t_b} \cdot e^{\frac{ipr_{bc}}{\hbar}} \int_{-\infty}^{\infty} e^{\frac{im}{2\hbar}r_{bc}^2 u^2}\, du$$

$$= \left(\frac{m}{2\pi i\hbar}\right)^{3/2} e^{-i\frac{E_a}{\hbar}t_b} \cdot e^{\frac{ipr_{bc}}{\hbar}} \cdot \left(\frac{2\pi i\hbar}{m}\right)^{1/2} \cdot \frac{1}{r_{bc}}$$

$$= \frac{m}{2\pi i\hbar}\cdot\frac{1}{r_{bc}}\, e^{-i\frac{E_a}{\hbar}t_b} \cdot e^{\frac{ipr_{bc}}{\hbar}}$$

or $\psi(\bar r_b, t_b)$

$$= e^{-i\frac{E_a}{\hbar}t_b}\left\{e^{\frac{i\bar p_a\cdot\bar r_b}{\hbar}} - \frac{m}{2\pi\hbar^2}\int^{\bar r_c} \frac{e^{\frac{ipr_{bc}}{\hbar}}}{r_{bc}} V(\bar r_c)\, e^{\frac{i}{\hbar}\bar p_a\cdot\bar r_c}\, d^3\bar r_c\right\}$$

In the limit of short distances r_c $\left(\frac{1}{r_b - r_c} \sim \frac{1}{r_b}\right)$ if we assume $V(\bar r_c) \to 0$, the 2nd term above becomes

$$-\frac{m}{2\pi\hbar^2}\frac{e^{\frac{ipr_b}{\hbar}}}{r_b}\int^{r_c} e^{\frac{i}{\hbar}(\bar p_a - \bar p)\cdot\bar r_c}\, V(\bar r_c)\, d^3 r_c$$

$$= -\frac{m}{2\pi\hbar^2}\frac{e^{\frac{ipr_b}{\hbar}}}{r_b}\, v(\bar q)$$

$$\therefore \psi(\bar r_b, t_b) = e^{-i\frac{E_a}{\hbar}t_b}\, e^{\frac{i\bar p_a\cdot\bar r_b}{\hbar}} + f(\bar q)\,\frac{e^{\frac{ipr_b}{\hbar}}}{r_b}$$

where $f = -\frac{m}{2\pi\hbar^2}v(\bar q)$

We interprete the above result as $r_0 \to \infty$ for $\psi(r_0, t_0)$ as follows.

The first term represents the w.f. of non scattered electrons $e^{i(\vec{k}_0)\cdot \vec{r}_0}$ and the 2nd piece represents the spatial w.f. of scattered electrons. It has the form of a spherical wave radiating outward from the center of the scattering atom.

To get the differential cross section we need the ratio of the probability flowing into a solid angle $d\Omega$ per second to the incident current density. So we find j_{sc} & j_{ins} as follows (consider motion along the z-axis).

$$\psi(z) = e^{ikz} + f(\theta,\varphi)\frac{e^{ikr}}{r}$$

It is seen thus as $r \to \infty$,

$\psi_{sc} = f(\theta,\varphi)\dfrac{e^{ikr}}{r} \to 0$ that we only consider the plane wave e^{ikz}

$$|\bar{j}_{inc}| = \left|\frac{\hbar}{2mi}\left(e^{-ikz}\nabla e^{ikz} - e^{ikz}\nabla e^{-ikz}\right)\right|$$

$$= \frac{\hbar k}{m}$$

But to calculate $\bar{j}_{sc}$ into $d\Omega$ we cannot use this trick since ψ_{sc} never dominates e^{ikz}. We rather say e^{ikz} is really an abs fraction for a wave that is limited in the transverse direction by some $r_{max} \gg r_0$.. Thus in any realistic description only ψ_{sc} will survive as $r \to \infty$, for $\theta \neq 0$ (Figure - Shankar.)

$$j_{sc} = \frac{\hbar}{2mi}\left(\psi_{sc}^{*}\nabla\psi_{sc} - \psi_{sc}\nabla\psi_{sc}^{*}\right)$$

$$\nabla = \bar{e}_r\frac{\partial}{\partial r} + \bar{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \bar{e}_\varphi\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}$$

As $r \to \infty$, we may neglect the last two pieces i.e.,

$$\frac{\partial}{\partial r} f(\theta,\varphi)\frac{e^{ikr}}{r} = f(\theta,\varphi)\, ik\frac{e^{ikr}}{r} + O\left(\frac{1}{r^2}\right)$$

whence

$$\bar{j}_{sc} = \frac{e_r}{r}|f|^2\frac{\hbar k}{m}$$

Probability flows into $d\Omega$ at the rate

$$R(d\Omega) = \bar{j}_{sc}\cdot\bar{e}_r r^2 d\Omega$$

$$= |f|^2\frac{\hbar k}{m}d\Omega$$

Since it arrives at the rate

$$j_{inc} = \frac{\hbar k}{m}\ \sec^{-1}\cdot\text{area}^{-1}$$

$$\frac{d\sigma}{d\Omega}d\Omega = \frac{R(d\Omega)}{j_{inc}} = |f|^2 d\Omega$$

i.e.,
$$\frac{d\sigma}{d\Omega} = |f(\theta,\varphi)|^2$$

which is the differential cross section in the time independent case.

6-14. Look at the back of this page

6-15. From prob. 5-4. We

have

$$\chi(x_2) = \int_{-\infty}^{\infty} \chi^*(x_2)\, \psi(x_2, t_2)\, dx_2$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \chi^*(x_2)\, k(x_2,t_2;x_1,t_1)\, \psi(x_1)\, dx_1\, dx_2$$

$\psi(x_1)$ being the initial w.f. at time t_1 and this the transition ampl. from the state n to the state m.

To zeroth order in V

$$\lambda_{mn} = \delta_{mn} e^{-\frac{iE_n}{\hbar}(t_2-t_1)}$$

$$= \int_{-\infty}^{\infty} \Phi_m^*(x_2,t_2)\, \phi_n(x_2,t_2)\, dx_2$$

To first order

$$\lambda_{mn}^{(1)} = -\frac{i}{\hbar}\int_{-\infty}^{\infty}\int_{t_1}^{t_2} \Phi_m^*(x_3)\, V(x_3,t_3)\, \Phi_n(x_3)\, dx_3\, e^{\frac{i}{\hbar}E_m(t_2-t_3)} \cdot e^{-\frac{iE_n}{\hbar}(t_3-t_1)}\, dt_3$$

$$= -\frac{i}{\hbar} e^{-i/\hbar(E_m t_2 - E_n t_1)} \int_{t_1}^{t_2} V_{mn}(t_3)\, e^{i/\hbar(E_m-E_n)t_3}\, dt_3$$

As seen the above two cases means that in the first case the system goes from the state n to the state m without any perturbation and in the 2nd case it goes to the state n to m under influence of the potential $V(x_3,t_3)$. So whatever order of V we consider λ_{mn} is a transition amplitude from the n^{th} state to the mth state i.e., it satisfies the def. given in prob. 5-4.

6-16. We have

$$V_{mn}(t_3) = \int_{-\infty}^{\infty} \Phi_m^*(x_3)\, V(x_3,t_3)\, \Phi_n(x_3)\, dx_3$$

The system might have come from an initial state $\Phi(x_1)$

i.e.,

$$V_{mn}(t_3) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi_m^*(x_3)\, V(x_3,t_3)\, k(x_3,t_3;x_1,t_1)\, \Phi_n(x_1)\, dx_1 dx_3$$

which is a sum over alternatives

6-17.

$$\lambda_{mn}^{(1)} = -\frac{i}{\hbar} e^{-\frac{i}{\hbar}E_m t_2} \cdot e^{+i/\hbar E_n t_1} \int_{t_1}^{t_2} V_{mn}(t_3)\, e^{i/\hbar(E_m-E_n)t_3}\, dt_3$$

The first two terms (factors) say resp. that the system has gone to the state m at t_2 and it was in the state n at time t_1. The integrand tells us that a transition has occurred from the state n to the state m at time t_3 due to a single scattering by a potential $V(x_3,t_3)$.

For the 2nd order coeff.

$$\lambda_{mn}^{(2)} = -\frac{1}{\hbar^2}\int_{t_1}^{t_2}\int_{t_1}^{t_2} \sum_k e^{-i/\hbar E_m(t_2-t_4)}\, V_{mk}(t_4) \cdot e^{-i/\hbar E_k(t_4-t_3)}\, V_{kn}(t_3)\, e^{-i/\hbar E_n(t_3-t_1)}\, dt_3\, dt_4.$$

$e^{-i/\hbar E_n(t_3-t_1)}$ is the amplitude that the system is in the state n at the time t_3. At t_3 a scattering has taken place and the amplitude it will end up in the state k and will continue up to the time t_4 is $e^{-i/\hbar E_k(t_4-t_3)}$, once more a scattering takes place at t_4 and the system remains in the state m until t_2 with a probability amplitude $e^{-i/\hbar E_m(t_2-t_4)}$ and in this last case k represents all the available states from which the system is scattered between t_3 & t_4.

Derivation of $\lambda_{km}^{(2)}$.

Of course it is the amplitude that after two scatterings the system will go to a state k from the initial state m.

$$K_V^{(2)}(2,1) = -\frac{1}{\hbar^2}\iint K_u(2,4)V(4)K_u(4,3) \cdot V(3)K_u(3,1)\,d\tau_3\,d\tau_4$$

$$[d\tau_3 = dx_3\,dt_3;\ d\tau_4 = dx_4\,dt_4]$$

But $K_u(j,i) = \sum_n \Phi_n(x_j)\Phi_n^*(x_i) \cdot e^{-i/\hbar E_n(t_j - t_i)}$

$$\Rightarrow K_V^{2}(2,1) = -\frac{1}{\hbar^2}\int_{t_1}^{t_2} dt_4 \int_{t_1}^{t_4} dt_3 \int_{-\infty}^{\infty} dx_3 \int_{-\infty}^{\infty} dx_4$$

$$\sum_m e^{-i/\hbar E_m(t_2 - t_4)} V(x_4, t_4)\Phi_m(x_2)\Phi_m^*(x_4)$$

$$\sum_k e^{-i/\hbar E_k(t_4 - t_3)} V(x_3, t_3)\Phi_k(x_4)\Phi_k^*(x_3)$$

$$\sum_n e^{-i/\hbar E_n(t_3 - t_1)} \Phi_n(x_3)\Phi_n^*(x_1)$$

$$= -\frac{1}{\hbar^2}\sum_m\sum_k\sum_n \Big\{ \int_{t_1}^{t_2}\int_{t_1}^{t_4} \Big(\int_{-\infty}^{\infty}\Phi_m^*(x_4)V(x_4,t_4)\Phi_k(x_4)dx_4\Big) \Big(\int_{-\infty}^{\infty}\Phi_k^*(x_3)V(x_3,t_3)\Phi_n(x_3)dx_3\Big)$$

$$\cdot e^{-i/\hbar E_m(t_2 - t_4)} \cdot e^{-i/\hbar E_k(t_4 - t_3)} \cdot e^{-i/\hbar E_n(t_3 - t_1)}\, dt_3\, dt_4 \Big\}\Phi_m(x_2)\Phi_n^*(x_1)$$

$$= -\frac{1}{\hbar^2}\sum_m\sum_k\sum_n \Big(\int_{t_1}^{t_2}\int_{t_1}^{t_4} V_{mk}(t_4)V_{kn}(t_3) \cdot e^{-i/\hbar E_m(t_2 - t_4)} \cdot e^{-i/\hbar E_k(t_4 - t_3)} \cdot e^{-i/\hbar E_n(t_3 - t_1)}\,dt_3\,dt_4\Big)$$

$$\Phi_m(x_2)\Phi_n^*(x_1)$$

$$= \sum_n\sum_m \Big\{ -\frac{1}{\hbar^2}\sum_k \int_{t_1}^{t_2}\Big[\int_{t_1}^{t_4} e^{-i/\hbar E_m(t_2 - t_4)} V_{mk}(t_4) \cdot e^{-i/\hbar E_k(t_4 - t_3)} V_{kn}(t_3)\, e^{-i/\hbar E_n(t_3 - t_1)}\,dt_3\Big]dt_4$$

$$\Phi_m(x_2)\Phi_n^*(x_1)$$

$$= \sum_n\sum_m \lambda_{mn}^{(2)}(t_2,t_1)\Phi_m(x_2)\Phi_n^*(x_1)$$

i.e., $\lambda_{mn}^{(2)}(t_2,t_1)$

$$= -\frac{1}{\hbar^2}\int_{t_1}^{t_2}\Big[\int_{t_1}^{t_4}\sum_k e^{-i/\hbar E_m(t_2 - t_4)} V_{mk}(t_4)\, e^{-i/\hbar E_k(t_4 - t_3)} V_{kn}(t_3)\, e^{-i/\hbar E_n(t_3 - t_1)}\,dt_3\Big]dt_4$$

6-18. We have

$$A_{mn}(t_2,t_1)$$

$$= \delta_{mn} e^{-i/\hbar E_n(t_2 - t_1)} + \lambda_{mn}^{(1)} + \lambda_{mn}^{(2)} + \cdots$$

$$= \delta_{mn} e^{-i/\hbar E_n(t_2 - t_1)} - \frac{i}{\hbar}\int_{t_1}^{t_2}\int_{-\infty}^{\infty}\Phi_m^*(x_3)V(x_3,t_3)\Phi_n(x_3)\,dx_3\, e^{i/\hbar[E_m(t_3 - t_2) - E_n(t_3 - t_1)]}\,dt_3$$

$$-\frac{1}{\hbar^2}\int_{t_1}^{t_2}\Big[\int_{t_1}^{t_4}\sum_k e^{-i/\hbar E_m(t_2 - t_4)} V_{mk}(t_4)\cdot e^{-i/\hbar E_k(t_4 - t_3)} V_{kn}(t_3)\, e^{-i/\hbar E_n(t_3 - t_1)}\,dt_3\Big]dt_4 + \cdots$$

$$\Lambda_{mn}(t_2,t_1)$$

$$= \delta_{mn}\, e^{-\frac{i}{\hbar}E_n(t_2-t_1)}$$

$$-\frac{i}{\hbar}\int_{t_1}^{t_2} V_{mn}(t_3)\, e^{\frac{i}{\hbar}[E_m(t_3-t_2)-E_n(t_3-t_1)]}\, dt_3$$

$$-\frac{1}{\hbar^2}\int_{t_1}^{t_2}\Big[\int_{t_1}^{t_3}\sum_k e^{-\frac{i}{\hbar}E_m(t_2-t_3)}\, V_{mk}(t_3)\, e^{-\frac{i}{\hbar}E_k(t_3-t_4)} \cdot V_{kn}(t_4)\, e^{-\frac{i}{\hbar}E_n(t_4-t_1)}\, dt_4\Big] dt_3$$

$$+ \ldots$$

Where in Λ^2_{mn} we interchanged t_3 & t_4 assuming t_4 is an earlier time.

$$\Lambda_{mn}(t_2,t_1)$$

$$= \delta_{mn}\, e^{-\frac{i}{\hbar}E_n(t_2-t_1)}$$

$$-\frac{i}{\hbar}\int_{t_1}^{t_2} e^{-\frac{i}{\hbar}E_m(t_2-t_3)} \cdot \Big\{ V_{mn}(t_3)\, e^{-\frac{i}{\hbar}E_n(t_3-t_1)}$$

$$-\frac{i}{\hbar}\int_{t_1}^{t_3}\sum_k V_{mk}(t_3)\, e^{-\frac{i}{\hbar}E_k(t_3-t_4)} \cdot V_{kn}(t_4)\, e^{-\frac{i}{\hbar}E_n(t_4-t_1)}\, dt_4 + \ldots\Big\} dt_3$$

$$= \delta_{mn}\, e^{-\frac{i}{\hbar}E_n(t_2-t_1)}$$

$$- \frac{i}{\hbar}\int_{t_1}^{t_2} e^{-\frac{i}{\hbar}E_m(t_2-t_3)} \cdot \Big\{ V_{mn}(t_3)\, e^{-\frac{i}{\hbar}E_n(t_3-t_1)}$$

$$-\frac{i}{\hbar}\sum_k V_{mk}(t_3)\int_{t_1}^{t_3} V_{kn}(t_4)\, e^{-\frac{i}{\hbar}E_k(t_3-t_4)} \cdot e^{-\frac{i}{\hbar}E_n(t_4-t_1)}\, dt_4 + \ldots\Big\} dt_3$$

or $\Lambda_{mn}(t_2,t_1)$

$$= \delta_{mn}\, e^{-\frac{i}{\hbar}E_n(t_2-t_1)}$$

$$-\frac{i}{\hbar}\int_{t_1}^{t_2} e^{-\frac{i}{\hbar}E_m(t_2-t_3)} \cdot \Big\{ \sum_k V_{mk}\, \delta_{kn}\, e^{-\frac{i}{\hbar}E_n(t_3-t_1)}$$

$$-\frac{i}{\hbar}\sum_k V_{mk}(t_3)\int_{t_1}^{t_3} V_{kn}(t_4)\, e^{-\frac{i}{\hbar}E_k(t_3-t_4)} \cdot e^{-\frac{i}{\hbar}E_n(t_4-t_1)}\, dt_4 + \ldots\Big\} dt_3$$

$$= \delta_{mn}\, e^{-\frac{i}{\hbar}E_n(t_2-t_1)}$$

$$-\frac{i}{\hbar}\int_{t_1}^{t_2} e^{-\frac{i}{\hbar}E_m(t_2-t_3)} \cdot \Big\{ \sum_k V_{mk}(t_3)\Big[\delta_{kn}\, e^{-\frac{i}{\hbar}E_n(t_3-t_1)}$$

$$-\frac{i}{\hbar}\int_{t_1}^{t_3} V_{kn}(t_4)\, e^{-\frac{i}{\hbar}E_k(t_3-t_4)} \cdot e^{-\frac{i}{\hbar}E_n(t_4-t_1)}\, dt_4 + \ldots\Big]\Big\} dt_3$$

$$\therefore \Lambda_{mn}(t_2,t_1) = \delta_{mn}\, e^{-\frac{i}{\hbar}E_n(t_2-t_1)} - \frac{i}{\hbar}\int_{t_1}^{t_2}\Big[e^{-\frac{i}{\hbar}E_m(t_2-t_3)} \cdot \sum_k V_{mk}(t_3)\, \Lambda_{kn}(t_3,t_1)\Big] dt_3$$

This means the system initial (at time t_1) in state n can go to the final state m without undergoing any scattering as predicted by the first or according to the 2nd alternative the transition takes place with one or more scatterings and at the point (x_3, t_3) the last scattering takes place. Here the amplitude that starting from t_3 the system remains up in the state n up to the final time t_2 is given by the factor $e^{-i/\hbar E_m (t_2 - t_3)}$.

6-19. From prob. 6-18,

$$\frac{d\Lambda_{mn}(t_2)}{dt_2} = -\frac{iE_m}{\hbar}\delta_{mn}\, e^{-i/\hbar E_m (t_2-t_1)}$$

$$-\left(\frac{iE_m}{\hbar}\right)\frac{i}{\hbar}\int_{t_1}^{t_2}\Big\{e^{-i/\hbar E_m (t_2-t_3)} \cdot \sum_k V_{mk}(t_3)\Lambda_{kn}(t_3,t_1)\Big\}dt_3$$

$$-\frac{i}{\hbar}e^{-\frac{i}{\hbar}E_m t_2}\cdot\frac{d}{dt_2}\int_{t_1}^{t_2}\Big\{e^{\frac{i}{\hbar}E_m t_3} \cdot \sum_k V_{mk}(t_3)\Lambda_{kn}(t_3,t_1)\Big\}dt_3$$

$$= -\frac{i}{\hbar}E_m \Lambda_{mn}(t_2)$$

$$-\frac{i}{\hbar}e^{-\frac{i}{\hbar}E_m t_2}\frac{d}{dt_2}\int_{t_1}^{t_2}\Big\{e^{\frac{i}{\hbar}E_m t_3} \cdot \sum_k V_{mk}(t_3)\Lambda_{kn}(t_3,t_1)\Big\}dt_3$$

Let $I = \dfrac{d}{dt_2}\displaystyle\int_{t_1}^{t_2} e^{\frac{i}{\hbar}E_m t_3}\sum_k V_{mk}(t_3)\Lambda_{kn}(t_3,t_1)\,dt_3$

Here after integrating wrt t_3 we substitute the value t_2 in place of t_3. (of course the expression with t_1 substituted is just constant.) Thus the differentiation wrt t_2 gives exactly the expression we have already in the integral (of course this is a function of t_2).

$$\Rightarrow I = e^{\frac{i}{\hbar}E_m t_2}\sum_k V_{mk}(t_2)\Lambda_{kn}(t_2)$$

$$\frac{d\Lambda_{mn}(t_2)}{dt_2} = -\frac{i}{\hbar}E_m\Lambda_{mn}(t_2) - \frac{i}{\hbar}\sum_k V_{mk}(t_2)\Lambda_{kn}(t_2)$$

6-20. It requires integration of $\Lambda_{mn}^{(1)}$ from $-\infty$ to $+\infty$ according to the interval $-\infty - 0$, $0 - T/2$, $T/2 - T$ & $T \to \infty$.

6-21.

$$\Lambda_{mn} = \delta_{mn} e^{-i/\hbar E_n (t_2 - t_1)} + \Lambda_{mn}^{(1)} + \Lambda_{mn}^{(2)} + \dots$$

Here, $t_1 = 0$, $t_2 = T$.

$$\Rightarrow \Lambda_{11}^{(0)} = \delta_{11} e^{-i/\hbar E_1 T}$$

$$\Lambda_{11}^{(1)} = -\frac{i}{\hbar} e^{-i/\hbar (E_1 t_2 - E_1 t_1)} \int_{t_1}^{t_2} V_{11}(t) e^{-i/\hbar (E_1 - E_1) t} dt$$

$= 0$, since $V_{11} = 0$

$$\Lambda_{11}^{(2)} = -\frac{1}{\hbar^2} \int_{t_1}^{t_2} \left[\int_{t_1}^{t_4} \sum_k e^{-i/\hbar E_1 (t_2 - t_4)} V_{1k}(t_4) \cdot e^{-i/\hbar E_k (t_4 - t_3)} V_{k1}(t_3) e^{-i/\hbar E_1 (t_3 - t_1)} dt_3 \right] dt_4$$

$$= -\frac{1}{\hbar^2} \int_{t_1}^{t_2} \left[\int_{t_1}^{t_4} e^{-\frac{i}{\hbar} E_1 (t_2 - t_4)} V_{12}(t_4) \cdot e^{-\frac{i}{\hbar} E_2 (t_4 - t_3)} V_{21}(t_3) \cdot e^{-\frac{i}{\hbar} E_1 (t_3 - t_1)} dt_3 \right] dt_4$$

$$= -\frac{1}{\hbar^2} \int_{t_1}^{t_2} \int_{t_1}^{t_4} e^{-\frac{i}{\hbar} E_1 (t_2 - t_1)} v^2 dt_3 \, dt_4$$

Since $V_{12}(t_4) = V_{21}(t_3) = v$ and $E_2 = E_1$.

$$\Rightarrow \Lambda_{11}^{(2)} = -\frac{e^{-\frac{i}{\hbar} E_1 (t_2 - t_1)}}{\hbar^2} v^2 \int_{t_1}^{t_2} (t_4 - t_1) dt_4$$

$$= -\frac{e^{-\frac{i}{\hbar} E_1 (t_2 - t_1)}}{\hbar^2} v^2 \left(\frac{t_4^2}{2} - t_1 t_4 \right)_{t_1}^{t_2}$$

$$= -\frac{e^{-\frac{i}{\hbar} E_1 (t_2 - t_1)}}{\hbar^2} v^2 \left(\frac{t_2^2}{2} - t_1 t_2 - \frac{t_1^2}{2} + t_1^2 \right)$$

$$= -\frac{e^{-\frac{i}{\hbar} E_1 (t_2 - t_1)}}{2\hbar^2} v^2 (t_2 - t_1)^2$$

$$= -\frac{e^{-\frac{i}{\hbar} E_1 T}}{2\hbar^2} v^2 T^2$$

We can similarly calculate the higher orders

$$\Rightarrow \Lambda_{11} = e^{-\frac{i E_1}{\hbar} T} \left(1 - \frac{v^2 T^2}{2\hbar^2} + \frac{v^4 T^4}{4!} + \dots \right)$$

$$= e^{-\frac{i E_1}{\hbar} T} \cos \frac{vT}{\hbar}$$

Since the phase is unimp.

$$\Lambda_{11} = \cos \frac{vT}{\hbar}$$

Also, $\Lambda_{12}^{(0)} = 0$, since $\delta_{12} = 0$

$$\Lambda_{12}^{(1)} = -\frac{i}{\hbar} e^{-i/\hbar E_1 T} v \int_0^T dt$$

$$= -\frac{i e^{-\frac{i}{\hbar} E_1 T}}{\hbar} vT.$$

Calculating in a similar way

$$\Lambda_{12}^{(2)} = i \frac{v^3 T^3}{6\hbar^3}$$

i.e., leaving the phase factor

$$\lambda_{12} = -i\,\frac{vT}{\hbar} + i\,\frac{v^3T^3}{3!\,\hbar^3} \cdots\cdots$$

$$= -i \sin \frac{vT}{\hbar}$$

6-22. In prob 6-21 we have $V_{12} = V_{21} = v$ i.e V_{12} is real. But if V_{12} is complex, $|V_{12}| = |V_{21}| = v$ ~~[illegible]~~ and hence the physical results in these two cases are the same since what matters the mod square of the λ_{ij} s.

6-23. If the w.f. is normalized for arbitrary volume V, the no. of states which have their momenta in phase space of element of volume $dV\,d^3\bar{P}_2$ is $V\,\frac{d^3P_2}{(2\pi\hbar)^3} = V p^2 dp\, d\Omega/(2\pi\hbar)^3$ (integrating over V).

The density of mom. states for particles traveling into the element solid angle $d\Omega$ is

$$d\rho(E) = \frac{1}{dE}\,\frac{V d^3\bar{P}_2}{(2\pi\hbar)^3} = \frac{Vmp\, d\Omega}{(2\pi\hbar)^3}$$

$$= \rho(E)\, d\Omega$$

$$dP/dt = \frac{2\pi}{\hbar}\,|v(\xi)|^2 \rho(E)$$

$$\Rightarrow \frac{dP}{dt} = \left(\frac{1}{2\pi\hbar^2}\right)^2 Vmp\,|v(\xi)|^2$$

The no. of particles in a volume V that will hit an effective area $d\sigma$ per unit time is

$$V u_1\, d\sigma$$

where u_1 is the initial velocity.

$$\text{But } \frac{dP}{dt}\, d\Omega = V u_1\, d\sigma = V\,\frac{p}{m}\, d\sigma$$

$$\text{i.e., } \frac{d\sigma}{d\Omega} = \left(\frac{m}{2\pi\hbar^2}\right)^2 |v(\xi)|^2$$

6-24.

$$\lambda_{mn}^{(1)} = -\frac{i}{\hbar}\, e^{-i/\hbar\,(E_n t_2 - E_m t_1)} \int_{t_1}^{t_2} 2i\, V_{mn} \left(e^{i\omega t} + e^{-i\omega t}\right) e^{i/\hbar\,(E_n - E_m)t}\, dt$$

$$= -\frac{i}{\hbar}\, e^{-\frac{i}{\hbar}(E_n t_2 - E_m t_1)}\, V_{mn} \int_{t_1}^{t_2} \left\{ e^{i\left(\frac{E_n - E_m}{\hbar} + \omega\right)t} + e^{i\left(\frac{E_n - E_m}{\hbar} - \omega\right)t} \right\} dt$$

$$= -\frac{i}{\hbar}\, e^{-\frac{i}{\hbar}(E_n t_2 - E_m t_1)}\, V_{mn} \left\{ \frac{e^{i\left(\frac{E_n - E_m}{\hbar} + \omega\right)t}}{i\left(\frac{E_n - E_m}{\hbar} + \omega\right)} + \frac{e^{i\left(\frac{E_n - E_m}{\hbar} - \omega\right)t}}{i\left(\frac{E_n - E_m}{\hbar} - \omega\right)} \right\}_{t_1}^{t_2}$$

with $t_1 = 0$, $t_2 = T$

$$\lambda_{mn}^{(1)} = -\frac{i}{\hbar}\, e^{-\frac{i}{\hbar} E_n T}\, V_{mn} \left\{ \frac{e^{i\left(\frac{E_n - E_m}{\hbar} + \omega\right)T}}{i\left(\frac{E_n - E_m}{\hbar} + \omega\right)} + \frac{e^{i\left(\frac{E_n - E_m}{\hbar} - \omega\right)T}}{i\left(\frac{E_n - E_m}{\hbar} - \omega\right)} - \frac{1}{i\left(\frac{E_n - E_m}{\hbar} + \omega\right)} - \frac{1}{i\left(\frac{E_n - E_m}{\hbar} - \omega\right)} \right\}$$

Let $\alpha = \frac{E_n - E_m}{\hbar}$

$$\Rightarrow \lambda_{mn}^{(1)} = -\frac{V_{mn}\, e^{-\frac{i}{\hbar}E_n T}}{\hbar} \left\{ \frac{e^{i(\alpha+\omega)T}}{\alpha+\omega} + \frac{e^{i(\alpha-\omega)T}}{\alpha-\omega} - \frac{2\alpha}{\alpha^2-\omega^2} \right\}$$

Whence,

$$|\lambda_{mn}^{(1)}|^2 = \frac{|V_{mn}|^2}{\hbar^2(\alpha^2-\omega^2)^2} \left\{ 4(\alpha^2+\omega^2) + 4(\alpha^2-\omega^2)^2 \cos\omega T - 4\alpha[(\alpha-\omega)\cos(\alpha+\omega)T + (\alpha+\omega)\cos(\alpha-\omega)T] \right\}$$

It is seen thus the probability of the transition is small unless $\alpha = \pm\omega$ (i.e.)

$$\frac{E_m - E_n}{\hbar} = \pm\omega$$

$$E_m - E_n = E_n \pm \hbar\omega$$

where the "+" corresponds to absorption and the "−" corresponds to emission.

6-25. Photo electric effect is impossible if matter obeys the quantum laws and light is still represented as a continuous wave. The transition rate

$$\frac{dP(n\to m)}{dt} = \frac{2\pi}{\hbar}|V_{nm}|^2 [\delta(E_m - E_n - \hbar\omega) + \delta(E_m - E_n + \hbar\omega)]$$

shows that the energy that under interaction with light a single electron gains or losses is $\pm\hbar\omega$ which means the energy spectrum of light is not continuous but it is rather quantized and therefore we resort to the quantum description electrodynamics.

6-26. Computing the first order transition amplitude and taking the absolute square we get

$$P(1\to 2) = |V_{12}|^2 |\Phi(\omega_0)|^2$$

Ref. "Quantum Mechanics and Path Integrals"
Some useful concepts and worked out steps.

Chapter 3.

3-1. The Free particle

It has been defined in the previous two chapters that for a particle starting from a point (x_a, t_a) and going to a final point (x_b, t_b) we can have a large no. of possible paths (for that matter infinite). To obtain the subset of all paths we divide the independent variable time into steps of width ϵ which will gives us a set of values t_i spaced a distance ϵ apart. Now at each time t_i we select a special point x_i and a path is constructed by joining the points so chosen by straight segments.

So we define the sum over all paths (path integral) by the multiple definite integral over all the points x_i, $i \neq 0, N$.

$$x_0 = x_a, \quad x_N = x_b$$

$$t_0 = t_a, \quad t_N = t_b$$

$$t_n = t_0 + n\epsilon$$

$$\epsilon = t_i - t_{i-1}$$

$$K(b,a) \sim \int \dots \int e^{i/\hbar\, S[b,a]} \, dx_1 \cdots dx_{n-1}$$

A more representative samples obtained by taking smaller and smaller values of ϵ. But the limit does not exist as we continuously take small values of ϵ. So we need a normalizing factor which depends on ϵ and for Lagrangian of the form

$$L = \frac{1}{2} m \dot{x}^2 - V(x,t)$$

this factor will be shown (Next sections & chapters) to be A^{-N} where $A = \left(\frac{2\pi i \hbar \epsilon}{m} \right)^{-1/2}$

In view of this factor the sum over all paths becomes

$$K(b,a) = \lim_{\epsilon \to 0} \frac{1}{A} \int \dots \int e^{i/\hbar\, S[b,a]} \frac{dx_1}{A} \cdots \frac{dx_{n-1}}{A}$$

$$\text{or } K(b,a) = \lim_{\epsilon \to 0} \int e^{i/\hbar\, S[b,a]} \, \mathcal{D}x(t)$$

Then for a free particle the Lagrangian is

$$L = \frac{m}{2} \dot{x}^2$$

and the corresponding kernel

$$K(b,a) = \lim_{\epsilon \to 0} \int \dots \int e^{i/\hbar\, S[b,a]} \, dx_1 \cdots dx_{n-1} \cdot \left(\frac{2\pi i \hbar \epsilon}{m} \right)^{-n/2}$$

$$S[b,a] = \int_{t_a}^{t_b} L(x, \dot{x}, t)\, dt$$

$$= \frac{m}{2} \int_{t_a}^{t_b} \dot{x}^2 dt$$

$$= \frac{m}{2} \left\{ \int_{t_0 = t_a}^{t_1} \dot{x}^2 dt + \int_{t_1}^{t_2} \dot{x}^2 dt + \dots + \int_{t_{i-1}}^{t_i} \dot{x}^2 dt + \dots + \int_{t_{n-1}}^{t_n = t_b} \dot{x}^2 dt \right\}$$

Since for each t_i we have a corresponding x_i,

$$S[b,a] = \frac{m}{2} \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \dot{x}_i^2 \, dt$$

$t_i - t_{i-1} = dt \,//\, \epsilon$ and in this interval $\dot{x}_i$ remains constant.

$$\Rightarrow S[b,a] = \frac{m}{2} \sum_{i=1}^{n} \left(\frac{x_i - x_{i-1}}{\epsilon} \right)^2 \epsilon$$

$$= \frac{m}{2\epsilon} \sum_{i=1}^{n} (x_i - x_{i-1})^2$$

$\therefore K(b,a) = \lim_{\epsilon \to 0}$

$$\int \cdots \int \exp\left\{\frac{im}{2\hbar\epsilon}\sum_{i=1}^{n}(x_i - x_{i-1})^2\right\}$$

$$dx_1 \cdots dx_{n-1}\left(\frac{2\pi i\hbar\epsilon}{m}\right)^{-n/2}$$

This represents a set of $(n-1)$ Gaussian integrals and we integrate over all the variables one after the other.

- over x_1

$$I_1 = \left(\frac{2\pi i\hbar\epsilon}{m}\right)^{-1/2}\int e^{\frac{im}{2\hbar\epsilon}\left[(x_2-x_1)^2+(x_1-x_0)^2\right]}dx_1$$

$$(x_2-x_1)^2+(x_1-x_0)^2$$

$$= x_2^2 - 2x_2x_1 + x_1^2 + x_1^2 - 2x_0x_1 + x_0^2$$

$$= x_2^2 + x_0^2 + 2(x_1^2 - (x_2+x_0)x_1)$$

$$= x_2^2 + x_0^2 + 2\left[x_1 - \left(\frac{x_2+x_0}{2}\right)\right]^2 - 2\left(\frac{x_2+x_0}{2}\right)^2$$

$$= \frac{x_2^2}{2} + \frac{x_0^2}{2} - x_0x_2 + 2\left[x_1 - \left(\frac{x_2+x_0}{2}\right)\right]^2$$

$$= \tfrac{1}{2}(x_2-x_0)^2 + 2\left[x_1 - \left(\frac{x_2+x_0}{2}\right)\right]^2$$

$$e^{\frac{im}{2\hbar\epsilon}\left[(x_2-x_1)^2+(x_1-x_0)^2\right]}$$

$$= e^{\frac{im}{2\cdot 2\hbar\epsilon}(x_2-x_0)^2} \cdot e^{\frac{im}{\hbar\epsilon}\left[x_1-\left(\frac{x_2+x_0}{2}\right)\right]^2}$$

$$\Rightarrow I_1 = \left(\frac{2\pi i\hbar\epsilon}{m}\right)^{-1/2} e^{\frac{im}{2\cdot 2\hbar\epsilon}(x_2-x_0)^2}\int_{-\infty}^{\infty} e^{\frac{im}{\hbar\epsilon}\left[x_1-\left(\frac{x_2+x_0}{2}\right)\right]^2}dx_1$$

Let $u = x_1 - \frac{x_2+x_0}{2}$

$du = dx_1$

i.e.,

$$I_1 = \left(\frac{2\pi i\hbar\epsilon}{m}\right)^{-1/2} e^{\frac{im}{2\cdot 2\hbar\epsilon}(x_2-x_0)^2}\int_{-\infty}^{\infty} e^{-\frac{m}{i\hbar\epsilon}u^2}du$$

$$= \left(\frac{2\pi i\hbar\epsilon}{m}\right)^{-1/2} e^{\frac{im}{2\cdot 2\hbar\epsilon}(x_2-x_0)^2}\sqrt{\frac{i\pi\hbar\epsilon}{m}}$$

$$= \left(\frac{2\pi i\hbar\epsilon}{m}\right)^{-1/2}\left(\frac{m}{\pi i\hbar\epsilon}\right)^{-1/2} e^{\frac{im}{2\cdot 2\hbar\epsilon}(x_2-x_0)^2}$$

$$= \frac{1}{\sqrt{2}}\, e^{\frac{im}{2\cdot 2\hbar\epsilon}(x_2-x_0)^2}$$

Next we multiply this by $\left(\frac{2\pi i\hbar\epsilon}{m}\right)^{-1/2} e^{\frac{im}{2\hbar\epsilon}(x_3-x_2)^2}$ to integrate over x_2.

$$I_2 = \left(\frac{2\pi i\hbar\epsilon\cdot 2}{m}\right)^{-1/2}\int_{-\infty}^{\infty} e^{\frac{im}{2\hbar\epsilon}(x_3-x_2)^2}\cdot e^{\frac{im}{2\cdot 2\hbar\epsilon}(x_2-x_0)^2}dx_2$$

$$= \left(\frac{2\pi i\hbar\epsilon\cdot 2}{m}\right)^{-1/2}\int_{-\infty}^{\infty} e^{\frac{im}{2\cdot 2\hbar\epsilon}\left\{2(x_3-x_2)^2+(x_2-x_0)^2\right\}}dx_2$$

$$2(x_3-x_2)^2 + (x_2-x_0)^2$$

$$= 2(x_3^2 + x_2^2 - 2x_3x_2) + x_2^2 + x_0^2 - 2x_0x_2$$

$$= 2x_3^2 + x_0^2 + 3x_2^2 - 2(x_0+2x_3)x_2$$

$$= 2x_3^2 + x_0^2 + 3\left[x_2 - \left(\frac{x_0+2x_3}{3}\right)\right]^2 - 3\left(\frac{x_0+2x_3}{3}\right)^2$$

$$= \frac{2}{3}x_3^2 + \frac{2}{3}x_0^2 - \frac{4}{3}x_0 x_3 + 3\left[x_2 - \left(\frac{x_0+2x_3}{3}\right)\right]^2$$

$$= \frac{2}{3}(x_3 - x_0)^2 + 3\left[x_2 - \left(\frac{x_0+2x_3}{3}\right)\right]^2$$

$$\Rightarrow I_2 = \left(\frac{2\pi i\hbar\epsilon\cdot 2}{m}\right)^{-1/2} e^{\frac{im}{3\cdot 2\hbar\epsilon}(x_3-x_0)^2} \int_{-\infty}^{\infty} e^{\frac{3im}{2\cdot 2\hbar\epsilon}\left[x_2 - \left(\frac{x_0+2x_3}{3}\right)\right]^2} dx_2$$

once again $u = x_2 - \frac{x_0+2x_3}{3}$

$du = dx_2$

$$\Rightarrow I_2 = \left(\frac{2\pi i\hbar\epsilon\cdot 2}{m}\right)^{-1/2} e^{\frac{im}{3\cdot 2\hbar\epsilon}(x_3-x_0)^2} \int_{-\infty}^{\infty} e^{-\frac{3m}{i4\hbar\epsilon}u^2} du$$

$$= \left(\frac{2\pi i\hbar\epsilon\cdot 2}{m}\right)^{-1/2} e^{\frac{im}{3\cdot 2\hbar\epsilon}(x_3-x_0)^2} \left[\frac{4\pi i\hbar\epsilon}{3m}\right]^{1/2}$$

$$= \left(\frac{2\pi i\hbar\epsilon\cdot 2}{m}\cdot\frac{3m}{4\pi i\hbar\epsilon}\right)^{-1/2} e^{\frac{im}{3\cdot 2\hbar\epsilon}(x_3-x_0)^2}$$

$$= \frac{1}{\sqrt{3}} e^{\frac{im}{3\cdot 2\hbar\epsilon}(x_3-x_0)^2}$$

It is seen thus there is a pattern and after $n-1$ steps we will get

$$I_{n-1} = \frac{1}{\sqrt{n}} e^{\frac{im}{n\cdot 2\hbar\epsilon}(x_n-x_0)^2}$$

which is the integral over x_{n-1}. So now together with the nth factor we obtain for the kernel that

$$K(b,a) = \left(\frac{2\pi i\hbar\epsilon}{m}\right)^{-1/2}\frac{1}{\sqrt{n}} e^{\frac{im}{n\cdot 2\hbar\epsilon}(x_n-x_0)^2}$$

$$= \left(\frac{2\pi i\hbar\epsilon\cdot n}{m}\right)^{-1/2} e^{\frac{im}{n\cdot 2\hbar\epsilon}(x_n-x_0)^2}$$

$$t_n = t_0 + n\epsilon$$

or $t_b = t_a + n\epsilon$

$$\Rightarrow n = \frac{t_b - t_a}{\epsilon}$$

$$\therefore K(b,a) = \left(\frac{2\pi i\hbar(t_b-t_a)}{m}\right)^{-1/2} e^{\frac{im}{2\hbar}\frac{(x_b-x_a)^2}{t_b-t_a}}$$

3-2. Diffraction Through a Slit

A particle starts from a point $x_a = 0$, $t_a = 0$. Classically, after a time T, the particle will reach a point x_0 of space coordinate. However, according to the results of quantum mechanics we are uncertain about its space coordinate at the time T. So we claim it will be at the position $\pm b$ of x_0 at this time. Now we would like to determine the probability amplitude as to where it will be after an additional time τ. Let this final point be $(x_0+x, T+\tau)$. The amplitude then can be written as

$$\psi(x) = \int_{-b}^{b} K(x+x_0, T+\tau; x_0+y, T)\cdot K(x_0+y, T; 0, 0)\, dy$$

where $-b \le y \le b$.

If we introduce a function $G(y)$ s.t. $G(y) = 1$ for $y \leq |b|$ and $G(y) = 0$ for $y > |b|$, we can extend the limit of integration to $\pm\infty$. Instead let us introduce a Gaussian function $G(y) = e^{-y^2/2b}$ practically, after $y = \pm b$, $G(y) \to 0$ and hence we write

$$\psi(x) = \int_{-\infty}^{\infty} K(x_0+x, T+\tau; x_0+y, T)\, K(x_0+y, T; 0 0)\, dy$$

$$= \int_{-\infty}^{\infty} \left(\frac{2\pi i\hbar\tau}{m}\right)^{-1/2} e^{\frac{im}{2\hbar\tau}(x-y)^2} \left(\frac{2\pi i\hbar T}{m}\right)^{-1/2} e^{\frac{im}{2\hbar T}(x_0+y)^2}\, dy$$

$$= \int_{-\infty}^{\infty} \frac{m}{2\pi i\hbar\sqrt{\tau T}}\, e^{\frac{im}{2\hbar}\left(\frac{x^2}{\tau}+\frac{x_0^2}{T}\right)}\, e^{\frac{im}{\hbar}\left(-\frac{x}{\tau}+\frac{x_0}{T}\right)y + \left(\frac{im}{2\hbar\tau}+\frac{im}{2\hbar T}-\frac{1}{2b^2}\right)y^2}\, dy$$

By completing the square

$$\int_{-\infty}^{\infty} e^{\alpha x^2+\beta x}\, dx = \sqrt{\frac{\pi}{-\alpha}}\, e^{-\beta^2/4\alpha}$$

$$\Rightarrow \psi(x) = \frac{m}{2\pi i\hbar\sqrt{\tau T}}\, e^{\frac{im}{2\hbar}\left(\frac{x^2}{\tau}+\frac{x_0^2}{T}\right)} \left[\frac{\pi}{-\frac{im}{2\hbar}\left(\frac{1}{\tau}+\frac{1}{T}+\frac{\hbar i}{mb^2}\right)}\right]^{1/2} e^{\frac{-\left(\frac{im}{\hbar}\right)^2\left(-\frac{x}{\tau}+\frac{x_0}{T}\right)^2}{4\frac{im}{2\hbar}\left(\frac{1}{\tau}+\frac{1}{T}+\frac{\hbar i}{mb^2}\right)}}$$

$$= \sqrt{\frac{m}{2\pi i\hbar}}\left[T\tau\left(\frac{1}{\tau}+\frac{1}{T}+\frac{\hbar i}{mb^2}\right)\right]^{-1/2} \exp\left\{\frac{im}{2\hbar}\left(\frac{x^2}{\tau}+\frac{x_0^2}{T}\right) - \frac{\left(\frac{im}{\hbar}\right)^2\left(-\frac{x}{\tau}+\frac{x_0}{T}\right)^2}{\frac{2im}{\hbar}\left(\frac{1}{\tau}+\frac{1}{T}+\frac{\hbar i}{mb^2}\right)}\right\}$$

$$= \sqrt{\frac{m}{2\pi i\hbar}}\left[T\tau\left(\frac{1}{\tau}+\frac{1}{T}+\frac{\hbar i}{mb^2}\right)\right]^{-1/2} \exp\left\{\frac{im}{2\hbar}\left[\frac{x^2}{\tau}+\frac{x_0^2}{T} - \frac{\left(-\frac{x}{\tau}+\frac{x_0}{T}\right)^2}{2im\hbar\left(\frac{1}{\tau}+\frac{1}{T}+\frac{\hbar i}{mb^2}\right)}\right]\right\}$$

In terms of $v_0 = x_0/T$ and rearranging we obtain

$$\psi(x) = \sqrt{\frac{m}{2\pi i\hbar}}\left(T+\tau+T\tau\frac{\hbar i}{mb^2}\right)^{-1/2} \exp\left\{\frac{im}{2\hbar}\left[\frac{x^2}{\tau}+v_0^2 T - \frac{\frac{1}{\tau^2}(v_0\tau-x)^2}{\left(\frac{1}{\tau}+\frac{1}{T}+\frac{\hbar i}{mb^2}\right)}\right]\right\} \quad \ldots\ldots (*)$$

Relative probability for the particle to arrive at various points along x-axis

$$P(x)\,dx = \frac{m}{2\pi\hbar}\left[\frac{1}{(\tau+T)^2+\frac{T^2\tau^2\hbar^2}{m^2b^4}}\right]^{1/2} e^{(\,)}$$

Let us rationalize the 2nd term in the exponent of (*).

$$-\frac{\frac{1}{\tau^2}(v_0\tau-x)}{\left(\frac{1}{\tau}+\frac{1}{T}+\frac{\hbar i}{mb^2}\right)}\;\frac{\left(\frac{1}{\tau}+\frac{1}{T}-\frac{\hbar i}{mb^2}\right)}{\left(\frac{1}{\tau}+\frac{1}{T}-\frac{\hbar i}{mb^2}\right)}$$

$$= \frac{-\frac{1}{\tau^2}(v_0\tau-x)^2\left(\frac{1}{\tau}+\frac{1}{T}-\frac{\hbar i}{mb^2}\right)}{\left(\frac{1}{\tau}+\frac{1}{T}\right)^2+\hbar^2/m^2b^4}$$

$$= \frac{-\frac{1}{\tau^2}(v_0\tau-x)^2\left(\frac{1}{\tau}+\frac{1}{T}\right)+\frac{\hbar i}{\tau^2 mb^2}(v_0\tau-x)^2}{\left(\frac{1}{\tau}+\frac{1}{T}\right)^2+\hbar^2/m^2b^4}$$

The exponent of (*) then becomes

$$\frac{im}{2\hbar}\underbrace{\left[\frac{x^2}{\tau}+v_0^2T-\frac{\frac{1}{\tau^2}(v_0\tau-x)^2\left(\frac{1}{\tau}+\frac{1}{T}\right)}{\left(\frac{1}{\tau}+\frac{1}{T}\right)^2+\hbar^2/m^2b^4}\right]}_{\alpha}$$

$$\underbrace{-\frac{1}{2\tau^2b^2}(v_0\tau-x)^2\Big/\left[\left(\frac{1}{\tau}+\frac{1}{T}\right)^2+\frac{\hbar^2}{m^2b^4}\right]}_{\beta}$$

$$e^{i\alpha}\cdot e^{i\beta}\;\left(e^{i\alpha}\right)^*\left(e^{i\beta}\right)^*$$

$$= e^{-\frac{1}{\tau^2 b^2}\frac{(v_0\tau - x)^2}{(1/\tau + 1/T)^2 + \hbar^2/m^2b^4}}$$

$$\therefore P(x)\,dx = \frac{m}{2\pi\hbar}\left[\frac{b^2/T^2}{b^2(1+\tau/T)^2 + \frac{\tau^2\hbar^2}{m^2b^2}}\right]^{1/2} e^{-\frac{(v_0\tau - x)^2}{b^2(1+\tau/T)^2 + \frac{\hbar^2\tau^2}{m^2b^2}}}\,dx$$

$$= \frac{m}{2\pi\hbar T}\frac{b}{\left[b^2(1+\tau/T)^2 + \frac{\tau^2\hbar^2}{m^2b^2}\right]^{1/2}} e^{-\frac{(v_0\tau - x)^2}{b^2(1+\tau/T)^2 + \hbar^2\tau^2/m^2b^2}}\,dx$$

$$\text{or}\quad P(x)\,dx = \frac{m}{2\pi\hbar T}\,\frac{b}{\Delta x}\,e^{-\frac{(v_0\tau - x)^2}{(\Delta x)^2}}\,dx$$

Where $(\Delta x)^2 = b^2(1+\tau/T)^2 + \frac{\hbar^2\tau^2}{m^2b^2}$

and, $x_1 = v_0\tau$

$b_1 = b(1+\tau/T)$

is a classical result that shows the dist. moved by the particle after a time τ and the spread of the distribution after the time τ resp.

Now we would like to get some limiting results. What will the amplitude be if the momentum is definite?

Quite apart from any quantum mechanical considerations, there is a classical uncertainity of b/T in the velocity. We make this uncertainity very small by choosing T very large. Of course, we can also make x_0 extremely large so that the average velocity $v_0 = x_0/T$ does not go to zero. In this limit (v_0 & τ constant) the amplitude becomes

$$\psi(x) = \sqrt{\frac{m}{2\pi\hbar T}}\left(1 + \frac{i\hbar\tau}{mb^2}\right)^{-1/2} \exp\left\{\frac{imx^2}{2\hbar\tau} - \frac{\frac{im}{2\hbar}(v_0\tau - x)^2}{(\tau + \hbar i/mb^2)}\right\}$$

$$= \frac{\text{const}}{\sqrt{1 + \frac{i\hbar\tau}{mb^2}}} \exp\left\{\frac{imx^2}{2\hbar\tau} + \frac{m(v_0\tau - x)^2}{2\hbar(i\tau - \hbar/mb^2)}\right\}$$

$$\text{or}\quad \psi(x) = \frac{\text{const}}{\sqrt{1 + \frac{i\hbar\tau}{mb^2}}} \cdot \exp\left\{\frac{imx^2}{2\hbar\tau} + \frac{m(v_0\tau - x)^2}{\frac{2\hbar^2}{m}\left(\frac{im\tau}{\hbar} - \frac{1}{b^2}\right)}\right\}$$

Next we assume that the quantum mechanical uncertainity in the mom. $\hbar/b$ is very small i.e.,

$$\psi(x) \simeq \text{const}\ \exp\left\{\frac{imv_0 x}{\hbar} - \frac{imv_0^2}{2\hbar}\tau\right\}$$

$$\equiv \text{const}\ \exp\left\{\frac{ipx}{\hbar} - \frac{i}{\hbar}\frac{p^2}{2m}\tau\right\}$$

That means if the mom. of a particle is known to be definite, p, then the amplitude implies, for the particle to arrive at the point x at time t varies as e^{ikx} i.e., $\psi(x)$ is a wave of definite w. no. $k = p/\hbar$. Furthermore it has a definite frequency $\omega = p^2/2m\hbar$. Then we say that a free particle of mom. p has a definite quantum mechanical energy (defined as $\hbar$ times frequency) which is $p^2/2m$ just as in classical mechanics.

3-11. Evaluation of path integrals by Fourier series

$$\lim_{N\to\infty}\prod_{n=1}^{N}\left(1 - \frac{\omega^2T^2}{n^2\pi^2}\right)^{-1/2} = \left(\frac{\sin\omega T}{\omega T}\right)^{-1/2}$$

Quantum Mechanics I (Phys. 542)
Problems:

1.1.1 Consider the set of all entities of the form (a,b,c) where a,b,c are real numbers. These form a vector space with addition and scalar multiplication defined as follows:

$$(a,b,c) + (d,e,f) = (a+d,\ b+e,\ c+f)$$
$$\alpha(a,b,c) = (\alpha a,\ \alpha b,\ \alpha c)$$

Write down the null vector and the inverse of (a,b,c). Verify that axioms (i)-(iv) are met. Do we have a vector space if a,b,c are required to be positive numbers? Show that vectors of the form $(a,b,1)$ do not form a linear vector space.

1.1.2 By using the axioms prove the following:

(1) $0\bar{V} = \bar{0}$ Hint: add $0\bar{V}$ to $\alpha\bar{V}$

(2) $\alpha\bar{0} = \bar{0}$ Hint: add $\alpha\bar{0}$ to $\alpha\bar{V}$

(3) $(-1)\bar{V} = -\bar{V}$ Hint: add $\bar{V}$ to $(-1)\bar{V}$

1.1.3 - Show that any set of vectors containing the null vector $\bar{0}$ is liniarly dependent.

1.1.4. Consider the vector space discussed in exercise (1). Show that the elements $(1,1,0)$, $(1,0,1)$, and $(3,2,1)$ are linearly dependent. [Assume that one of them is a linear combination of the other two, and find the (non trivial) coefficients of the expansion.] Show likewise that $(1,1,0)$; $(1,0,1)$ and $(0,1,1)$ are linearly independent.

1.2.1- By going through the proof of theorem 3, show that the inequality becomes an equality if $\bar{V}_i = \lambda \underline{V}_j$, where λ is an arbitrary scalar.
for the proof of theorem

1.2.2. Show likewise by analyzing the proof of theorem 4 that the inequality becomes an equality if $\bar{V}_i = \lambda\bar{V}_j$, where λ is a real positive scalar. (There are two inequalities here, $\mathrm{Re}\langle V_i|V_j\rangle \le |\langle V_i|V_j\rangle|$, and $|\langle V_i|V_j\rangle| \le |V_i|\,|V_j|$, both of which must become equalities.

1.3.1 Nowhere in the above proof of the Gram-Schmidt theorem did we refer to the LI of the set $\{\bar{V}_1,\dots\bar{V}_n\}$. Why was it stipulated in the statement of the theorem? (Hint: It can happen that a vector we construct, say $|m'\rangle$, vanishes, bringing the proof to a halt. Show that the vanishing of $|m'\rangle$ implies that we can express $|V_m\rangle$ in terms of $|V_1\rangle, \dots |V_{m-1}\rangle$, contrary to the assumption of LI.)

1.3.2. Consider the vectors

$$|V_1\rangle \leftrightarrow \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}, \quad |V_2\rangle \leftrightarrow \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \quad |V_3\rangle \leftrightarrow \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix}$$

Use the Gram-Schmidt procedure to get the following orthonormal basis.

$$|1\rangle \leftrightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad |2\rangle \leftrightarrow \begin{bmatrix} 0 \\ 1/5^{1/2} \\ 2/5^{1/2} \end{bmatrix}, \quad |3\rangle \leftrightarrow \begin{bmatrix} 0 \\ -2/5^{1/2} \\ 1/5^{1/2} \end{bmatrix}$$

Is this the only orthonormal basis you can get in this case? (What if you change the sign of the components of $|1\rangle$?

1.4.1. In a space V^n, prove that the set of all vectors $\{|V_\perp^1\rangle, |V_\perp^2\rangle, \cdots\}$, orthogonal to any $|V\rangle \neq 0$ for a subspace V^{n-1}.

1.4.2. Suppose $V_1^{n_1}$ and $V_2^{n_2}$ are two subspaces s.t. any element of V_1 is orthogonal to any element of V_2. Show that the dimensionality of $V_1 \oplus V_2$ is $n_1 + n_2$. (Hint: Theorem 6.)

1.6.1. An operator Ω is given by the matrix

$$\Omega = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

What is its action?

1.6.2. Given Ω & Λ are Hermitian what can you say about (i) $\Omega\Lambda$; (ii) $\Omega\Lambda + \Lambda\Omega$; (iii) $[\Omega, \Lambda]$; and (iv) $i[\Omega, \Lambda]$

1.6.3. Show that a product of unitary operators is unitary.

1.6.4. It is assumed that you know (i) what a determinant is, (ii) that $\det \Omega^T = \det \Omega$ (T denotes transpose), (iii) that the determinant of a product of matrices is the product of the determinants. Prove that the determinant of a unitary matrix is a complex no. of unit modulus.

1.6.5. Verify that $R(\frac{1}{2}\pi i)$ is unitary (orthogonal) by examining its matrix.

1.6.6. Verify that the following matrices are unitary

$$\frac{1}{2^{1/2}} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}, \quad \frac{1}{2} \begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix}$$

Verify that the determinant is of the form $e^{i\theta}$ in each case. Are any of the above matrices Hermitian.

1.7.1. The trace of a matrix is defined to be the sum of its diagonal matrix elements:

$$Tr\ \Omega = \sum_i \Omega_{ii}$$

Show that a, $Tr(\Omega\Lambda) = Tr(\Lambda\Omega)$

b, $Tr(\Omega\Lambda\theta) = Tr(\Lambda\theta\Omega) = Tr(\theta\Omega\Lambda)$

(The permutations are cyclic)

c, The trace of an operator is unaffected by a unitary change of basis $|i\rangle \rightarrow U|i\rangle$. [Equivalently show that $Tr\ \Omega = Tr(U^{+}\Omega U)$.]

1.7.2. Show that the determinant of a matrix is unaffected by a unitary change of basis. [Equivalently show $\det \Omega = \det(U^{+}\Omega U)$]

1.8.1. a, Find the eigenvalues and normalized eigenvectors of the matrix

$$\Omega = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 2 & 0 \\ 0 & 1 & 4 \end{bmatrix}$$

b, Is the matrix Hermitian? Are the eigenvectors orthogonal?

1.8.2. Consider the matrix

$$\Omega = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

a, Is it Hermitian?

b, Find its eigenvalues and eigenvectors

c, verify that $U^{+}\Omega U$ is diagonal, U being the matrix of the eigenvectors of Ω.

1.8.3. Consider the Hermitian matrix

$$\Omega = \frac{1}{2}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$

a, Show that $\omega_1 = \omega_2 = 1,\ \omega_3 = 2$

b, Show that $|\omega = 2\rangle$ is any vector of the form

$$\frac{1}{\sqrt{2}\,a}\begin{bmatrix} 0 \\ a \\ -a \end{bmatrix}$$

c, Show that the $\omega = 1$ eigenspace contains all vectors of The form

$$\frac{1}{\sqrt{b^2+2c^2}} \begin{bmatrix} b \\ c \\ c \end{bmatrix}$$

either by feeding $\omega = 1$ into the eqs or by requiring that the $\omega = 1$ eigenspace be orthogonal to $|\omega = 2\rangle$.

1.8.4. An arbitrary $n \times n$ matrix need not have n eigenvectors. Consider as an example

$$\Omega = \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix}$$

a, Show that $\omega_1 = \omega_2 = 3$

b, By feeding in this value show that we get only one eigenvector of the form

$$\frac{1}{\sqrt{2a^2}} \begin{bmatrix} +a \\ -a \end{bmatrix}$$

We cannot find another one that is L.I.

1.8.5. Consider the matrix

$$\Omega = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

a, Show that it is unitary

b, Show that its eigenvalues are $e^{i\theta}$ and $e^{-i\theta}$

c, find the corresponding eigenvectors; show that they are orthogonal.

d, Verify that $U^{\dagger}\Omega U =$ (diagonal matrix), where U is the matrix of eigenvectors of Ω.

1.8.6. a, We have seen that the determinant of a matrix is unchanged under a unitary change of basis. Argue now that

$$\det \Omega = \text{product of eigenvalues of } \Omega = \prod_{i=1}^{n} \omega_i$$

for a Hermitian or unitary Ω.

b, Using the invariance of the trace under the same transformation show that

$$\mathrm{Tr}\,\Omega = \sum_{i=1}^{n} \omega_i$$

1.8.7. By using the results on the trace and determinant from the last prob. show that the eigenvalues of the matrix

$$\Omega = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

are 3 & -1. Verify this by explicit computation. Note that the Hermitian nature of the matrix is an essential ingredient.

1.8.8. Consider Hermitian Matrices M^1, M^2, M^3, M^4 ~~etc.~~ that obey

$$M^i M^j + M^j M^i = 2\delta^{ij} I \quad , \; i,j = 1, \ldots 4$$

a, Show that the eigenvalues of M^i are ± 1. (Hint: go to the eigenbasis of M^i, and use the eq. for $i=j$.)

b, By considering the relation

$$M^i M^j = -M^j M^i \quad \text{for } i \neq j$$

Show that M^i are traceless [Hint: $Tr(ACB) = Tr(CBA)$]

c, Show that they cannot be odd-dimensional matrices.

1.8.9. A collection of masses m_α, located at $\bar{r}_\alpha$ and rotating with angular velocity $\bar{\omega}$ around a common axis has a angular mom.

$$\bar{L} = \sum_\alpha m_\alpha (\bar{r}_\alpha \times \bar{v}_\alpha)$$

Where $\bar{v}_\alpha = \bar{\omega} \times \bar{r}_\alpha$ is the velocity of m_α. By using the identity

$$\bar{A} \times (\bar{B} \times \bar{C}) = \bar{B}(\bar{A}\cdot\bar{C}) - \bar{C}(\bar{A}\cdot\bar{B})$$

show that each Cartesian component L_i of $\bar{L}$ is given by

$$L_i = \sum_j M_{ij}\, \omega_j$$

where $M_{ij} = \sum_\alpha m_\alpha [r_\alpha^2 \delta_{ij} - (r_\alpha)_i (r_\alpha)_j]$ ~~or in~~

or in Dirac notation

$$|L\rangle = M|\omega\rangle$$

a, Will the angular mom. and angular velocity always be parallel?

b, Show that the moment of inertial matrix M_{ij} is Hermitian

c, Argue now that there exist three directions for $\bar{\omega}$ s.t. $\bar{L}$ and $\bar{\omega}$ will be parallel. How are these directions to be found?

d, Consider the moment of inertia matrix of a sphere. Due to the complete symmetry of the sphere, it is clear that every direction is its eigendirection for rotation. What does this say about three eigenvalues of the matrix M?

~~1.1.9.11. We know that the series~~

~~$f(x) = \sum_{n=0}^{\infty} (x^n)$~~

~~May we expand the function $f(x) = (1-x)^{-1}$ in [illegible]~~

1.8.10. By considering the commutator, show that the following Hermitian matrices may be simultaneously diagonalized. Find the eigenvectors common to both and verify that under a unitary transformation to this basis, both matrices are diagonalized.

$$\Omega = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

Since Ω is degenerate and Λ is not, you must be prudent in deciding which matrix dictates the choice of basis.

1.8.11. Consider the coupled mass prob. discussed above.
(i) Given that the initial state is $|1\rangle$, in which the first mass is displaced by unity and the second mass is left alone, calculate $|1(t)\rangle$ by following the alogorithm.
(ii) compare your result with that following from eq. 1.8.39.

1.8.12. Consider once again the prob. discussed in the example of the Harmonic oscillator. (i) Assuming that

$$|\ddot{x}\rangle = \Omega|x\rangle$$

has a solution

$$|x(t)\rangle = U(t)|x(0)\rangle$$

find the differential eq. satisfied by $U(t)$. Use the fact that $|x(0)\rangle$ is arbitrary.

(ii) Assuming (as is the case) that Ω and U can be simultaneously diagonalized, solve for the elements of the matrix U in this common basis and regain eq. 1.8.43. Assume $|\dot{x}(0)\rangle = 0$.

1.9.1. We know that the series

$$f(x) = \sum_{n=0}^{\infty} x^n$$

may be equated to the function $f(x) = (1-x)^{-1}$ if $|x| < 1$. By going to the eigenbasis, examine when the q-no. power series of a Hermitian operator Ω may be identified with $(1-\Omega)^{-1}$.

1.9.2. If H is a Hermitian operator, show that $U = e^{iH}$ is unitary. (Notice the analogy with c nos: if θ is real, $U = e^{i\theta}$ is a no. of unit modulus.)

1.9.3. For the case above, show that $\det U = e^{i \operatorname{Tr} H}$.

1.10.1. Show that $\delta(ax) = \delta(x)/|a|$. [Consider $\int \delta(ax)\, d(ax)$. Remember that $\delta(x) = \delta(-x)$.]

1.10.2. Show that

$$\delta(f(x)) = \sum_i \frac{\delta(x_i - x)}{|df/dx_i|}$$

Where x_i are the zeros of $f(x)$. Hint: where does $\delta(f(x))$ blow up? Expand $f(x)$ near such points in a Taylor series, keeping the first nonzero term.

1.10.3. Consider the theta function $\theta(x-x')$ which vanishes if $x-x' < 0$ and equals 1 if $x-x' > 0$. Show that the theta function is the integral of the delta function.

1.10.4. A string is displaced as follows at $t=0$:

$$\psi(x,0) = \begin{cases} \frac{2xh}{L}, & 0 \le x \le L/2 \\ \frac{2h}{L}(L-x), & \frac{L}{2} \le x \le L \end{cases}$$

Show that

$$\psi(x,t) = \sum_{m=1}^{\infty} \sin\left(\frac{m\pi x}{L}\right) \cos \omega_m t \left(\frac{8h}{\pi^2 m^2}\right) \sin\left(\frac{\pi m}{2}\right)$$

Chapter four

4.2.1. Consider the following operators on a Hilbert space $V^3(C)$:

$$L_x = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad L_y = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix}, \quad L_z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

a) What are the possible values one can obtain if L_z is measured?

b) Take the state in which $L_z = 1$. In this state what are $\langle L_x \rangle$, $\langle L_x^2 \rangle$ and ΔL_x

c) Find the normalized eigenstates and eigenvalues of L_x in L_z basis.

d, If the particle is in the state with $L_z = -1$, and L_x is measured, what are the possible outcomes and their probabilities.

e, Consider the state $|\psi\rangle = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/\sqrt{2} \end{bmatrix}$.

in the L_z basis. If L_z^2 measured in this state and a result +1 is obtained, what is the state after the measurement? How probable was this result? If L_z is measured what are the outcomes and their respective probabilities?

f, A particle is in a state for which the probabilities are $P(L_z = 1) = 1/4$, $P(L_z = 0) = 1/2$ & $P(L_z = -1) = 1/4$. Convince yourself that the most general, normalized state with this property is

$$|\psi\rangle = \frac{e^{i\delta_1}}{2}|L_z = 1\rangle + \frac{e^{i\delta_2}}{\sqrt{2}}|L_z = 0\rangle + \frac{e^{i\delta_3}}{2}|L_z = -1\rangle.$$

It was stated earlier on that if $|\psi\rangle$ is a normalized state then the state $e^{i\theta}|\psi\rangle$ is a physically equivalent normalized state. Does this mean that the factor $e^{i\delta_i}$ multiplying the L_z eigenstates are irrelevant? (calculate for example $P(L_z=0)$.)

4.2.2. Show that for a real wave function $\psi(x)$, the expectation value of the mom. $\langle P\rangle = 0$. (Hint: show that the probabilities for the momenta $\pm P$ are equal). Generalize this result to the case $\psi = c\psi_r$, where ψ_r is real and c an arbitrary (real or complex) constant. (Recall that $|\psi\rangle$ and $\alpha|\psi\rangle$ are physically equivalent.)

4.2.3. Show that if $\psi(x)$ has mean mom. $\langle p\rangle$, $e^{iP_0 x/\hbar}\psi(x)$ has mean mom. $\langle p\rangle + P_0$.

Chapter 5

5.1.1. Show that the eq. $U(t) = \int_{-\infty}^{\infty} |p\rangle\langle p| e^{-i(p^2/2m\hbar)t}\, dp$ for the propagator of the free particle may be rewritten as an integral over E and a sum over the $\pm$ index as

$$U(t) = \sum_{\alpha=\pm} \int_0^{\infty} \left[\frac{m}{\sqrt{2mE}}\right] |E,\alpha\rangle\langle E,\alpha| e^{-iEt/\hbar}\, dE$$

5.1.2 By solving the eigenvalue eq $\frac{P^2}{2m}|E\rangle = E|E\rangle$ in the X-basis obtain the eq. $|E\rangle = \beta|p = \sqrt{2mE}\rangle + \gamma|p = -\sqrt{2mE}\rangle$

i.e.) Show that the general solution of energy E is

$$\psi_E(x) = \beta \frac{\exp[i\sqrt{2mE}\, x/\hbar]}{\sqrt{2\pi\hbar}} + \gamma \frac{\exp[-i\sqrt{2mE}\, x/\hbar]}{\sqrt{2\pi\hbar}}$$

(The factor $1/\sqrt{2\pi\hbar}$ is arbitrary and may be absorbed into β & γ. Though $\psi_E(x)$ will satisfy the equation even if $E<0$, are these functions in the Hilbert space?

5.1.3. We know that there exists another formula for $U(t)$, namely, $U(t) = e^{-iHt/\hbar}$. For a free particle this becomes

$$U(t) = \exp\left\{\frac{i}{\hbar}\left(\frac{\hbar^2 t}{2m}\frac{d^2}{dx^2}\right)\right\}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{i\hbar t}{2m}\right)^n \frac{d^{2n}}{dx^{2n}}$$

Consider the initial state with $P_0 = 0$, and set $\Delta = 1$ $t' = 0$ i.e.)

$$\psi(x',0) = e^{iP_0 x'/\hbar} \frac{e^{-x'^2/2\Delta^2}}{(\pi\Delta^2)^{1/4}}$$ reduces to

$$\psi(x,0) = \frac{e^{-x^2/2}}{(\pi)^{1/4}}$$

Find $\psi(x,t)$ and compare with

$$\psi(x,t) = \left[\frac{1}{\sqrt{\pi}}\left(\Delta + \frac{i\hbar t}{m\Delta}\right)\right]^{-1/2} \exp\left[\frac{-(x - P_0 t/m)^2}{2\Delta^2(1 + i\hbar t/m\Delta^2)}\right]$$

$$\times \exp\left[\frac{iP_0}{\hbar}(x - P_0 t/2m)\right]$$

for the original initial wave packet $\psi(x',0)$.

Hints:

(i) write $\psi(x,0)$ as a power series:

$$\psi(x,0) = (\pi)^{-1/4} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!\,(2)^n}$$

(ii) Find the action of a few terms

1. $\frac{i\hbar t}{2m}\frac{d^2}{dx^2}$, $\frac{1}{2!}\left(\frac{i\hbar t}{2m}\right)^2 \frac{d^4}{dx^4}$, etc. on this power series.

(iii) Collect terms with the same powers of x.

(iv) Look for the following series expansion in the coeff. of x^{2n}:

$$\left(1+\frac{it\hbar}{m}\right)^{-n-1/2} = 1-(n+1/2)\left(\frac{it\hbar}{m}\right)+\frac{(n+1/2)(n+3/2)}{2!}\left(\frac{it\hbar}{m}\right)^2+\ldots$$

(v) juggle around till you get the answer.

5.1.4. A famous counter example. Consider the wave function

$$\psi(x,0) = \sin\frac{\pi x}{L} \quad |x| \le L/2$$

$$= 0, \quad |x| \geq L/2$$

It is clear that when this function is differentiated any no. of times we get another function confined to the interval $|x| \le L/2$. Consequently the action of

$$U(t) = \exp\left[i/\hbar \left(\frac{\hbar^2 t}{2m}\right)\frac{d^2}{dx^2}\right]$$

On this function is to give a function confined to $|x| \le L/2$. What about the spreading of the wave packet?

5.2.1. A particle is in the ground state of a box of length L. Suddenly the box expands (symmetrically) to twice its size, leaving the wave function undisturbed. Show that the probability of finding the particle in the ground state of the new box is $(8/3\pi)^2$.

5.2.2. a, Show that for any normalized $|\psi\rangle$, $\langle\psi|H|\psi\rangle \geq E_0$ where E_0 is the lowest-energy eigenvalue. (Hint: Expand $|\psi\rangle$ in the eigenbasis of H.)

b, Prove the following theorem: Every attractive potential in 1-D has at least one bound state. Hint: Since V is attractive, if we define $V(\infty)=0$, it follows that $V(x) = -|V(x)|$ for all x. To show that there exists a bound state with $E<0$ consider

$$\psi_\alpha(x) = \left(\frac{\alpha}{\pi}\right)^{1/4} e^{-\alpha x^2/2}$$

and calculate $E(\alpha) = \langle\psi_\alpha|H|\psi_\alpha\rangle$, $H = \frac{-\hbar^2}{2m}\frac{d^2}{dx^2} - |V(x)|$

Show that $E(\alpha)$ can be made negative by a suitable choice of α. The desired result follows from the application of the theorem proved above.

5.2.3. Consider $V(x) = -aV_0\delta(x)$. Show that it admits a bound state of energy $E = -ma^2V_0^2/2\hbar^2$. Are there any other bound states? ~~of energy~~. Hint: Solve Schrödinger's eq. outside the potential for $E<0$, and keep only the solution that has the right behaviour at infinity and is continuous at $x=0$. Draw the W.F. and see how there is a cusp, or a discontinuous change of slope at $x=0$. Calculate the change in slope and equate it to

$$\int_{-\epsilon}^{\epsilon} \left(\frac{d^2\psi}{dx^2}\right) dx$$

Where ϵ is infinitesimal. determined from Schrödinger's eq.

5.2.4. Consider a particle of mass m in the state $|n\rangle$ of a box of length L. Find the force $F = -\partial E/\partial L$ encountered when the walls are slowly pushed in, assuming the particle remains in the nth state of the box as its size changes. Consider a classical particle of energy E_n in this box. Find its velocity, the frequency of ~~oscillation~~ collision on a given wall, the mom. transfer per collision, and hence the average force. Compare it to $\partial E/\partial L$ computed above.

5.2.5. If the box extends from $x=0$ to $x=L$ (instead of $-L/2$ to $L/2$) show that $\psi_n(x) = \sqrt{2/L}\,\sin(n\pi x/L)$, $n=1,2,\dots\infty$ and $E_n = \hbar^2\pi^2n^2/2mL^2$.

5.2.6. Square well potential. Consider a particle in a square well potential:

$$V(x) = \begin{cases} 0, & |x| \le a \\ V_0, & |x| > a \end{cases}$$

Since when $V_0 \to \infty$, we have a box, let us guess what the lowering of the walls does to the states. First of all, all the bound states (which alone we are interested in), will have $E \le V_0$. Second the W.F.s of the low lying-levels will look like those of the particle in the box, with the obvious difference that ψ will not vanish at the walls but instead spill out with an exceptional tail. The eigenfunctions will still be even, odd, even etc.

(i) Show that the even solutions have energies that satisfy the transcendental eq. $k\tan ka = \varkappa$

While the odd ones will have energies that satisfy
$k\cot ka = -\varkappa$
where k and $i\varkappa$ are the real & complex W.F.s wave nos inside and outside the well resp. Note that k and $\varkappa$ are related by

$$k^2 + \varkappa^2 = 2mV_0/\hbar^2$$

verify that as $V_0 \to \infty$, we regain the levels in the box.

(ii) The first two eqs must be solved graphically. In the ($\alpha = a$, $\beta = \kappa a$) plane, imagine a circle that obeys the last eq. above. The bound states are then given by the intersection of the curve $\alpha \tan \alpha = \beta$ or $\alpha \cot \alpha = -\beta$ with the circle. (Remember α & β are positive.)

5.3.1. Consider the case $V = V_r - iV_i$, where the imaginary part V_i is a constant. Is the Hamiltonian Hermitian? Go through the derivation of the continuity eq. and show that the total probability for finding the particle decreases exponentially as $e^{-2V_i t/\hbar}$. Such complex potentials are used to describe processes in which particles are absorbed by a sink.

5.3.2. Convince yourself that if $\psi = c\tilde{\psi}$, where c is constant (real or complex) and $\tilde{\psi}$ is real, the corresponding $\mathbf{J}$ vanishes.

5.3.3. Consider

$$\psi_{\vec{p}} = \left(\frac{1}{2\pi\hbar}\right)^{3/2} e^{i(\vec{p}\cdot\vec{r})/\hbar}$$

Find $\mathbf{J}$ and ρ and compare the relation between them to the electromagnetic eq. $\bar{J} = \rho\bar{v}$, $\bar{v}$ being the velocity. Since ρ and $\mathbf{J}$ are const, note that the continuity eq. is trivially satisfied.

5.3.4. Consider $\psi = Ae^{ipx/\hbar} + Be^{-ipx/\hbar}$ in 1-D. Show that $J = (|A|^2 - |B|^2)p/m$. The absence of cross terms between the right and left-moving pieces in ψ allows us to associate the two parts of J with corresponding parts of ψ.

~~Chapter 17~~

5.4.1 x

5.4.2 a, Calculate R and T for scattering of a potential $V(x) = V_0 a\delta(x)$. b, Do the same for the case $V = 0$ for $|x| > a$ and $V = V_0$ for $|x| < a$. Assume that the energy is positive but less than V_0.

5.4.3. Consider a particle subject to a constant force f in one dimension. Solve for the propagator in the mom. space and get

$$U(p,t;p',0) = \delta(p - p' - ft)\, e^{i(p'^3 - p^3)/6m\hbar f}$$

Transform back to coordinate space and obtain

$$U(x,t;x',0) = \left(\frac{m}{2\pi\hbar i t}\right)^{1/2} \exp\left\{\frac{i}{\hbar}\left[\frac{m(x-x')^2}{2t} + \tfrac{1}{2}ft(x+x') - \frac{f^2t^3}{24m}\right]\right\}$$

Chapter 7

7.3.3. If $\psi(x)$ is even and $\phi(x)$ is odd under $x \to -x$, show that $\int_{-\infty}^{\infty} \psi(x)\Phi(x)\,dx = 0$

This can be used in showing the orthogonality property of the oscillator eigenfunctions.

7.3.4. Using the recursion relations show that

$$\langle n'|X|n\rangle = \left(\frac{\hbar}{2m\omega}\right)^{1/2}\left[\delta_{n',n+1}(n+1)^{1/2} + \delta_{n',n-1}\, n^{1/2}\right]$$

$$\langle n'|P|n\rangle = \left(\frac{m\omega\hbar}{2}\right)^{1/2}\left[\delta_{n',n+1}(n+1)^{1/2} - \delta_{n',n-1}\, n^{1/2}\right]$$

7.3.5. Using the symmetry arguments from 7.3.3. show that $\langle n|X|n\rangle = \langle n|P|n\rangle = 0$ and thus that $\langle X^2\rangle = (\Delta X)^2$ and $\langle P^2\rangle = (\Delta P)^2$ in these states. Show that $\langle 1|X^2|1\rangle = 3\hbar/2m\omega$ and $\langle 1|P^2|1\rangle = \tfrac{3}{2}m\omega\hbar$. Show that $\psi_0(x)$ saturates the uncertainty bound $\Delta X \cdot \Delta P \geq \hbar/2$.

7.3.6. Consider a particle in a potential

$$V(x) = \tfrac{1}{2}m\omega x^2, \quad x > 0$$
$$= \infty, \quad x \leq 0$$

What are the boundary ~~functions~~ conditions on the wave functions now. Find the eigenvalues and eigenfunctions.

Quantum Mechanics I (phys. 542)
Solution to some problems

1.1.1. - In this vector space the null vector in the form given is (0,0,0).

- suppose the inverse of (a,b,c) is (a',b',c') then by axiom (iv)

$$(a,b,c) + [(a',b',c')] = (0,0,0)$$

$$(a+a', b+b', c+c') = (0,0,0) \quad \text{, by the def.}$$

i.e., $a + a' = 0$
$b + b' = 0$
$c + c' = 0$

$\Leftrightarrow$ $a' = -a, \; b' = -b, \; c' = -c$

$$(a', b', c') = (-a, -b, -c)$$

or $(a', b', c') = -(a, b, c)$, by the def.

- If a, b, c are required to be positive nos we could not have a vector space as we could not have inverse elements of the vectors as well as the identity element (0,0,0).

- ~~[illegible]~~

~~[illegible]~~

Suppose the identity element ~~[illegible]~~ is (d,e,1)
By axiom (iii)

$$(a,b,1) + (d,e,1) \overset{?}{=} (a,b,1)$$

$$(a+d, b+e, 2) \overset{?}{=} (a,b,1)$$

$\Rightarrow$ $a+d = a, \; d = 0$
$b+e = b, \; e = 0$
$2 = 1$, impossible

$\therefore$ (a, b, 1) do not form a linear vector space as at lease axiom (iii) is not satisfied.

1.1.2. (1) $\alpha \bar{V} + 0\bar{V} = (\alpha + 0)\bar{V}$, by axiom (vi)
$= \alpha \bar{V}$

But $\alpha \bar{V} + \bar{0} = \alpha \bar{V}$, by axiom (iii)
Comparing the L.H.S. of the two equations we obtain
$$0\bar{V} = \bar{0}$$

(2) $\alpha\bar{v} + \alpha\bar{0} = \alpha(\bar{v}+\bar{0})$, by axiom (v)

$= \alpha\bar{v}$, by axiom (iii)

But $\alpha\bar{v} + \bar{0} = \alpha\bar{v}$, Again by axiom (iii)

Comparing the L.H.S. of the equations

$\alpha\bar{0} = \bar{0}$

(3) $(-1)\bar{v} + \bar{v} = (-1+1)\bar{v}$, by axiom (vi)

$= 0\bar{v}$

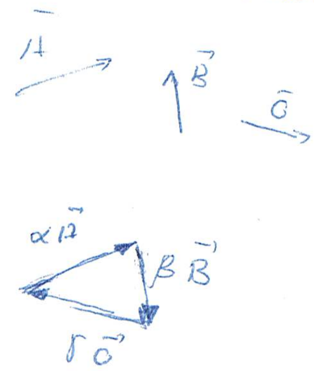
$= \bar{0}$ by (1) of this exercise

$\bar{v} + (-\bar{v}) = \bar{0}$, by axiom (iv)

i.e. $(-1)\bar{v} = -\bar{v}$, by comparing the two eqs above.

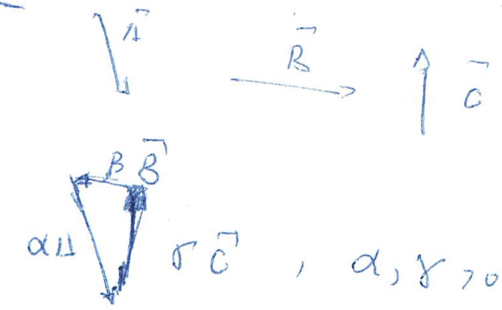
1.1.3. For vectors in a plane this statement can be proved by geometric reasonings.

–



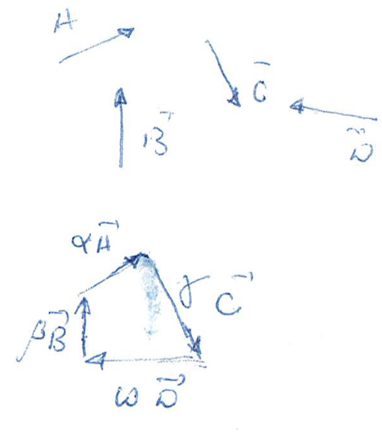
$\alpha > 0$, $\beta, \gamma < 0$

–



, $\alpha, \gamma > 0$, $\beta < 0$

–



1.1.4. – If the elements are linearly dependent we may express one of them as the linear combination of the remaining ones. Suppose then

$$(3,2,1) = \alpha(1,1,0) + \beta(1,0,1)$$

$$(3,2,1) = (\alpha,\alpha,0) + (\beta,0,\beta)$$

$$= (\alpha+\beta, \alpha, \beta)$$

i.e., $\alpha + \beta = 3$

$\alpha = 2$

$\beta = 1$

$\Leftrightarrow \alpha(1,1,0) + \beta(1,0,1) + \gamma(3,2,1) = 0$

with general solution

$$\alpha, \ \beta = \frac{\alpha}{2}, \ \gamma = -\frac{\alpha}{2}$$

so that we can select any nonzero real no. α.

Thus the vectors are indeed linearly dependent.

– suppose we can express one of the vectors say, (1,1,0) as

$$(1,1,0) = \alpha(1,0,1) + \beta(0,1,1)$$

$= (\alpha, 0, \alpha) + (0, \beta, \beta)$ by the def of prob. 1

$= (\alpha, \beta, \alpha+\beta)$, Again by def.

$\Rightarrow \alpha = 1$

$\beta = 1$

$\alpha+\beta = 0$ i.e., $\beta = -\alpha$

The last result contradicts the first two. That means we cannot express any one of the three vectors as a linear combination of the remaining two. Consequently the vectors are linearly dependent.

1.2.1. For the proof of theorem 3 we constructed a vector

$$\bar{v} = \bar{v}_i - \frac{\langle v_j | v_i \rangle \bar{v}_j}{|\bar{v}_j|^2}$$

$\langle v | v \rangle \geq 0$

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

$$\langle v|v\rangle = \left\langle v_i - \frac{\langle v_j|v_i\rangle v_j}{|v_j|^2} \,\middle|\, v_i - \frac{\langle v_j|v_i\rangle v_j}{|v_j|^2} \right\rangle$$

$$= \langle v_i|v_i\rangle - \left\langle v_i \,\middle|\, \frac{(\langle v_j|v_i\rangle) v_j}{|v_j|^2} \right\rangle$$

$$- \left\langle \frac{(\langle v_j|v_i\rangle) v_j}{|v_j|^2} \,\middle|\, v_i \right\rangle + \left\langle \frac{(\langle v_j|v_i\rangle)}{|v_j|^2} v_j \,\middle|\, \frac{(\langle v_j|v_i\rangle) v_j}{|v_j|^2} \right\rangle$$

$$= |v_i|^2 - \frac{\langle v_j|v_i\rangle\langle v_i|v_j\rangle}{|v_j|^2} - \frac{\langle v_j|v_i\rangle^*\langle v_j|v_i\rangle}{|v_j|^2}$$

$$+ \frac{\langle v_j|v_i\rangle^*\langle v_j|v_i\rangle\langle v_j|v_j\rangle}{|v_j|^4}$$

$$= |v_i|^2 - \frac{\langle v_j|v_i\rangle\langle v_i|v_j\rangle}{|v_j|^2} - \frac{\langle v_i|v_j\rangle\langle v_j|v_i\rangle}{|v_j|^2}$$

$$+ \frac{\langle v_i|v_j\rangle\langle v_j|v_i\rangle}{|v_j|^2}$$

$$= |v_i|^2 - \frac{\langle v_j|v_i\rangle\langle v_i|v_j\rangle}{|v_j|^2}$$

$$= |v_i|^2 - \frac{\langle v_i|v_j\rangle^*\langle v_i|v_j\rangle}{|v_j|^2}$$

$$= |v_i|^2 - \frac{|\langle v_i|v_j\rangle|^2}{|v_j|^2} \geq 0$$

i.e., $|\langle v_i|v_j\rangle|^2 \leq |v_i|^2|v_j|^2$

Now $\langle v|v\rangle = |v_i|^2 - \dfrac{|\langle v_i|v_j\rangle|^2}{|v_j|^2} \geq 0$

Suppose $\bar{v}_i = \lambda\bar{v}_j$. Then

$$\bar{v} = \lambda\bar{v}_j - \frac{\langle v_j|v_i\rangle \bar{v}_j}{|v_j|^2}$$

$$= \lambda\bar{v}_j - \lambda\frac{|v_j|^2}{|v_j|^2}\bar{v}_j$$

$$= \lambda\bar{v}_j - \lambda\bar{v}_j$$

$$= 0$$

$\Rightarrow \langle v|v\rangle = |v_i|^2 - \dfrac{|\langle v_i|v_j\rangle|^2}{|v_j|^2} = 0$

$\therefore |\langle v_i|v_j\rangle| = |v_i|^2|v_j|^2$ when $\bar{v}_i = \lambda\bar{v}_j$.

1.2.2.

$$|v_i+v_j|^2 = \langle v_i+v_j | v_i+v_j \rangle$$
$$= \langle v_i|v_i\rangle + \langle v_i|v_j\rangle + \langle v_j|v_i\rangle + \langle v_j|v_j\rangle$$
$$= |v_i|^2 + |v_j|^2 + \langle v_i|v_j\rangle + \langle v_i|v_j\rangle^*$$
$$= |v_i|^2 + |v_j|^2 + 2\,\mathrm{Re}\langle v_i|v_j\rangle$$

or $|v_i+v_j|^2 \le |v_i|^2 + |v_j|^2 + 2|\langle v_i|v_j\rangle|$ as $\mathrm{Re}\langle v_i|v_j\rangle \le |\langle v_i|v_j\rangle|$

$\le |v_i|^2 + |v_j|^2 + 2|v_i||v_j|$, as $|\langle v_i|v_j\rangle| \le |v_i||v_j|$ by the Schwarz inequality.

$\le (|v_i| + |v_j|)^2$

i.e., $|v_i+v_j| \le |v_i| + |v_j|$

But if $\bar{v}_i = \lambda \bar{v}_j$, where λ is a real positive no

(i) $\mathrm{Re}\langle v_i|v_j\rangle = \mathrm{Re}\langle \lambda v_j|v_j\rangle = \lambda\,\mathrm{Re}\langle v_j|v_j\rangle = \lambda|v_j|^2$

$|\langle v_i|v_j\rangle| = |\langle \lambda v_j|v_j\rangle| = \lambda|\langle v_j|v_j\rangle| = \lambda|v_j|^2$

$\Rightarrow$

$\mathrm{Re}\langle v_i|v_j\rangle = |\langle v_i|v_j\rangle|$

(ii) $|v_i||v_j| = |\lambda v_j||v_j| = \lambda|v_j|^2$

From (i) and this result

$|\langle v_i|v_j\rangle| = |v_i||v_j|$

So now by the transitivity of equality

$\mathrm{Re}\langle v_i|v_j\rangle = |\langle v_i|v_j\rangle| = |v_i||v_j|$ where $\bar{v}_i = \lambda\bar{v}_j$

and

$$|v_i+v_j|^2 = |v_i|^2 + |v_j|^2 + 2\,\mathrm{Re}\langle v_i|v_j\rangle$$
$$= |v_i|^2 + |v_j|^2 + 2|v_i||v_j|$$
$$= (|v_i| + |v_j|)^2$$

or

$$\boxed{|v_i+v_j| = |v_i| + |v_j| \quad \bar{v}_i = \lambda\bar{v}_j}$$

Q.E.D.

1.3.1. We construct a set of mutually orthogonal vectors the $(m')^{th}$ of which is given as

$$|m'\rangle = v_m - \sum_{i'=1}^{m-1} \frac{|i'\rangle\langle i'|v_m\rangle}{\langle i'|i'\rangle}$$

And orthogonal vectors are LI.

Suppose then we have the null vector in our set and let it be given by

$$|m'\rangle = |V_m\rangle - \sum_{i'=1}^{m-1} \frac{|i'\rangle\langle i'|V_m\rangle}{\langle i'|i'\rangle} = 0$$

$$\Rightarrow |V_m\rangle = \sum_{i'=1}^{m-1} \frac{|i'\rangle\langle i'|V_m\rangle}{\langle i'|i'\rangle}$$

$$= \sum_{i'=1}^{m-1} \frac{|i'\rangle\langle i'|V_m\rangle}{|i'|^2}$$

But $\frac{\langle i'|V_m\rangle}{\langle i'|i'\rangle} = \frac{\langle i'|V_m\rangle}{|i'|^2}$ is the projection of $|V_m\rangle$ along $|i'\rangle$.

$$|V_m\rangle = \sum_{i'=1}^{m-1} |i'\rangle v_{i'}$$

$$= \sum_{i'=1}^{m-1} \alpha_{i'} |V_{i'}\rangle$$

$$= \sum_{j=1}^{m-1} \alpha_j |V_j\rangle, \quad i' \text{ is a dummy index.}$$

That means we express $|V_m\rangle$ in terms of $|V_1\rangle, \dots |V_{m-1}\rangle$.

1.3.2- From the proof of the Gram-Schmidt theorem we have that

$$|m'\rangle = |V_m\rangle - \sum_{i'=1}^{m-1} \frac{|i'\rangle\langle i'|V_m\rangle}{\langle i'|i'\rangle}$$

$$|1'\rangle = |V_1\rangle \Rightarrow |1\rangle = \frac{|1'\rangle}{|1'|} = \frac{|V_1\rangle}{|V_1|} = \frac{1}{3}\begin{bmatrix}3\\0\\0\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix}$$

$$|2'\rangle = |V_2\rangle - \frac{|1'\rangle\langle 1'|V_2\rangle}{\langle 1'|1'\rangle}$$

$$= \begin{bmatrix}0\\1\\2\end{bmatrix} - \frac{1}{9}\begin{bmatrix}3\\0\\0\end{bmatrix}\cdot 0 = \begin{bmatrix}0\\1\\2\end{bmatrix}$$

$$|2\rangle = \frac{|2'\rangle}{|2'|} = \frac{1}{\sqrt{5}}\begin{bmatrix}0\\1\\2\end{bmatrix} = \begin{bmatrix}0\\1/\sqrt{5}\\2/\sqrt{5}\end{bmatrix}$$

$$|3'\rangle = |v_3\rangle - \frac{|1'\rangle\langle 1'|v_3\rangle}{\langle 1'|1'\rangle} - \frac{|2'\rangle\langle 2'|v_3\rangle}{\langle 2'|2'\rangle}$$

$$\langle 1'|1'\rangle = (3,0,0)\begin{bmatrix}3\\0\\0\end{bmatrix} = (9)$$

$$\langle 1'|v_3\rangle = (3,0,0)\begin{bmatrix}0\\2\\5\end{bmatrix} = (0)$$

$$\langle 2'|v_3\rangle = (0,1,2)\begin{bmatrix}0\\2\\5\end{bmatrix} = (12)$$

$$\langle 2'|2'\rangle = (0,1,2)\begin{bmatrix}0\\1\\2\end{bmatrix} = (5)$$

$$\Rightarrow |3'\rangle = \begin{bmatrix}0\\2\\5\end{bmatrix} - \frac{12}{5}\begin{bmatrix}0\\1\\2\end{bmatrix} = \begin{bmatrix}0\\2\\5\end{bmatrix} - \begin{bmatrix}0\\2.4\\4.8\end{bmatrix} = \begin{bmatrix}0\\-0.4\\0.2\end{bmatrix}$$

$$|3\rangle = \frac{|3'\rangle}{|3'|} \; ; \quad |3'| = \sqrt{(-0.4)^2 + (0.2)^2} = \sqrt{0.2} = \sqrt{\frac{2}{10}} = \frac{1}{\sqrt{5}}$$

$$|3\rangle = \frac{1}{\sqrt{5}}\begin{bmatrix}0\\-0.4\\0.2\end{bmatrix} = \begin{bmatrix}0\\-0.4/\sqrt{5}\\0.2/\sqrt{5}\end{bmatrix} = \begin{bmatrix}0\\-2/\sqrt{5}\\1/\sqrt{5}\end{bmatrix}$$

In general

$$|1\rangle \leftrightarrow \begin{bmatrix}1\\0\\0\end{bmatrix}, \quad |2\rangle \leftrightarrow \begin{bmatrix}0\\1/\sqrt{5}\\2/\sqrt{5}\end{bmatrix}, \quad |3\rangle \leftrightarrow \begin{bmatrix}0\\-2/\sqrt{5}\\1/\sqrt{5}\end{bmatrix}$$

1.4.1. V^n is an n-D vector space. It contains all possible vectors that can be expressed as the linear combinations of the n LI vectors in V^n. Also, by the Gram-Schmidt theorem we have n mutually orthogonal vectors in V^n which are obtained by a mathematical procedure from these n LI vectors. Clearly the set $\{|V_1^1\rangle, |V_1^2\rangle, \ldots\}$ of orthogonal vectors to any $|V\rangle \neq 0$ ~~of N/M~~ is a subset of V^n since its elements are expressible in terms of the n mutually orthogonal vectors of V^n.

It must then be true that the basis vectors of the above set must be constructed in accordance with the procedure in the Gram-Schmidt theorem. Thus, for the last vector of the set we must have

$$|V_1^{n'}\rangle = |V_n\rangle - \sum_{i'=1}^{n-1} \frac{|i'\rangle\langle i'|V_n\rangle}{\langle i'|i'\rangle}$$

But this last vector vanishes (Exercise 1.3.1) i.e., we have n-1 mutual orthogonal vectors in $\{|V_1^1\rangle, \ldots\}$ and these are LI by the Lemma of theorem 6. Thus the above set is (n-1)-D vector space and hence it is a subspace V^{n-1} of V^n.

1.4.2. The dimension of V_1 is n_1 whereas that of V_2 is n_2. $V_1 \oplus V_2$ consists of the set of a, all elements of V_1 b, all elements of V_2 c, all possible linear combinations of the above. The basis vectors of V_1 are orthogonal to the n_2 basis orthogonal vectors of V_2 - according to the prob. So we have n_1+n_2 mutually orthogonal vectors in $V_1 \oplus V_2$ which by theorem 6 are LI. Where by the dimensionality of $V_1 \oplus V_2$ is n_1+n_2.

1.6.1. The action of Ω on any vector is seen by its action on the basis vectors

$$\Omega|1\rangle = |1'\rangle = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = |2\rangle$$

$$\Omega|2\rangle = |2'\rangle = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = |3\rangle$$

$$\Omega|3\rangle = |3'\rangle = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = |1\rangle$$

1.6.2. $\Omega^+ = \Omega, \; \Lambda^+ = \Lambda$

(i) $(\Omega\Lambda)^+ = \Lambda^+\Omega^+ = \Lambda\Omega$ ~~#Hermit~~

~~(ii) ΩΛ is not Hermitian~~

$$(\Lambda\Omega)^+ = \Omega^+\Lambda^+ = \Omega\Lambda$$

In general for operators $\Omega\Lambda \neq \Lambda\Omega$ and hence $\Omega\Lambda$ is not Hermitian.

(ii) $$(\Omega\Lambda + \Lambda\Omega)^+ = (\Omega\Lambda)^+ + (\Lambda\Omega)^+$$
$$= \Lambda^+\Omega^+ + \Omega^+\Lambda^+$$
$$= \Lambda\Omega + \Omega\Lambda$$
$$= \Omega\Lambda + \Lambda\Omega$$

i.e., $\Omega\Lambda + \Lambda\Omega$ is Hermitian.

(iii). $$[\Omega, \Lambda] = \Omega\Lambda - \Lambda\Omega$$

$$[\Omega,\Lambda]^+ = (\Omega\Lambda - \Lambda\Omega)^+$$
$$= (\Omega\Lambda)^+ - (\Lambda\Omega)^+$$
$$= \Lambda^+\Omega^+ - \Omega^+\Lambda^+$$
$$= \Lambda\Omega - \Omega\Lambda$$
$$= -[\Omega\Lambda - \Lambda\Omega]$$
$$= -[\Omega, \Lambda]$$

i.e., $[\Omega, \Lambda]$ is anti Hermitian.

(iv) $i[\Omega, \Lambda] = i\{\Omega\Lambda - \Lambda\Omega\}$

$\{i[\Omega,\Lambda]\}^{+} = -i\{\Omega\Lambda - \Lambda\Omega\}^{+}$

$= -i\{(\Omega\Lambda)^{+} - (\Lambda\Omega)^{+}\}$

$= -i\{\Lambda^{+}\Omega^{+} - \Omega^{+}\Lambda^{+}\}$

$= -i\{\Lambda\Omega - \Omega\Lambda\}$

$= i\{\Omega\Lambda - \Lambda\Omega\}$

$= i[\Omega, \Lambda]$

i.e., $i[\Omega, \Lambda]$ is Hermitian.

1.6.3. Show that a product of unitary operators is unitary.

Consider two unitary operators U_1 & U_2

$(U_1 U_2)(U_1 U_2)^{+}$

$= U_1 U_2 U_2^{+} U_1^{+}$

$= U_1 I U_1^{+}$, U_2 is unitary

$= U_1 U_1^{+}$

$= I$, U_1 is unitary.

∴ $U_1 U_2$ is unitary.

1.6.4. For a unitary matrix U

$UU^{+} = I$

$\det(UU^{+}) = \det I = 1$ $\qquad \det U = e^{i\theta}$

$\det[U(U^{T})^{*}] = 1$

$\det U \det(U^{T})^{*} = 1$

$\det U (\det U^{T})^{*} = 1$

$\det U (\det U)^{*} = 1$

$|\det U|^{2} = 1$

or $|\det U| = 1$

i.e., we may write

1.6.5.
$$R(\tfrac{1}{2}\pi\hat{i}) = \begin{bmatrix} \langle 1|R(\tfrac{1}{2}\pi\hat{i})|1\rangle & \langle 1|R(\tfrac{1}{2}\pi\hat{i})|2\rangle & \langle 1|R(\tfrac{1}{2}\pi\hat{i})|3\rangle \\ \langle 2|R(\tfrac{1}{2}\pi\hat{i})|1\rangle & \langle 2|R(\tfrac{1}{2}\pi\hat{i})|2\rangle & \langle 2|R(\tfrac{1}{2}\pi\hat{i})|3\rangle \\ \langle 3|R(\tfrac{1}{2}\pi\hat{i})|1\rangle & \langle 3|R(\tfrac{1}{2}\pi\hat{i})|2\rangle & \langle 3|R(\tfrac{1}{2}\pi\hat{i})|3\rangle \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$R(\tfrac{1}{2}\pi\hat{i})$ is defined on a real vector space so that its adjoint is just its transpose.

$$[R(\tfrac{1}{2}\pi\hat{i})]^{+} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

$$[R(\tfrac{1}{2}\pi\hat{i})][R(\tfrac{1}{2}\pi\hat{i})]^{+}$$
$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Hence $R(\tfrac{1}{2}\pi\hat{i})$ is unitary (orthogonal).

1.6.6. – $\Omega = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$, $\Omega^{+} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}$

$$\Omega\Omega^{+} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

i.e., $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$ is unitary,

– $\Lambda = \frac{1}{2}\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix}$, $\Lambda^{+} = \frac{1}{2}\begin{bmatrix} 1-i & 1+i \\ 1+i & 1-i \end{bmatrix}$

$$\Lambda\cdot\Lambda^{+} = \frac{1}{2}\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix}\frac{1}{2}\begin{bmatrix} 1-i & 1+i \\ 1+i & 1-i \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

i.e., $\frac{1}{2}\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix}$ is unitary.

Determinants

$$\det\Omega = \frac{1}{\sqrt{2}}\begin{vmatrix} 1 & i \\ i & 1 \end{vmatrix} = \frac{1}{\sqrt{2}}(1+1) = \frac{2}{\sqrt{2}} = \sqrt{2}\,e^{i0} \quad \text{i.e., } \theta = 0$$

$$\det\Lambda = \frac{1}{2}\begin{vmatrix} 1+i & 1-i \\ 1-i & 1+i \end{vmatrix} = \frac{1}{2}\left((1+i)^2 - (1-i)^2\right) = \frac{1}{2}(4i) = 2i = 2e^{i\frac{\pi}{2}}$$

i.e., $\theta = \pi/2$ rad.

They are not Hermitian: $\Omega^{+} \neq \Omega$, $\Lambda^{+} \neq \Lambda$.

2.7.1. a, Let $\Psi = \Omega\Lambda$

$$Tr\ \Omega\Lambda = \sum_i \Psi_{ii}$$

But from matrix product $\Psi_{ij} = \sum_k \Omega_{ik}\Lambda_{kj}$

Whence,

$$\Psi_{ii} = \sum_k \Omega_{ik}\Lambda_{ki}$$

$$Tr(\Omega\Lambda) = \sum_i \Psi_{ii} = \sum_i \sum_k \Omega_{ik}\Lambda_{ki}$$

$$= \sum_i \sum_k \Lambda_{ki}\Omega_{ik}$$

$$= \sum_k \sum_i \Lambda_{ki}\Omega_{ik}$$

$$= Tr(\Lambda\Omega)$$

b, 1. $Tr(\Omega\Lambda\theta) = Tr[\Omega(\Lambda\theta)]$, Matrix product is associative

$= Tr[(\Lambda\theta)\Omega]$, by (a) treating $\Lambda\theta$ as one operator

$= Tr(\Lambda\theta\Omega)$

2. $Tr(\Omega\Lambda\theta) = Tr[(\Omega\Lambda)\theta]$

$= Tr[\theta(\Omega\Lambda)]$, by (a) above

$= Tr(\theta\Omega\Lambda)$

∴ From (1) and (2),

$$Tr(\Omega\Lambda\theta) = Tr(\Lambda\theta\Omega) = Tr(\theta\Omega\Lambda)$$

c, If $|i\rangle \to U|i\rangle$, then $\langle i'|\Omega|i\rangle \to \langle Ui'|\Omega|Ui\rangle$

$= \langle i'|U^+\Omega U|i\rangle$

i.e., $\Omega \to U^+\Omega U$

$Tr(U^+\Omega U) = Tr(\Omega UU^+)$, by (b)

$= Tr(\Omega I)$

$= Tr(\Omega)$, thus the trace is unaffected.

Also, $Tr\ \Omega = Tr(I\Omega) = Tr(UU^+\Omega)$

= ~~[illegible]~~

$= Tr(U^+\Omega U)$, by (b).

Q.E.D.

1.7.2. $\det(U^+ \Omega U) = \det[U^+ (\Omega U)]$

$= \det U^+ \det(\Omega U)$ - the det. of a product of matrices equals the product of the determinants

$= \det(\Omega U) \det U^+$, product of nos is commutative

$= \det(\Omega U U^+)$

$= \det[\Omega (U U^+)]$

$= \det(\Omega I)$.

$= \det \Omega$

And $\det \Omega = \det(I \Omega)$

$= \det(U U^+ \Omega)$

$= \det[U (U^+ \Omega)]$

$= \det U \det(U^+ \Omega)$

$= \det(U^+ \Omega) \det U$

$= \det(U^+ \Omega U)$, Q.E.D.

1.8.1. a, The characteristic eq. which gives the eigenvalues may be written as

$$\det(\Omega - \omega I) = 0$$

$$\Rightarrow \begin{vmatrix} 1-\omega & 3 & 1 \\ 0 & 2-\omega & 0 \\ 0 & 1 & 4-\omega \end{vmatrix} = 0$$

$$(1-\omega)(2-\omega)(4-\omega) = 0$$

i.e., $\omega = 1, 2, 4$ are the eigenvalues

the comps of the eigenvectors satisfy the following eqs.

— $\omega_1 = 1$

$$\begin{bmatrix} 0 & 3 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ $3x_2 + x_3 = 0$ (x_1 is arbitrary)

$x_2 = 0$

$x_2 + 3x_3 = 0$

Then $x_2 = 0 \Rightarrow x_3 = 0$

Soln x_1 arbitrary

$x_2 = x_3 = 0$

The corresponding normalized eigenvector would be

$$|\omega = 1\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- $\omega = 2$

$$\begin{bmatrix} -1 & 3 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-x_1 + 3x_2 + x_3 = 0$

$0 = 0$

$x_2 + 2x_3 = 0 \Rightarrow x_2 = -2x_3$

Then $-x_1 + 3(-2x_3) + x_3 = 0$

$-x_1 - 5x_3 = 0$

$x_1 = -5x_3$

Soln. $x_1 = -5x_3$, x_3 arbitrary

$x_2 = -2x_3$

The corresponding normalized eigenvector would be

$$|\omega = 2\rangle = \frac{1}{\sqrt{30}} \begin{bmatrix} -5 \\ -2 \\ 1 \end{bmatrix}$$

- $\omega = 4$

$$\begin{bmatrix} -3 & 3 & 1 \\ 0 & -2 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-3x_1 + 3x_2 + x_3 = 0$

$-2x_2 = 0$

$x_2 = 0$

$-3x_1 + x_3 = 0$

$x_3 = 3x_1$

Soln. x_1 - arbitrary

$x_2 = 0$

$x_3 = 3x_1$

$$|\omega = 4\rangle = \frac{1}{\sqrt{10}} \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

b) $\Omega^{+} = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 2 & 1 \\ 1 & 0 & 4 \end{bmatrix} \neq \Omega$, hence it is not Hermitian.

orthogonality:

$$\langle \omega=1|\omega=2\rangle = (1,0,0)\frac{1}{\sqrt{30}}\begin{bmatrix}-5\\-2\\1\end{bmatrix} = \frac{1}{\sqrt{30}}(-5) = \frac{-5}{\sqrt{30}} \neq 0$$

$$\langle \omega=1|\omega=4\rangle = (1,0,0)\frac{1}{2}\begin{pmatrix}1\\0\\3\end{pmatrix} = \frac{1}{2}(1+0+0) = \frac{1}{2} \neq 0$$

$$\langle \omega=2|\omega=4\rangle = \frac{1}{\sqrt{30}}(-5,-2,1)\frac{1}{2}\begin{pmatrix}1\\0\\3\end{pmatrix} = \frac{1}{2\sqrt{30}}(-5+3) = -\frac{1}{\sqrt{30}} \neq 0$$

Hence the eigenvectors are not orthogonal.

1.8.2. a) $\Omega^+ = \begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix} = \Omega$, hence Ω is Hermitian

b) characteristic eq.

$$\begin{vmatrix}-\omega&0&1\\0&-\omega&0\\1&0&-\omega\end{vmatrix} = 0$$

$$\Rightarrow -\omega(\omega^2) + \omega = 0$$
$$\omega(\omega^2-1) = 0$$
$$\omega(\omega-1)(\omega+1) = 0$$

Soln. $\omega=0,\ \omega=+1,\ \omega=-1$

- $\omega = 0$

$$\begin{bmatrix}0&0&1\\0&0&0\\1&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$x_3 = 0$
$0 = 0$
$x_1 = 0$

Soln $x_1 = 0$, x_2 arbitrary, $x_3 = 0$

$$|\omega=0\rangle = \begin{bmatrix}0\\1\\0\end{bmatrix}$$

- $\omega = 1$

$$\begin{bmatrix}-1&0&1\\0&-1&0\\1&0&-1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$-x_1 + x_3 = 0$
$-x_2 = 0$
$x_1 - x_3 = 0$

Soln x_1 arbitrary
$x_2 = 0$
$x_3 = x_1$

$$|\omega=1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\end{bmatrix}$$

- $\omega = -1$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 + x_3 = 0$$
$$x_2 = 0$$
$$x_1 + x_3 = 0$$

Soln x_1 arbitrary
$x_2 = 0$
$x_3 = -x_1$

$$|\omega = -1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

c, U, the matrix of the eigenvectors of Ω is

$$U = \begin{bmatrix} 0 & 1/\sqrt{2} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

, of course other choices are possible.

$$U^{+} = \begin{bmatrix} 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix}$$

$$U^{+}\Omega U = U^{+}(\Omega U)$$

$$= \begin{pmatrix} 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{pmatrix}\left\{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}\right\}$$

$$= \begin{pmatrix} 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{pmatrix}\begin{pmatrix} 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

i.e., $U^{+}\Omega U$ is diagonal.

1.8.3. a, characteristic eq.

$$\left|\begin{matrix} 1-\omega & 0 & 0 \\ 0 & \frac{3}{2}-\omega & -1/2 \\ 0 & -1/2 & \frac{3}{2}-\omega \end{matrix}\right| = 0$$

$$(1-\omega)\left[\left(\tfrac{3}{2}-\omega\right)^2 - \tfrac{1}{4}\right] = 0$$

$$(1-\omega)\left[\left(\tfrac{3}{2}-\omega\right)^2 - \tfrac{1}{4}\right] = 0$$

$(1-\omega)[(9-12\omega+4\omega^2-1)+1] = 0$

~~$9-12\omega+4\omega^2-9\omega+12\omega^2-4\omega^3+1+10$~~

~~$8-12\omega+16\omega^2-4\omega^3+\omega$~~

~~$-4\omega^3+16\omega^2+2\omega-8+\omega$~~

$(1-\omega)\{8-12\omega+4\omega^2\} = 0$

$(1-\omega)(2-3\omega+\omega^2) = 0$

$(1-\omega)(2-2\omega-\omega+\omega^2) = 0$

$(1-\omega)\{2(1-\omega)-\omega(1-\omega)\} = 0$

$(1-\omega)(2-\omega)(1-\omega) = 0$

$(1-\omega)^2(2-\omega) = 0$

i.e, $\omega_1 = \omega_2 = 1, \ \omega_3 = 2.$

b, Taking $\omega = 2$

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1/2 & -1/2 \\ 0 & -1/2 & -1/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-x_1 = 0$

$-\left(\frac{x_2}{2}+\frac{x_3}{2}\right) = 0$

$\left(\frac{x_2}{2}+\frac{x_3}{2}\right) = 0$

Soln. $x_1 = 0$

$x_2 = a$ arbitrary

$x_3 = -x_2 = -a$

$$|\omega = 2\rangle = \frac{1}{\sqrt{2}\,a}\begin{bmatrix} 0 \\ a \\ -a \end{bmatrix}$$

c, $\omega = 1$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & +1/2 & -1/2 \\ 0 & -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$0 = 0$

$\frac{1}{2}(x_2 - x_3) = 0$

$\frac{1}{2}(x_3 - x_2) = 0$

Soln x_1 arbitrary $= b$, $\quad |\omega = 1\rangle = \frac{1}{\sqrt{b^2+2c^2}}\begin{bmatrix} b \\ c \\ c \end{bmatrix}$

$x_2 = x_3 = c$.

1.8.4 : a, Characteristic eq.

$$\begin{vmatrix} 4-\omega & 1 \\ -1 & 2-\omega \end{vmatrix} = 0$$

$$(4-\omega)(2-\omega) + 1 = 0$$
$$8 - 4\omega - 2\omega + \omega^2 + 1 = 0$$
$$\omega^2 - 6\omega + 9 = 0$$
$$(\omega - 3)^2 = 0$$

i.e.) $\omega_1 = \omega_2 = 3$.

b) for $\omega = 3$

$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_1 + x_2 = 0$$
$$-(x_1 + x_2) = 0$$

i.e.) $x_2 = -x_1$, or $x_1 = a$, arbitrary

∴ We have as an eigenvector

$$\frac{1}{\sqrt{2}a} \begin{bmatrix} a \\ -a \end{bmatrix}$$

1.8.5 a, If Ω is unitary $\Omega \Omega^\dagger = I$.

$$\Omega^\dagger = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\Omega \Omega^\dagger = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & -\sin\theta\cos\theta + \cos\theta\sin\theta \\ -\sin\theta\cos\theta + \cos\theta\sin\theta & \sin^2\theta + \cos^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

i.e.) Ω is unitary

b) characteristic eq.

$$\begin{vmatrix} \cos\theta - \omega & \sin\theta \\ -\sin\theta & \cos\theta - \omega \end{vmatrix} = 0$$

$$(\cos\theta - \omega)^2 + \sin^2\theta = 0$$
$$\cos^2\theta - 2\omega\cos\theta + \omega^2 + \sin^2\theta = 0$$
$$\omega^2 - 2\omega\cos\theta + 1 = 0$$
$$\omega^2 - (2\cos\theta)\omega + 1 = 0$$
$$\omega = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2}$$

$\omega = \cos\theta \pm i\sin\theta$

i.e. $\omega = \omega_1 = \cos\theta + i\sin\theta = e^{i\theta}$ or $\omega = \omega_2 = \cos\theta - i\sin\theta = e^{-i\theta}$

Thus the eigenvalues are $e^{i\theta}$ and $e^{-i\theta}$.

C). $\omega_1 = e^{i\theta}$

$$\begin{bmatrix} \cos\theta - e^{i\theta} & \sin\theta \\ -\sin\theta & \cos\theta - e^{i\theta} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow (\cos\theta - e^{i\theta}) x_1 + \sin\theta\, x_2 = 0 \quad \cdots (1)$

$(-\sin\theta) x_1 + (\cos\theta - e^{i\theta}) x_2 = 0 \quad \cdots (2)$

From (1) $\quad x_2 = \dfrac{e^{i\theta} - \cos\theta}{\sin\theta} = i$

$(-\sin\theta) x_1 + (\cos\theta - e^{i\theta}) i = 0$

$(-\sin\theta) x_1 + (-i\sin\theta) i = 0$

$-x_1 + 1 = 0$

or $\quad x_1 = 1$

$|\omega = e^{i\theta}] = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ i \end{bmatrix}$

$\omega_2 = e^{-i\theta}$

$$\begin{bmatrix} \cos\theta - e^{-i\theta} & \sin\theta \\ -\sin\theta & \cos\theta - e^{-i\theta} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow (\cos\theta - e^{-i\theta}) x_1 + (\sin\theta) x_2 = 0 \quad \cdots (1)'$

$(-\sin\theta) x_1 + (\cos\theta - e^{-i\theta}) x_2 = 0 \quad \cdots (2)'$

From (1)' $\quad x_2 = \left(\dfrac{e^{-i\theta} - \cos\theta}{\sin\theta}\right) = -i$

$(-\sin\theta) x_1 + (\cos\theta - e^{-i\theta})(-i) = 0$

$(-\sin\theta) x_1 + (i\sin\theta)(-i) = 0$

$-x_1 + 1 = 0$

or $\quad x_1 = 1$

$|\omega = e^{-i\theta}] = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -i \end{bmatrix}$

Now, $\langle \omega = e^{i\theta} | \omega = e^{-i\theta}] = \frac{1}{\sqrt{2}} (1, -i) \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -i \end{bmatrix}$

$= \frac{1}{2} (1 - 1) = 0$

$\therefore$ the eigenvectors are orthogonal.

d) $U = \begin{bmatrix} \frac{i}{\sqrt2} & \frac{1}{\sqrt2} \\ \frac{i}{\sqrt2} & \frac{-i}{\sqrt2} \end{bmatrix} \Rightarrow U^+ = \begin{bmatrix} -i/\sqrt2 & -i/\sqrt2 \\ 1/\sqrt2 & i/\sqrt2 \end{bmatrix}$

$$U^+ \Omega U = U^+(\Omega U) = \begin{bmatrix} \frac{-i}{\sqrt2} & \frac{-i}{\sqrt2} \\ 1/\sqrt2 & i/\sqrt2 \end{bmatrix} \left\{ \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} i/\sqrt2 & 1/\sqrt2 \\ i/\sqrt2 & -i/\sqrt2 \end{bmatrix} \right\}$$

$$= \begin{bmatrix} \frac{-i}{\sqrt2} & \frac{-i}{\sqrt2} \\ 1/\sqrt2 & i/\sqrt2 \end{bmatrix} \begin{bmatrix} \frac{\cos\theta}{\sqrt2} + \frac{i\sin\theta}{\sqrt2} & \frac{\cos\theta}{\sqrt2} - \frac{i\sin\theta}{\sqrt2} \\ -\frac{\sin\theta}{\sqrt2} + \frac{i\cos\theta}{\sqrt2} & -\frac{\sin\theta}{\sqrt2} - \frac{i\cos\theta}{\sqrt2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\cos\theta}{2} + \frac{i\sin\theta}{2} + \frac{i\sin\theta}{2} + \frac{\cos\theta}{2} & \frac{\cos\theta}{2} - \frac{i\sin\theta}{2} + \frac{i\sin\theta}{2} - \frac{\cos\theta}{2} \\ \frac{\cos\theta}{2} + \frac{i\sin\theta}{2} - \frac{i\sin\theta}{2} - \frac{\cos\theta}{2} & \frac{\cos\theta}{2} - \frac{i\sin\theta}{2} - \frac{i\sin\theta}{2} + \frac{\cos\theta}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta + i\sin\theta & 0 \\ 0 & \cos\theta - i\sin\theta \end{bmatrix}$$

$$= \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}$$

i.e., $U^+\Omega U$ is diagonal

1.8.6. a) From prob. 1.7.1, under a unitary change of the basis we have that

$$\Omega \rightarrow U^+\Omega U = \Omega'$$

and $\det \Omega = \det U^+\Omega U = \det \Omega'$

So under such a change Ω' will be diagonal with its elements being its eigenvalues i.e.,

$$\Omega \longrightarrow \Omega' = \begin{bmatrix} \omega_1 & & \cdots & 0 \\ & \omega_2 & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \cdots & \omega_n \end{bmatrix}$$

$$\det \Omega = \det \Omega' = \begin{vmatrix} \omega_1 & \cdots & & 0 \\ & \omega_2 & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \cdots & \omega_n \end{vmatrix} = \prod_{i=1}^{n} \omega_i \quad , \text{Q.E.D.}$$

b) Again from prob. 1.7.1

$$\operatorname{Tr} \Omega = \operatorname{Tr} \Omega' = \sum_{i=1}^{n} \Omega'_{ii}$$

But since $\Omega'_{ii} = \omega_i$, we obtain

$$\boxed{\operatorname{Tr} \Omega = \sum_i \omega_i} \quad \text{Q.E.D.}$$

1.8.7.

In a basis in which Ω is diagonal we may write

$$\Omega \rightarrow \begin{bmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{bmatrix}$$

$\therefore \det \Omega = \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = \omega_1\omega_2$ or $\omega_1\omega_2 = -3$

2. $\text{Tr}\, \Omega = \text{Tr}\, \Omega'$

i.e., $\sum_{i=1}^{2} \omega_i = \sum_{i=1}^{2} \omega_i^*$ and $2 = \omega_1 + \omega_2$

$$\begin{cases} \omega_1 \omega_2 = -3 \\ \omega_1 + \omega_2 = 2 \end{cases}$$

$\Rightarrow \omega_1(2-\omega_1) = -3$

$2\omega_1 - \omega_1^2 = -3$

$\omega_1^2 - 2\omega_1 - 3 = 0$

$\omega_1^2 + \omega_1 - 3\omega_1 - 3 = 0$

$\omega_1(\omega_1+1) - 3(\omega_1+1) = 0$

$(\omega_1 - 3)(\omega_1+1) = 0$

or $\omega_1 = -1, 3$

Thus $\omega_1 = -1 \Rightarrow \omega_2 = 3$

$\omega_1 = 3 \Rightarrow \omega_2 = -1$

In any case the eigenvalues are -1 and 3.

18.8. a. $M^i M^j + M^j M^i = 2\delta^{ij} I$

In the eigenbasis of M_i we have a diagonal M^i (since they are Hermitian). Further assume that $i = j$.

$M^{i\,2} + M^{i\,2} = 2I$, $\delta^{ii} = 1$

$2M^{i\,2} = 2I$

$M^{i\,2} = I$

$$\begin{bmatrix} \omega_1^2 & \cdots & & 0 \\ \cdot & \omega_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \omega_n^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

In general for the i^{th} element of M_i^2 we obtain

$\omega_i^2 = 1$

or $\omega_i = \pm 1$, for each ~~distinct~~ eigenvalue ω_i. Q.E.D.

b. $M^i M^j = -M^j M^i$

$M^i I M^j = -M^j M^i$

$\text{Tr}(M^i I M^j) = -\text{Tr}(M^j M^i)$

$\text{Tr}(I M^j M^i) = -\text{Tr}(M^j M^i)$

$\text{Tr}(M^j M^i) = -\text{Tr}(M^j M^i)$

$2\,\text{Tr}(M^j M^i) = 0$

$\text{Tr}(M^j M^i) = 0$

~~Trace is invariant under a unitary change of basis.~~ ~~$\sum \omega_k \ldots$~~

c, The M^i are assumed to be Hermitian and hence they can be diagonalized by a unitary change of their basis i.e. in their eigenspace. We have obtained from (a) that their eigen values are ±1 and from (b) that they are traceless. This would mean that the elements on the diagonal are the alternate succession of +1 and -1 and thus for the trace to be zero, the no. of the diagonal elements must be ~~zero~~ even i.e. the M^i are not odd dimensional.

Note : We have implicitly assumed that the trace remains the same under diagonalization.

1.8.9

$$\bar{L} = \sum_\alpha M_\alpha (\bar{r}_\alpha \times \bar{v}_\alpha)$$

$$= \sum_\alpha M_\alpha [\bar{r}_\alpha \times (\bar{\omega} \times \bar{r}_\alpha)]$$

$$= \sum_\alpha M_\alpha [\bar{\omega} r_\alpha^2 - \bar{r}_\alpha (\bar{r}_\alpha \cdot \bar{\omega})]$$

$$\Rightarrow L_x = \sum_\alpha M_\alpha \{\omega_x r_\alpha^2 - (r_\alpha)_x [(r_\alpha)_x \omega_x + (r_\alpha)_y \omega_y + (r_\alpha)_z \omega_z]\}$$

$$L_y = \sum_\alpha M_\alpha \{\omega_y r_\alpha^2 - (r_\alpha)_y [(r_\alpha)_x \omega_x + (r_\alpha)_y \omega_y + (r_\alpha)_z \omega_z]\}$$

$$L_z = \sum_\alpha M_\alpha \{\omega_z r_\alpha^2 - (r_\alpha)_z [(r_\alpha)_x \omega_x + (r_\alpha)_y \omega_y + (r_\alpha)_z \omega_z]\}$$

In general

$$L_i = \sum_\alpha M_\alpha \{\omega_i r_\alpha^2 - (r_\alpha)_i \sum_j (r_\alpha)_j \omega_j\}$$

$$i = x, y, z$$

or

$$L_i = \sum_\alpha M_\alpha [r_\alpha^2 \sum_j \omega_j \delta_{ij} - (r_\alpha)_i \sum_j (r_\alpha)_j \omega_j]$$

$$= \sum_j \sum_\alpha M_\alpha [r_\alpha^2 \delta_{ij} - (r_\alpha)_i (r_\alpha)_j] \omega_j$$

$$= \sum_j M_{ij} \omega_j$$

Where $M_{ij} = \sum_\alpha M_\alpha [r_\alpha^2 \delta_{ij} - (r_\alpha)_i (r_\alpha)_j]$

In Dirac notation,

$$|L\rangle = M |\omega\rangle$$

Where

$$|L\rangle \rightarrow \begin{bmatrix} L_x \\ L_y \\ L_z \end{bmatrix} \equiv \begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix}$$

$$|\omega\rangle \rightarrow \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \equiv \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$$

$$M_{ij} \rightarrow \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix}$$

and

$$M_{11} = \sum_\alpha M_\alpha (r_\alpha^2 - (r_\alpha)_1 (r_\alpha)_1)$$

$$= \sum_\alpha M_\alpha (r_\alpha^2 - [(r_\alpha)_1]^2)$$

$$M_{12} = -\sum_\alpha M_\alpha (r_\alpha)_1 (r_\alpha)_2$$

$$M_{13} = -\sum_\alpha M_\alpha (r_\alpha)_1 (r_\alpha)_3$$

$$M_{21} = -\sum_\alpha M_\alpha (r_\alpha)_2 (r_\alpha)_1 = M_{21}$$

$$M_{22} = \sum_\alpha M_\alpha [r_\alpha^2 - [(r_\alpha)_2]^2]$$

$$M_{23} = -\sum_\alpha M_\alpha (r_\alpha)_2 (r_\alpha)_3$$

$$M_{13} = M_{31}$$

$$M_{23} = M_{32}$$

$$M_{33} = \sum_\alpha M_\alpha [r_\alpha^2 - [(r_\alpha)_3]^2]$$

a) If $\bar{L} \parallel \bar{\omega}$, $\bar{\omega} \times \bar{L} = 0$.

$$\bar{\omega} \times \bar{L} = \sum_\alpha m_\alpha \, \bar{\omega} \times [\bar{r}_\alpha \times (\bar{\omega} \times \bar{r}_\alpha)]$$

$$= \sum_\alpha m_\alpha \bar{\omega} \times [\bar{\omega} r_\alpha^2 - \bar{r}_\alpha \, \bar{r}_\alpha \cdot \bar{\omega})$$

$$= \sum_\alpha m_\alpha (r_\alpha^2 \, \bar{\omega} \times \bar{\omega} - \bar{r}_\alpha \cdot \bar{\omega} \; \bar{\omega} \times \bar{r}_\alpha)$$

$$= - \sum_\alpha m_\alpha (\bar{r}_\alpha \cdot \bar{\omega}) \, \bar{\omega} \times \bar{r}_\alpha$$

$\neq 0$, in general

So $\bar{L}$ and $\bar{\omega}$ will not be always parallel.

b.

From the above result.

$$M = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{12} & M_{22} & M_{23} \\ M_{31} & M_{23} & M_{33} \end{bmatrix}$$

After all the elements of M are real. It is seen from the matrix that

$M_{ij}^{*} = M_{ji}$

$\Rightarrow M^{+} = M$

$\therefore$ M is Hermitian

c) If $\bar{L}$ is parallel to $\bar{\omega}$, we should have

$$|L\rangle = M|\omega\rangle = \lambda|\omega\rangle$$

i.e., we have to solve the eigen-value prob. of M. The three directions will then be given by the eigenvectors of M; which means $\bar{\omega}$ has to lie along the eigen vectors.

d.

1.8.10. Since $[\Omega, \Lambda] = 0$, they can be simultaneously diagonalized.

Characteristic eq. of

(i) Ω

$$\begin{vmatrix} 1-\omega & 0 & 1 \\ 0 & -\omega & 0 \\ 1 & 0 & 1-\omega \end{vmatrix} = 0$$

$$-\omega(1-\omega)^2 + \omega = 0$$

$$\omega[\,1 - (1-\omega)^2\,] = 0$$

$$\omega = 0$$

$$\text{or } (1-\omega)^2 = 1$$

$$1-\omega = \pm 1$$

$$\omega = 1 \pm 1$$

i.e., $\omega = 0, 2$

Then $\omega = 0$ is a degenerate eigenvalue.

(ii) Λ

$$\begin{vmatrix} 2-\omega & 1 & 1 \\ 1 & -\omega & -1 \\ 1 & -1 & 2-\omega \end{vmatrix} = 0$$

$$(2-\omega)[-\omega(2-\omega) - 1]$$

$$-[(2-\omega) + 1] + (-1 + \omega) = 0$$

$$(2-\omega)[-\omega(2-\omega) - 1 - 1] - 1 - 1 + \omega = 0$$

$$(2-\omega)(-\omega(2-\omega) - 2) - (2-\omega) = 0$$

$$(2-\omega)\,(\not{2-\omega})\,(-\omega(2-\omega) - 2 - 1) = 0$$

$$(2-\omega)\,(\not{2-\omega})\,(-\omega(2-\omega) - 3) = 0$$

$$2-\omega = 0 \;:\; \omega = 2$$

$$-\omega(2-\omega) - 3 = 0$$

$$\omega^2 - 2\omega - 3 = 0$$

$$\omega^2 + \omega - 3\omega - 3 = 0$$

$$\omega(\omega+1) - 3(\omega+1) = 0$$

$$(\omega+1)(\omega-3) = 0$$

$$\omega = -1, 3$$

i.e., Λ is nondegenerate

So Λ dictates the choice of the basis.

Let us then find the eigenvectors Λ which will also be of Ω.

$$\lambda = -1, 2, 3$$

– $\lambda = -1$

$$\begin{bmatrix} 3 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$3x_1 + x_2 + x_3 = 0 \quad \text{--- } 1$$

$$x_1 + x_2 - x_3 = 0 \quad \text{--- } 2$$

$$x_1 - x_2 + 3x_3 = 0 \quad \text{--- } 3$$

Adding (2) & (3)

$$2x_1 + 2x_3 = 0 \quad \text{--- } 4$$

~~[illegible]~~

~~[illegible]~~

$$x_3 = -x_1$$

from (1)

$$3x_1 + x_2 - x_1 = 0$$

$$x_2 = -2x_1$$

i.e., x_1 is arbitrary

$$x_2 = -2x_1$$

$$x_3 = -x_1$$

So the normalized eigenvector would be

$$\frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$$

– $\lambda = 2$

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & -2 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x_2 + x_3 = 0 \Rightarrow x_3 = -x_2$

$x_1 - 2x_2 - x_3 = 0 \Rightarrow x_1 = x_2$

$x_1 - x_2 = 0 \Rightarrow x_1 = x_2$

i.e., x_1 is arbitrary

$x_2 = -x_3 = x_1$

the corresponding eigenvector is

$$\frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}.$$

— $\lambda = 3$

$$\begin{bmatrix} -1 & 1 & 1 \\ 1 & -3 & -1 \\ 1 & -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$

$-x_1 + x_2 + x_3 = 0$

$x_1 - 3x_2 - x_3 = 0$

$x_1 - x_2 - x_3 = 0$

The last two eqs would give

$x_2 = 0$

and from the first $x_2 = x_1$

i.e., $x_1 = x_3$, $x_2 = 0$ with the corresponding normalized eigenvector being

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

In this basis of the three eigenvectors we should have

$$\Omega \to \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\Lambda \to \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

with

$$|\omega = 0, \lambda = -1\rangle = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$$

$$|\omega = 0, \lambda = 2\rangle = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$|\omega = 2, \lambda = 3\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Now U is the matrix of these eigenvectors

i.e.,

$$U = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{-2}{\sqrt{6}} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & -1/\sqrt{3} & 1/\sqrt{2} \end{bmatrix}$$

$$U^+ = \begin{bmatrix} \frac{1}{\sqrt{6}} & -2/\sqrt{6} & -1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

To show

$U^+ \Omega U$ and $U^+ \Lambda U$ are diagonal

$$\Omega U = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{-2}{\sqrt{6}} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & -1/\sqrt{3} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 2/\sqrt{2} \\ 0 & 0 & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix}$$

$$U^+ \Omega U = \begin{bmatrix} 1/\sqrt{6} & -2/\sqrt{6} & -1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

$$\frac{2}{\sqrt{2}}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \frac{2}{\sqrt{2}} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\Lambda U = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \\ -2/\sqrt{6} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & -1/\sqrt{3} & 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} -1/\sqrt{6} & 2/\sqrt{3} & 3/\sqrt{2} \\ 2/\sqrt{6} & 2/\sqrt{3} & 0 \\ 1/\sqrt{6} & -2/\sqrt{3} & 3/\sqrt{2} \end{bmatrix}$$

$$U^{+} \Lambda U = \begin{bmatrix} 1/\sqrt{6} & -2/\sqrt{6} & -1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \\ +1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \cdot \begin{bmatrix} -1/\sqrt{6} & 2/\sqrt{3} & 3/\sqrt{2} \\ 2/\sqrt{6} & 2/\sqrt{3} & 0 \\ 1/\sqrt{6} & -2/\sqrt{3} & 3/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$\therefore$ $U^{+} \Omega U$ & $U^{+} \Lambda U$ are diagonal.

1.8.11

2-8-11 . Algorithm

1. Solve the eigenvalue prob. of Ω.

2. Find the coefficients

$$x_I(0) = \langle I | x(0) \rangle$$

$$x_{II}(0) = \langle II | x(0) \rangle .$$

From $|x(0)\rangle = |I\rangle x_I(0) + |II\rangle x_{II}(0)$

3. Append to each coeft. ($i = I, II$) a time dependence $\cos \omega_i t$ to get the coefts in the expansion of $|x(t)\rangle$.

Solution:

(i) If x_1 & x_2 are the displacements of the masses from their equil. points from Newton's second law

$$\ddot{x}_1 = -\frac{2k}{m} x_1 + \frac{k}{m} x_2$$

$$\ddot{x}_2 = k/m\, x_1 - \frac{2k}{m} x_2$$

In matrix form

$$\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} = \begin{bmatrix} -\frac{2k}{m} & \frac{k}{m} \\ k/m & -2k/m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{--- } (*)$$

The vector $|x\rangle$ has two real comps.

$$\Omega = \begin{bmatrix} -\frac{2k}{m} & -\frac{2k}{m} \\ k/m & k/m \end{bmatrix}$$ is a Hermitian operator in $V^2(R)$.

The abstract form of the above eq. is

$$|\ddot{x}(t)\rangle = \Omega |x(t)\rangle \quad \text{--- } (**)$$

so that $(*)$ is obtained by projecting $(**)$ on the basis vectors $|1\rangle, |2\rangle$ which have the following physical significance

$|1\rangle \leftrightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix} \leftrightarrow$ [first mass displaced by unity, second mass undisplaced]

$|2\rangle \leftrightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix} \leftrightarrow$ [first mass undisplaced, 2nd mass displaced by unity]

An arbitrary state in which the masses are displaced by x_1 and x_2 is given by (in this basis)

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x_2$$

whose abstract counterpart is

$$|x(t)\rangle = |1\rangle x_1 + |2\rangle x_2$$

It is in this basis that Ω is represented by the matrix in $(*)$.

So now to solve the differential eq. easily we go to a basis in which Ω is diagonal since it is in this basis that the comps of $|x\rangle$ obey uncoupled differential equations. Later we will return to $|1\rangle, |2\rangle$ basis to obtain x_1, x_2.

The basis that diagonalizes Ω is its eigenbasis.

— step i:

$$\Omega |\omega\rangle = \omega |\omega\rangle$$

Characteristic eq.

$$\begin{vmatrix} -\frac{2k}{m} - \omega & k/m \\ k/m & -\frac{2k}{m} - \omega \end{vmatrix} = 0$$

$$\left(\omega + \frac{2k}{m}\right)^2 - k^2/m^2 = 0$$

$$\Rightarrow \left(\omega + \frac{2k}{m}\right)^2 = k^2/m^2$$

$$\omega + \frac{2k}{m} = \pm k/m$$

$$\omega = -\frac{2k}{m} \pm k/m$$

$$\omega_{1,2} = -k/m, \; -\frac{3k}{m}$$

Let $\omega_1 = -\omega_I^2 = -k/m$

$\omega_2 = -\omega_{II}^2 = -3k/m$

i.e., $\omega_I = \sqrt{k/m}$, $\omega_{II} = \sqrt{\frac{3k}{m}}$

$-\omega_I^2 = -\frac{3K}{m}$

$$\begin{bmatrix} -K/m & K/m \\ K/m & -K/m \end{bmatrix} \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow -K/m\, x_1' + K/m\, x_2' = 0$

$K/m\, x_1' - K/m\, x_2' = 0$

i.e., $x_1' = x_2'$

So the normalized eigenvector corresponding to $-K/m$ is say

$$|I\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Similarly, corresponding to $-3K/m$

$$|II\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

In this basis

$$\Omega \rightarrow \begin{bmatrix} -K/m & 0 \\ 0 & -3K/m \end{bmatrix}$$

Note: In this basis $|x(t)\rangle = |I\rangle x_I(t) + |II\rangle x_{II}(t)$

and $\begin{bmatrix} \ddot{x}_I \\ \ddot{x}_{II} \end{bmatrix} = \begin{bmatrix} -\omega_I^2 & 0 \\ 0 & -\omega_{II}^2 \end{bmatrix} \begin{bmatrix} x_I \\ x_{II} \end{bmatrix}$

so that we can solve for $x_i(t), i = I, II$

- Step ii:

$x_I(0) = \langle I | x(0)\rangle$

$= \frac{1}{\sqrt{2}}(1,1)\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}}$

$x_{II}(0) = \langle II | x(0)\rangle$

$= \frac{1}{\sqrt{2}}(1,-1)\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}}$

- Step iii:

$|x(t)\rangle = |I\rangle \frac{1}{\sqrt{2}} \cos\omega_I(t)$

$+ |II\rangle \cos\omega_{II} t$

Coming to the $|1\rangle, |2\rangle$ basis

$x_1(t) = \langle 1|x(t)\rangle = \langle 1|I\rangle \frac{\cos\omega_I t}{\sqrt{2}}$

$\langle 1|II\rangle \frac{\cos\omega_{II} t}{\sqrt{2}}$

$x_2(t) = \langle 2|x(t)\rangle$

$= \langle 2|I\rangle \frac{\cos\omega_I t}{\sqrt{2}}$

$+ \langle 2|II\rangle \frac{\cos\omega_{II} t}{\sqrt{2}}$

Now, $\langle 1|I\rangle = (1,0)\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}$

$\langle 1|II\rangle = (1,0)\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}}$

$\langle 2|I\rangle = (0,1)\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}$

$\langle 2|II\rangle = (0,1)\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = -1/\sqrt{2}$

i.e.,

$x_1(t) = \frac{\cos\omega_I t}{2} + \frac{\cos\omega_{II} t}{2}$

$x_2(t) = \frac{\cos\omega_I t}{2} - \frac{\cos\omega_{II} t}{2}$

But $\omega_I = \sqrt{K/m}$, $\omega_{II} = \sqrt{3K/m}$

and $|1(t)\rangle = |x(t)\rangle$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{\cos\omega_I t}{2} & \frac{\cos\omega_{II} t}{2} \\ \frac{\cos\omega_I t}{2} & -\frac{\cos\omega_{II} t}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\cos\omega_I t}{2} & \frac{\cos\omega_{II} t}{2} \\ \frac{\cos\omega_I t}{2} & -\frac{\cos\omega_{II} t}{2} \end{bmatrix} \left[\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right]$$

$\Rightarrow |x(t)\rangle = U(t)\, [\,|1\rangle + |2\rangle\,]$

or $|1(t)\rangle = U(t)\,|1\rangle$

$$\begin{bmatrix} \frac{\cos\omega_I t}{2} & \frac{\cos\omega_{II} t}{2} \\ \frac{\cos\omega_I t}{2} & -\frac{\cos\omega_{II} t}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{\cos\omega_I t}{2} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$\therefore |1(t)\rangle = \frac{\cos \omega_I t}{2} |1\rangle$

(ii) From 1.8.39

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{\cos \omega_I t + \cos \omega_{II} t}{2} \\ \frac{\cos \omega_I t - \cos \omega_{II} t}{2} \end{bmatrix}$$

for $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

or $$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{\cos \omega_I t}{2} & \frac{\cos \omega_{II} t}{2} \\ \frac{\cos \omega_I t}{2} & -\frac{\cos \omega_{II} t}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Again

$|1(t)\rangle = \frac{\cos \omega_I t}{2} |1\rangle$, as before.

1.8.12.

(i) $|x(t)\rangle = U(t)|x(0)\rangle$

$\Rightarrow |\ddot{x}(t)\rangle = \ddot{U}(t)|x(0)\rangle$

Since $|x(0)\rangle$ is arbitrary and assuming it consists of constant elements. But

$|\ddot{x}(t)\rangle = \Omega |x(t)\rangle$

i.e., $\ddot{U}(t)|x(0)\rangle = \Omega |x(t)\rangle$

$$\Rightarrow \begin{bmatrix} \ddot{u}_{11} & \ddot{u}_{12} \\ \ddot{u}_{21} & \ddot{u}_{22} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} -\frac{2k}{m} & k/m \\ k/m & -2k/m \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

or $\ddot{u}_{11} x_1(0) + \ddot{u}_{12} x_2(0) = -\frac{2k}{m} x_1 + \frac{k}{m} x_2$ (ω_I^2)

$\ddot{u}_{21} x_1(0) + \ddot{u}_{22} x_2(0) = \frac{k}{m} x_1 - \frac{2k}{m} x_2$

Also $|x(t)\rangle = U(t)|x(0)\rangle$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$$

$x_1 = u_{11} x_1(0) + u_{12} x_2(0)$

$x_2 = u_{21} x_1(0) + u_{22} x_2(0)$

i.e.,

$$\ddot{u}_{11} x_1(0) + \ddot{u}_{12} x_2(0) = -2\omega_I^2 (u_{11} x_1(0) + u_{12} x_2(0)) + \omega_I^2 (u_{21} x_1(0) + u_{22} x_2(0))$$

$$\ddot{u}_{21} x_1(0) + \ddot{u}_{22} x_2(0) = \omega_I^2 (u_{11} x_1(0) + u_{12} x_2(0)) - 2\omega_I^2 (u_{21} x_1(0) + u_{22} x_2(0))$$

or

$$(\ddot{u}_{11} + 2\omega_I^2 u_{11} - \omega_I^2 u_{21}) x_1(0) + (\ddot{u}_{12} + 2\omega_I^2 u_{12} - \omega_I^2 u_{22}) x_2(0) = 0$$

$$(\ddot{u}_{21} + 2\omega_I^2 u_{21} - \omega_I^2 u_{11}) x_1(0) + (\ddot{u}_{22} + 2\omega_I^2 u_{22} - \omega_I^2 u_{12}) x_2(0) = 0$$

Since $|x(0)\rangle$ is arbitrary we may set $x_1(0) = 1$, $x_2(0) = 1$

$$\ddot{u}_{11} + 2\omega_I^2 u_{11} - \omega_I^2 u_{21} + \ddot{u}_{12} + 2\omega_I^2 u_{12} - \omega_I^2 u_{22} = 0$$

and

$$\ddot{u}_{21} + 2\omega_I^2 u_{21} - \omega_I^2 u_{11} + \ddot{u}_{22} + 2\omega_I^2 u_{22} - \omega_I^2 u_{12} = 0$$

Adding these two

$$\ddot{u}_{11} + \ddot{u}_{22} + \omega_I^2 (u_{11} + u_{22}) + \omega_I^2 (u_{12} + u_{21}) + \ddot{u}_{12} + \ddot{u}_{21} = 0$$

... the D.E. satisfied by U.

$\therefore U \xrightarrow[\text{basis}]{I,II} \begin{bmatrix} \cos\omega_I t & 0 \\ 0 & \cos\omega_{II} t \end{bmatrix}$

(ii) The eigenvalue prob. has been solved and the eigenvalues of Ω are $-\omega_I^2, -\omega_{II}^2$

$$\Omega \to \begin{bmatrix} -\omega_I^2 & 0 \\ 0 & -\omega_{II}^2 \end{bmatrix}$$

The corresponding normalized eigenvectors are

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

which are common to Ω & U

$$\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} |I\rangle = \alpha |I\rangle$$

$$U |II\rangle = \beta |II\rangle$$

$$\Rightarrow \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{\alpha}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{\beta}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

i.e.,
$$u_{11} + u_{12} = \alpha$$
$$u_{21} + u_{22} = \alpha$$
$$u_{11} - u_{12} = \beta$$
$$u_{21} - u_{22} = -\beta$$

$$\Rightarrow \alpha = \frac{u_{11} + u_{22} + u_{12} + u_{21}}{2}$$

$$\beta = \frac{u_{11} + u_{22} - (u_{12} + u_{21})}{2}$$

Then by using the matrix elements of U

$$\alpha = \cos\sqrt{k/m}\,t, \quad \beta = \cos\sqrt{\frac{3k}{m}}\,t$$

1.9.1. If Ω is Hermitian, going to its eigenbasis we may have

$$\Omega = \begin{bmatrix} \omega_1 & 0 & \cdots & 0 \\ 0 & \omega_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \omega_m \end{bmatrix}$$

$$\Omega^n = \begin{bmatrix} \omega_1^n & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \omega_m^n \end{bmatrix}$$

$$\sum_{n=0}^{\infty} \Omega^n = \begin{bmatrix} \sum_{n=0}^{\infty} \omega_1^n & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sum_{n=0}^{\infty} \omega_m^n \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{1-\omega_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{1}{1-\omega_m} \end{bmatrix}$$

i.e., $f(\Omega) = \sum_{n=0}^{\infty} \Omega^n = \frac{1}{1-\Omega}$

1.9.2. Since H is Hermitian going to its eigenbasis we have that

$$iH = \begin{bmatrix} ih_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & ih_m \end{bmatrix}$$

$$(iH)^n = \begin{bmatrix} (ih_1)^n & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & (ih_m)^n \end{bmatrix}$$

$$\sum_{n=0}^{\infty} \frac{(iH)^n}{n!} = \begin{bmatrix} \sum_{n=0}^{\infty} \frac{(ih_1)^n}{n!} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sum_{n=0}^{\infty} \frac{(ih_m)^n}{n!} \end{bmatrix}$$

$$e^{iH} = \begin{bmatrix} e^{ih_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{ih_m} \end{bmatrix}$$

$$\Rightarrow U = e^{iH} = \begin{bmatrix} e^{ih_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{ih_m} \end{bmatrix}$$

$$U^{+} = \begin{bmatrix} e^{-ih_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{-ih_m} \end{bmatrix}$$

The h_is are real

$$U^{+}U = \begin{bmatrix} e^{-ih_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{-ih_m} \end{bmatrix} \begin{bmatrix} e^{ih_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{ih_m} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$= I$$

$\therefore$ $U = e^{iH}$ is unitary for H - Hermitian.

~~$Tr|U| = |\sum_{j=1}^{m} e^{ih_j}|$~~

~~$\leq |\sum_{j=1}|$~~

1.9.3 $\det U = \prod_m e^{ih_m}$

$$= e^{i\sum_m h_m}$$

$$= e^{i \operatorname{Tr} H} \quad \text{, Q.E.D.}$$

1.10.1.

If $a > 0$

$$\int \delta(ax)\, d(ax) = 1 \quad \text{---- (1)}$$

If $a < 0$

$$\int \delta(-ax)\, d(-ax) = 1 \quad \text{---- (2)}$$

Also, $$\int \delta(x)\, dx = 1 \quad \text{---- (3)}$$

Comparing (1) and (3)

$$a\,\delta(ax) = \delta(x)$$

$$\Rightarrow \delta(ax) = \frac{\delta(x)}{a} \quad \text{---- (*)}$$

Comparing (2) and (3)

$$-a\,\delta(-ax) = \delta(x)$$

$\Leftrightarrow -a\delta(ax) = \delta(x)$, δ is even function.

$$\Rightarrow \delta(ax) = \frac{\delta(x)}{-a} \quad \text{---- (**)}$$

$\therefore$ From (*) and (**)

$$\boxed{\delta(ax) = \frac{\delta(x)}{|a|}}$$

1.10.2. $\delta(f(x))$ blows up at the roots of $f(x)$. If x_i is a root of $f(x)$ we expand $f(x)$ near this point by

$$f(x) = f(x_i) + \left.\frac{df}{dx}\right|_{x=x_i}(x_i - x) + \frac{1}{2!}\left.\frac{d^2f}{dx^2}\right|_{x=x_i}(x_i - x)^2 + \ldots$$

The first term is obviously zero as x_i is a root of $f(x)$. Further we neglect the higher terms since the deviation from x_i is small.

Then,

$$f(x) = \frac{df}{dx_i}(x_i - x)$$

$$\frac{1}{f(x)} = \frac{1}{\frac{df}{dx_i}(x_i - x)}$$

$$= \sum_i \frac{1}{\left|\frac{df}{dx_i}\right|(x_i - x)}$$

Considering all the roots. Then

$$\delta f(x) = \sum_i \frac{\delta(x_i - x)}{\left|df/dx_i\right|}$$

1.10.3.

$$\theta(x - x') = \begin{cases} 0 & \text{if } x - x' < 0 \\ 1 & \text{if } x - x' > 0 \end{cases}$$

$$\delta(x - x') = \begin{cases} 0 & \text{if } x' \neq x \\ \infty, & \text{if } x' = x \end{cases}$$

$$\Rightarrow \int \delta(x - x')\, dx' = \begin{cases} 0 & \text{if } x' \neq x \\ 1, & \text{if } x' = x \end{cases}$$

i.e., $\int \delta(x - x')\, dx' = \theta(x - x')$.

Chapter Four

4.2.1 a, the possible values that can be obtained if a variable is measured are the eigenvalues. so now we solve ~~for~~ the eigenvalue prob. of L_z:

$$L_z |x_z\rangle = L_z |L_z\rangle$$

Characteristic eq.

$$|L_z - L_z I| = 0 \Rightarrow \begin{vmatrix} 1-L_z & 0 & 0 \\ 0 & -L_z & 0 \\ 0 & 0 & -1-L_z \end{vmatrix} = 0$$

$$(1-L_z)\, L_z\, (1+L_z) = 0$$

i.e., $L_z = -1, 0, 1$ are the possible results of the measurement.

b, the normalized states of $L_z = 1$.

$$[L_z - L_z I][X] = [0]$$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$0 = 0, \; -x_2 = 0, \; -2x_3 = 0$

$\Rightarrow$ x_1 is arbitrary, $x_2 = x_3 = 0$

$$\text{i.e., } |L_z = 1\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

– $$\langle L_x \rangle = \langle L_z = 1 | L_x | L_z = 1\rangle$$

$$= (1,0,0)\, \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} (1,0,0) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$= 0$$

– $$L_x^2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\langle L_x^2 \rangle = \langle L_z = 1 | L_x^2 | L_z = 1 \rangle$$

$$= (1,0,0)\, \frac{1}{2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{2} (1,0,0) \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$= \frac{1}{2}$$

$(\Delta L_x) = [\langle (L_x - \langle L_x\rangle)^2\rangle]^{1/2}$

$= \langle\psi|(L_x - \langle L_x\rangle)^2|\psi\rangle^{1/2}$

$= \langle\psi| L_x^2 - 2L_x\langle L_x\rangle + \langle L_x\rangle^2|\psi\rangle^{1/2}$

~~$= \langle\psi| L_x^2|\psi\rangle + ...$~~

$= \langle\psi| L_x^2 - \langle L_x\rangle^2|\psi\rangle^{1/2}$ since $\langle\psi|L_x|\psi\rangle = \langle L_x\rangle$

$= [\langle L_x^2\rangle - \langle L_x\rangle^2]^{1/2}$

$= (1/2 - 0)^{1/2}$

$= \frac{1}{\sqrt{2}}$

c,

The eigenvectors of L_z

$|L_z = 1\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, as found in (b)

- $L_z = 0$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow$ $x_1 = 0$, $0 = 0$, $-x_3 = 0$ $\Big\}$ $x_1 = 0$, x_2 arbitrary, $x_3 = 0$

$\therefore |L_z = 0\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

- $L_z = -1$

$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$-2x_1 = 0$, $x_2 = 0$, $0 = 0$ $\Big\}$ $x_1 = 0$, $x_2 = 0$, x_3 arbitrary

$\therefore |L_z = -1\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Now let us solve the eigenvalue prob. of L_x to find its eigenvectors and eigenvalues.

$L_x|L_x\rangle = L_x|L_x\rangle$

Characteristic eq:

$\Rightarrow \begin{vmatrix} -L_x & 1 & 0 \\ 1 & -L_x & 1 \\ 0 & 1 & -L_x \end{vmatrix} = 0$

$-L_x[L_x^2 - 1] + L_x = 0$

$L_x(1 - (L_x^2 - 1)) = 0$

$L_x = 0$

$1 - (L_x^2 - 1) = 0$

$L_x^2 = 2$

or $L_x = \pm\sqrt{2}$

So the eigenvalues of L_x are

$-\sqrt{2}, 0, +\sqrt{2}$

Eigenvectors:

- $L_x = -\sqrt{2}$

$\begin{bmatrix} \sqrt{2} & 1 & 0 \\ 1 & \sqrt{2} & 1 \\ 0 & 1 & \sqrt{2} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\sqrt{2}x_1 + x_2 = 0$

$x_1 + \sqrt{2}x_2 + x_3 = 0$

$x_2 + \sqrt{2}x_3 = 0$

$\Rightarrow$ $x_2 = -\sqrt{2}x_1$

$x_3 = x_1$

x_1 arbitrary

i.e. $|L_x = -\sqrt{2}\rangle = \frac{1}{2}\begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix}$

- $L_x = 0$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{array}{r} x_2 = 0 \\ x_1 + x_3 = 0 \\ x_2 = 0 \end{array}\right\} \begin{array}{l} x_1 \text{ arbitrary} \\ x_2 = 0 \\ x_3 = -x_1 \end{array}$$

i.e., $|L_x = 0\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

- $L_x = \sqrt{2}$

$$\begin{bmatrix} -\sqrt{2} & 1 & 0 \\ 1 & -\sqrt{2} & 1 \\ 0 & 1 & -\sqrt{2} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{array}{r} -\sqrt{2}\, x_1 + x_2 = 0 \\ x_1 - \sqrt{2}\, x_2 + x_3 = 0 \\ x_2 - \sqrt{2}\, x_3 = 0 \end{array}\right\} \begin{array}{l} x_2 = \sqrt{2} x_1 \\ x_3 = x_1 \\ x_1 \text{ arbitrary} \end{array}$$

i.e., $|L_x = \sqrt{2}\rangle = \frac{1}{2}\begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}$

So we write each vector $|L_x\rangle$ as a linear combination of $|L_z\rangle$s as follows.

$$|L_x = -\sqrt{2}\rangle = \tfrac{1}{2}|L_z = 1\rangle - \tfrac{\sqrt{2}}{2}|L_z = 0\rangle + \tfrac{1}{2}|L_z = -1\rangle$$

$$|L_x = 0\rangle = \tfrac{1}{\sqrt{2}}|L_z = 1\rangle + 0|L_z = 0\rangle - \tfrac{1}{\sqrt{2}}|L_z = -1\rangle$$

$$|L_x = \sqrt{2}\rangle = \tfrac{1}{2}|L_z = 1\rangle + \tfrac{\sqrt{2}}{2}|L_z = 0\rangle + \tfrac{1}{2}|L_z = -1\rangle$$

These are the normalized eigenstates of L_x in the L_z basis.

$$L_x \cdot |L_z = 1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Similarly

$$L_x|L_z = 0\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$L_x|L_z = -1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

i.e., none of the L_z eigenstates are eigenstates of L_x i.e., when L_x acts on them it changes them to diff. vectors. ∴ the eigenvalues of L_x in L_z basis are zero.

d) We express the $L_z = -1$ state in terms of the eigenvectors of L_x.

i.e., $|L_z = -1\rangle = \alpha|L_x = -\sqrt{2}\rangle + \beta|L_x = 0\rangle + \gamma|L_x = \sqrt{2}\rangle$

α, β, γ determined by the dot products.

$$|L_z = -1\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Since the kets are normalized

$$\alpha = \langle L_z = -1 | L_x = -\sqrt{2}\rangle = (0,0,1)\,\frac{1}{2}\begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix} = \frac{1}{2}$$

$$\beta = \langle L_z = -1 | L_x = 0\rangle = (0,0,1)\,\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = -1/\sqrt{2}$$

$$\gamma = \langle L_z = -1 | L_x = \sqrt{2}\rangle = (0,0,1)\,\tfrac{1}{2}\begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix} = \tfrac{1}{2}$$

$\therefore \quad |L_z = -1\rangle = \frac{1}{2}|L_x = -\sqrt{2}\rangle$
$-\frac{1}{\sqrt{2}}|L_x = 0\rangle$
$+\frac{1}{2}|L_x = \sqrt{2}\rangle$

So now if L_x is measured the possible outcomes will be
$L_x = -\sqrt{2}, 0, \sqrt{2}$
with respective probabilities $\frac{1}{4}, \frac{1}{2}, \frac{1}{4}$.

e. $L_z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

$\Rightarrow L_z^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Now a result +1 for L_z^2 means +1 is an eigenvalue of L_z^2.

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad 0 = 0, \quad x_2 = 0, \quad 0 = 0 \quad \Big| \quad x_1, x_3$ arbitrary, $x_2 = 0$

$|\psi\rangle =$ [illegible]
[illegible]
$\frac{1}{2}|L_z = 1\rangle + \frac{1}{2}|L_z = 0\rangle$
$+\frac{1}{\sqrt{2}}|L_z = -1\rangle \quad \cdots (*)$

$|L_z = 1\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, |L_z = 0\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$|L_z = -1\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$L_z^2 |L_z = 1\rangle = |L_z = 1\rangle$

$L_z^2 |L_z = 0\rangle = 0|L_z = 0\rangle$

$L_z^2 |L_z = -1\rangle = |L_z = -1\rangle$

i.e., $|L_z = 1\rangle, |L_z = -1\rangle$ are eigenvectors of L_z^2 with degenerate eigenvalue +1. So before L_z^2 is measured

$|\psi\rangle = \frac{1}{2}|L_z = 1\rangle + \frac{1}{2}|L_z = 0\rangle$
$+\frac{1}{\sqrt{2}}|L_z = -1\rangle$

$\equiv \frac{1}{2}|L_z^2 = 1, 1\rangle + \frac{1}{\sqrt{2}}|L_z^2 = 1, 2\rangle$
$+\frac{1}{2}|L_z^2 = 0\rangle$

Now, the state after the measurement is

$|\psi\rangle = \frac{1}{\sqrt{3}}|L_z^2 = 1, 1\rangle + \frac{2}{\sqrt{6}}|L_z^2 = 1, 2\rangle$

or $\quad \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \frac{2}{\sqrt{6}}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

projection operator

$P_1 = |L_z^2 = 1, 1\rangle\langle L_z^2 = 1, 1|$
$+ |L_z^2 = 1, 2\rangle\langle L_z^2 = 1, 2|$

The probability of this value +1 is $P(1)$
$= \langle\psi|P_1|\psi\rangle = \langle P_1\psi|P_1\psi\rangle$

or $P(1) = \frac{1}{3}$ [illegible]

– If L_z is measured the outcomes are 1, 0, −1 with respective probabilities 1/4, 1/4 and 1/2, from (*).

f,

Correction ?

~~what would ...~~

~~$|\psi\rangle = \frac{1}{2}$ ...~~

$P(L_z = 1) = 1/4$

$P(L_z = 0) = 1/2$, $P(L_z = -1) = 1/4$

Since the relative probabilities are the modulus squared of the coefts. we may write

$$|\psi\rangle = \frac{1}{2}|L_z=1\rangle + \frac{1}{\sqrt{2}}|L_z=0\rangle + \frac{1}{2}|L_z=-1\rangle$$

But we know if $|\psi\rangle$ is a state of a system so is $\alpha|\psi\rangle$ in particular $e^{i\theta}|\psi\rangle$ for a normalized state vector $|\psi\rangle$. This is so because the phase factor cannot change the probabilities of obtaining the various results by measuring any physical quantity. So we may rewrite the above expression as

$$|\psi\rangle = \frac{e^{i\delta_1}}{2}|L_z=1\rangle + \frac{e^{i\delta_2}}{\sqrt{2}}|L_z=0\rangle + \frac{e^{i\delta_3}}{2}|L_z=-1\rangle$$

where the δ_is are arbitrary phase factors s.t $|e^{i\delta_i}| = 1$.

It is not, however, to say these phase factors are not irrelevant. They rather give the freedom of choosing a particular state vector without changing the norm.

$$|L_z=1\rangle = \begin{bmatrix}1\\0\\0\end{bmatrix}, \quad |L_z=0\rangle = \begin{bmatrix}0\\1\\0\end{bmatrix}$$

$$|L_z=-1\rangle = \begin{bmatrix}0\\0\\1\end{bmatrix}$$

and

$$|L_x=\sqrt{2}\rangle = \frac{1}{2}\begin{bmatrix}1\\\sqrt{2}\\1\end{bmatrix}, \quad |L_x=0\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\-1\end{bmatrix}$$

$$|L_x=-\sqrt{2}\rangle = \frac{1}{2}\begin{bmatrix}1\\-\sqrt{2}\\1\end{bmatrix}$$

To express the $|L_z\rangle$s in terms of $|L_x\rangle$s.

$\langle L_z=1|L_x=\sqrt{2}\rangle^* = 1/2$

$\langle L_z=1|L_x=0\rangle^* = 1/\sqrt{2}$

$\langle L_z=1|L_x=-\sqrt{2}\rangle^* = +1/2$

$\langle L_z=0|L_x=\sqrt{2}\rangle^* = \sqrt{2}/2$

$\langle L_z=0|L_x=0\rangle^* = 0$

$\langle L_z=0|L_x=-\sqrt{2}\rangle^* = -\sqrt{2}/2$

$\langle L_z=-1|L_x=\sqrt{2}\rangle^* = 1/2$

$\langle L_z=-1|L_x=0\rangle^* = -1/\sqrt{2}$

$\langle L_z=-1|L_x=-\sqrt{2}\rangle^* = 1/2$

i.e.,

$$|\psi\rangle = \frac{e^{i\delta_1}}{2}\left(\tfrac{1}{2}|L_x=\sqrt{2}\rangle + \tfrac{1}{\sqrt{2}}|L_x=0\rangle + \tfrac{1}{2}|L_x=-\sqrt{2}\rangle\right)$$
$$+ \frac{e^{i\delta_2}}{\sqrt{2}}\left(\frac{\sqrt{2}}{2}|L_x=\sqrt{2}\rangle + 0|L_x=0\rangle - \frac{\sqrt{2}}{2}|L_x=-\sqrt{2}\rangle\right)$$
$$+ \frac{e^{i\delta_3}}{2}\left(\tfrac{1}{2}|L_x=\sqrt{2}\rangle - \tfrac{1}{\sqrt{2}}|L_x=0\rangle + \tfrac{1}{2}|L_x=-\sqrt{2}\rangle\right)$$

$$= \frac{1}{2}\left(\frac{e^{i\delta_1}}{2} + e^{i\delta_2} + \frac{e^{i\delta_3}}{2}\right)|L_x=\sqrt{2}\rangle$$
$$+ \frac{1}{\sqrt{2}}\left(\frac{e^{i\delta_1}}{2} + 0e^{i\delta_2} - \frac{e^{i\delta_3}}{2}\right)|L_x=0\rangle$$
$$+ \frac{1}{2}\left(\frac{e^{i\delta_1}}{2} - e^{i\delta_2} + \tfrac{1}{2}e^{i\delta_3}\right)|L_x=-\sqrt{2}\rangle$$

So if $\hat{L}_x$ is measured the probability of obtaining $L_x = 0$ is

$$P(L_x=0) = \frac{1}{2}\left| \frac{e^{i\delta_1}}{2} - \frac{e^{i\delta_2}}{2} \right|^2$$

$$= \frac{1}{2}\left(\frac{e^{i\delta_1}}{2} - \frac{e^{i\delta_2}}{2} \right)\left(\frac{e^{-i\delta_1}}{2} - \frac{e^{-i\delta_2}}{2} \right)$$

$$= \frac{1}{2}\left(\frac{1}{4} - \frac{e^{i(\delta_1-\delta_2)}}{4} - \frac{e^{-i(\delta_1-\delta_2)}}{2} + \frac{1}{4} \right)$$

$$= \frac{1}{2}\left(\frac{1}{2} - \frac{\cos(\delta_1-\delta_2)}{2} \right)$$

$$= \frac{1}{4}\left[1 - \cos(\delta_1-\delta_2)\right]$$

4.2.2.

$$\langle P \rangle = \langle \psi | P | \psi \rangle$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \langle \psi | x \rangle \langle x | P | x' \rangle \langle x' | \psi \rangle dx\, dx'$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \psi(x)\left[-i\hbar\, \delta'(x-x')\right]\psi(x')\, dx\, dx'$$

(since $\psi(x)$ is real)

$$= -i\hbar \int_{-\infty}^{\infty} \psi(x) \frac{d\psi(x)}{dx} dx$$

$$= -i\hbar \int_{-\infty}^{\infty} \frac{1}{2}\frac{d}{dx}\left[\psi(x)\right]^2 dx$$

$$= \lim_{\substack{t\to\infty \\ -t\to-\infty}} -\frac{i\hbar}{2}\left[\psi(t)\right]^2 \Big|_{-t}^{t}$$

$$= 0$$

Now if $\psi = c\psi_r$

$$\langle P \rangle = \langle c\psi_r | P | c\psi_r \rangle$$

$$= |c|^2 \langle \psi_r | P | \psi_r \rangle$$) as found above.

$$= |c|^2 \cdot 0$$

$$= 0$$

4.2.3. For $\psi(x)$

$$\langle P \rangle = \langle \psi | P | \psi \rangle$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \langle \psi | x \rangle \langle x | P | x' \rangle \langle x' | \psi \rangle\, dx\, dx'$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \psi^*(x)\left[-i\hbar\, \delta'(x-x')\right]\psi(x')\, dx\, dx'$$

$$= -i\hbar \int_{-\infty}^{\infty} \psi^*(x) \frac{d\psi(x)}{dx} dx$$

For $\psi'(x) = e^{\frac{ip_0}{\hbar}x}\psi(x)$

$$\langle P \rangle' = \langle \psi' | P | \psi' \rangle$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \langle \psi' | x \rangle \langle x | P | x' \rangle \langle x' | \psi' \rangle\, dx\, dx'$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \psi'^*(x)\left[-i\hbar\, \delta'(x-x')\right]\psi'(x')\, dx\, dx'$$

$$= -i\hbar \int_{-\infty}^{\infty} \psi'^*(x) \frac{d\psi'(x)}{dx} dx$$

$$= -i\hbar \int_{-\infty}^{\infty} e^{-\frac{ip_0}{\hbar}x}\psi^*(x) \frac{d}{dx}\left(e^{\frac{ip_0}{\hbar}x}\psi(x)\right) dx$$

$$= -i\hbar \int_{-\infty}^{\infty} e^{-\frac{ip_0}{\hbar}x}\psi^*(x)\left(\frac{ip_0}{\hbar} e^{\frac{ip_0}{\hbar}x}\psi(x) + e^{\frac{ip_0}{\hbar}x}\frac{d\psi(x)}{dx} \right) dx$$

$$= p_0 \int_{-\infty}^{\infty} |\psi(x)|^2 dx - i\hbar \int_{-\infty}^{\infty} \psi^*(x) \frac{d\psi(x)}{dx} dx$$

$$= p_0 + \langle P \rangle$$

$= \langle P \rangle + p_0$, which is the desired result.

5.1.1. $U(t) = \int_{-\infty}^{\infty} |p\rangle\langle p| e^{-i\left(\frac{p^2}{2m\hbar}\right)t}\, dp$

$|p\rangle \equiv |E, \pm\rangle$

i.e., to each eigenvalue E there corresponds two eigen functions.

$p = \pm\sqrt{2mE}$

$\Rightarrow dp = \pm \frac{m}{\sqrt{2mE}}\, dE$

So we can sum over the ± thereby the limit of integration ranging from 0 — ∞.

$\therefore U(t) = \sum_{\alpha=\pm} \int_0^{\infty} |E,\alpha\rangle\langle E,\alpha| e^{-iEt/\hbar} \frac{m}{\sqrt{2mE}}\, dE$

$= \sum_{\alpha=\pm} \int_0^{\infty} \left(\frac{m}{\sqrt{2mE}}\right) |E,\alpha\rangle\langle E,\alpha| e^{\frac{-iEt}{\hbar}}\, dE$

5.1.2. $\frac{P^2}{2m}|E\rangle = E|E\rangle$

$\langle x|\frac{P^2}{2m}|E\rangle = E\langle x|E\rangle$

$= -\frac{\hbar^2}{2m}\frac{d^2\psi_E(x)}{dx^2} = E\psi_E(x)$

$\frac{d^2\psi_E}{dx^2} + \frac{2mE}{\hbar^2}\psi_E = 0$

i.e., $\psi_E \sim e^{\pm i\sqrt{\frac{2mE}{\hbar^2}}\, x}$

Taking one of them and normalizing to the Dirac delta function we would find for ψ_E

$\psi_E(x) = \frac{\beta e^{i\sqrt{2mE}\, x/\hbar} + \gamma e^{-i\sqrt{2mE}\, x/\hbar}}{\sqrt{2\pi\hbar}}$

5.1.3. Following the scheme outlined

(i) $\psi(x,0) = \frac{1}{(\pi)^{1/4}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!\, 2^n}$

(ii) The first three terms from U(t)

(a) $\frac{i\hbar t}{2m}\frac{d^2}{dx^2}$

(b) $\frac{1}{2}\left(\frac{i\hbar t}{2m}\right)^2 \frac{d^4}{dx^4}$

(c) $\frac{1}{6}\left(\frac{i\hbar t}{2m}\right)^3 \frac{d^6}{dx^6}$

The action of the operator in

(i) (a)

$\frac{i\hbar t}{2m}\frac{d^2\psi(x,0)}{dx^2}$

$= (\pi)^{-1/4}\left(\frac{i\hbar t}{2m}\right) \sum_{n=1}^{\infty} \frac{(-1)^n (2n)(2n-1) x^{2n-2}}{n!\, 2^n}$

(ii) (b)

$\frac{1}{2}\left(\frac{i\hbar t}{2m}\right)^2 \frac{d^4\psi(x,0)}{dx^4}$

$= (\pi)^{-1/4}\frac{1}{2}\left(\frac{i\hbar t}{2m}\right)^2 \sum_{n=2}^{\infty} \frac{(-1)^n (2n)(2n-1)(2n-2)(2n-3) x^{2n-4}}{n!\, 2^n}$

$= (\pi)^{-1/4}\frac{1}{2}\left(\frac{i\hbar t}{2m}\right)^2 \sum_{n=2}^{\infty} \frac{(-1)^n (2n)(2n-1)(2n-2)(2n-3) x^{2n-4}}{n!\, 2^n}$

(iii) $\frac{1}{6}\left(\frac{i\hbar t}{2m}\right)^3 \frac{d^6\psi(x,0)}{dx^6}$

$= (\pi)^{-1/4}\frac{1}{6}\left(\frac{i\hbar t}{2m}\right)^3 \sum_{n=3}^{\infty} \frac{(-1)^n (2n)(2n-1)(2n-2)(2n-3)(2n-4)(2n-5) x^{2n-6}}{n!\, 2^n}$

(iii)

$$\psi(x,t) = U(t)\,\psi(x,0)$$

$$= \left\{ \frac{i\hbar t}{2m}\frac{d^2}{dx^2} + \frac{1}{2}\left(\frac{i\hbar t}{2m}\right)^2 \frac{d^4}{dx^4} + \frac{1}{6}\left(\frac{i\hbar t}{2m}\right)^3 \frac{d^6}{dx^6} \right\} \cdot \psi(x,0)$$

$$= (\pi)^{-1/4} \left\{ \frac{i\hbar t}{2m} \sum_{n=1}^{\infty} \frac{(-1)^n (2n)(2n-1)\, x^{2n-2}}{n!\, 2^n} + \frac{1}{2!}\left(\frac{i\hbar t}{2m}\right)^2 \sum_{n=2}^{\infty} \frac{(-1)^n (2n)(2n-1)(2n-2)(2n-3)\, x^{2n-4}}{n!\, 2^n} + \frac{1}{3!}\left(\frac{i\hbar t}{2m}\right)^3 \sum_{n=3}^{\infty} \frac{(-1)^n (2n)(2n-1)(2n-2)(2n-3)(2n-4)(2n-5)\, x^{2n-6}}{n!\, 2^n} + \cdots \right\}$$

$$= (\pi)^{-1/4} \left\{ \left(\frac{i\hbar t}{2m}\right) \sum_{n=0}^{\infty} \frac{(-1)^{n+1} (2n+2)(2n+1)\, x^{2n}}{(n+1)!\, 2^{n+1}} + \frac{1}{2!}\left(\frac{i\hbar t}{2m}\right)^2 \sum_{n=0}^{\infty} \frac{(-1)^{n+2} (2n+4)(2n+3)(2n+2)(2n+1)\, x^{2n}}{(n+2)!\, 2^{n+2}} + \frac{1}{3!}\left(\frac{i\hbar t}{2m}\right)^3 \sum_{n=0}^{\infty} \frac{(-1)^{n+3} (2n+6)(2n+5)(2n+4)(2n+3)(2n+2)(2n+1)\, x^{2n}}{(n+3)!\, 2^{n+3}} + \cdots \right\}$$

$$= (\pi)^{-1/4} \sum_{n=0}^{\infty} \left\{ \frac{(-1)^{n+1}(2n+2)(2n+1)}{(n+1)!\, 2^{n+1}} \left(\frac{i\hbar t}{2m}\right) + \frac{1}{2!}\left(\frac{i\hbar t}{2m}\right)^2 (-1)^{n+2} \frac{(2n+4)(2n+3)(2n+2)(2n+1)}{(n+2)!\, 2^{n+2}} + \frac{1}{3!}\left(\frac{i\hbar t}{2m}\right)^3 (-1)^{n+3} \frac{(2n+6)(2n+5)(2n+4)(2n+3)(2n+2)(2n+1)}{(n+3)!\, 2^{n+3}} + \cdots \right\} x^{2n}$$

$$= (\pi)^{-1/4} \sum_{n=0}^{\infty} \left\{ \left[\frac{i\hbar t}{m}\right] (-1)^{n+1} \frac{(n+\frac{1}{2})}{n!\, 2^n} + \frac{1}{2!}\left(\frac{i\hbar t}{m}\right)^2 (-1)^{n+2} \frac{(n+\frac{3}{2})(n+\frac{1}{2})}{n!\, 2^n} + \frac{1}{3!}\left(\frac{i\hbar t}{m}\right)^3 (-1)^{n+3} \frac{(n+\frac{5}{2})(n+\frac{3}{2})(n+\frac{1}{2})}{n!\, 2^n} + \cdots \right\} x^{2n}$$

$$= (\pi)^{-1/4} \sum_{n=0}^{\infty} \left\{ \frac{1}{n!\, 2^n} \left[(-1)^{n+1} \frac{i\hbar t}{m}(n+\tfrac{1}{2}) + \frac{1}{2!}(-1)^{n+2} \left(\frac{i\hbar t}{m}\right)^2 (n+\tfrac{1}{2})(n+\tfrac{3}{2}) + \frac{1}{3!}(-1)^{n+3} \left(\frac{i\hbar t}{m}\right)^3 (n+\tfrac{1}{2})(n+\tfrac{3}{2})(n+\tfrac{5}{2}) + \cdots \right\} x^{2n}$$

~~$= (\pi)^{-1/4} \sum_{n,m=0}^{\infty} (-1)^{n+m+1}$~~

$$= (\pi)^{-1/4} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\, 2^n} \left\{ -\frac{i\hbar t}{m}(n+\tfrac{1}{2}) + \frac{1}{2!}\left(\frac{i\hbar t}{m}\right)^2 (n+\tfrac{1}{2})(n+\tfrac{3}{2}) - \frac{1}{3!}\left(\frac{i\hbar t}{m}\right)^3 (n+\tfrac{1}{2})(n+\tfrac{3}{2})(n+\tfrac{5}{2}) + \cdots \right\}$$

or

$$\boxed{\psi(x,t) = (\pi)^{-1/4} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\, 2^n} \left\{ \left(1 + \frac{i\hbar t}{m}\right)^{-n-\frac{1}{2}} - 1 \right\}}$$

5.1.4. The answer is given in the book.

5.2.1. Before the walls expand the ground state W.F. is

$$\psi_0(x) = \sqrt{2/L}\cos\frac{\pi x}{L}$$

The eigen functions after the expansion of the walls are

$$\psi_n(x) = \frac{1}{\sqrt{L}}\sin\frac{n\pi x}{2L},\ n = 1,3,5\ldots$$

$$= \frac{1}{\sqrt{2}}\cos\frac{n\pi x}{2L},\ n = 2,4,6,\ldots$$

We then expand ψ_0 in terms of this set of orthonormal functions

$$\psi_0(x) = \sum_{n=1}^{\infty} a_n \psi_n(x)$$

When $a_n = \int_{-L/2}^{L/2} \psi_0(x)\,\psi_n(x)\,dx$

is the probability amplitude in the n^{th} state. Thus the probability that the particle is in the ground state after expansion is

$$a_0^2 = \left(\int_{-L/2}^{L/2} \sqrt{2/L}\cos\frac{\pi x}{L}\cdot\frac{1}{\sqrt{L}}\cos\frac{\pi x}{2L}\,dx\right)^2$$

$$= \left[\frac{\sqrt{2}}{2L}\int_{-L/2}^{L/2} \cos\frac{\pi x}{2L} + \cos\frac{3\pi x}{2L}\,dx\right]^2$$

$$= \left[\frac{\sqrt{2}}{2L}\left|\frac{2L}{\pi}\sin\frac{\pi x}{2L} + \frac{2L}{3\pi}\sin\frac{3\pi x}{2L}\right|_{-L/2}^{L/2}\right]^2$$

$$= \left[\frac{\sqrt{2}}{\pi}\left(\sin\frac{\pi}{4} + \frac{1}{3}\sin\frac{3\pi}{4} + \sin\frac{\pi}{4} + \frac{1}{3}\sin\frac{3\pi}{4}\right)\right]^2$$

$$= \left(\frac{\sqrt{2}}{\pi}\left(\frac{2}{\sqrt{2}} + \frac{2}{3\sqrt{2}}\right)\right)^2$$

$$= \left(\frac{\sqrt{2}}{\pi}\left(8/3\right)\right)^2$$

or

$$a_0^2 = \left(\frac{8}{3\pi}\right)^2$$

5.2.2.

a) In the eigenbasis of H

$$|\psi\rangle = \sum_E |E\rangle\langle E|\psi\rangle$$

$$\langle\psi|H|\psi\rangle = \int_{-\infty}^{\infty}\langle\psi|x\rangle\langle x|H|\psi\rangle dx$$

Inserting the expression of $|\psi\rangle$

$$\langle\psi|H|\psi\rangle = \int_{-\infty}^{\infty}\langle\psi|x\rangle\langle x|\sum_E H|E\rangle\langle E|\psi\rangle dx$$

$$= \int_{-\infty}^{\infty}\langle\psi|x\rangle\langle x|\sum_E E|E\rangle\langle E|\psi\rangle dx$$

But $E \geq E_0$

$$\Rightarrow \langle\psi|H|\psi\rangle \geq \int_{-\infty}^{\infty}\langle\psi|x\rangle E_0\langle x|\sum_E |E\rangle\langle E|\psi\rangle dx$$

But $\langle x|\psi\rangle = \psi(x) = \langle x|\sum_E |E\rangle\langle E|\psi\rangle$

$$\Rightarrow \langle\psi|H|\psi\rangle \geq \int_{-\infty}^{\infty}\langle\psi|x\rangle E_0\langle x|\psi\rangle dx$$

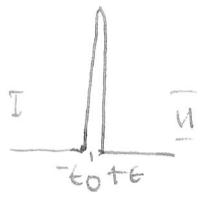
$$\geq E_0\int_{-\infty}^{\infty}\psi^*(x)\,\psi(x)\,dx$$

$$\geq E_0$$

Which is the desired result.

b)

5.2.3.



Schrödinger eq.

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} - aV_0\delta(x)\psi = E\psi$$

$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}(E + aV_0\delta(x))\psi = 0$$

To elliminate the delta function integrate from $-\epsilon$ to ϵ

$$\left[\frac{d\psi(\epsilon)}{dx} - \frac{d\psi(-\epsilon)}{dx}\right] + \frac{2m}{\hbar^2}E\int_{-\epsilon}^{\epsilon}\psi\,dx \;-\!-*$$
$$+\frac{2m\,a V_0}{\hbar^2} = 0$$

as $\epsilon \to 0$, the 2nd vanishes

$$\left[\frac{d\psi(\epsilon)}{dx} - \frac{d\psi(-\epsilon)}{dx}\right] + \frac{2maV_0\psi(0)}{\hbar^2} = 0 \quad --(*)$$

Outside the well

$$\frac{d^2\psi}{dx^2} + \frac{2mE}{\hbar^2}\psi = 0$$

$$\frac{d^2\psi}{dx^2} - \frac{2m|E|}{\hbar^2}\psi = 0, \; E < 0$$

In region I

$$\psi_I(x) = Ae^{\alpha x}, \; x < 0$$

In region II

$$\psi_{II}(x) = Be^{-\alpha x}, \; x > 0$$

Since $\epsilon \to 0$, ψ_I & ψ_{II} must be continuous at $x=0$

$$\Rightarrow A = B$$

Normalization:

$$\int_{-\infty}^{0}|\psi_I(x)|^2dx + \int_{0}^{\infty}|\psi_{II}(x)|^2dx$$

$$\int_{-\infty}^{0}A^2e^{+2\alpha x}dx + \int_{0}^{\infty}A^2e^{-2\alpha x}dx$$

$$A^2\left(\int_{-\infty}^{0}e^{2\alpha x}dx + \int_{0}^{\infty}e^{-2\alpha x}dx\right) = 1$$

$$A^2\left(\int_{0}^{\infty}e^{-2\alpha x}dx \int_{0}^{\infty}e^{-2\alpha x}dx\right) = 1$$

$$2A^2\int_{0}^{\infty}e^{-2\alpha x}dx = 1$$

$$\frac{2A^2}{-2\alpha}e^{-2\alpha x}\Big|_0^{\infty} = 1$$

$$\Rightarrow \frac{2A^2}{\alpha} = 1$$

or $A = \sqrt{\alpha}$

Now,

$$\psi_I(x) = \sqrt{\alpha}\,e^{+\alpha x}$$

$$\psi_{II}(x) = \sqrt{\alpha}\,e^{-\alpha x}$$

going to $(*)$

$$[-\alpha^{3/2} - \alpha^{3/2}] + \frac{2maV_0}{\hbar^2}\alpha^{1/2} = 0$$

$$-2\alpha + \frac{2maV_0}{\hbar^2} = 0$$

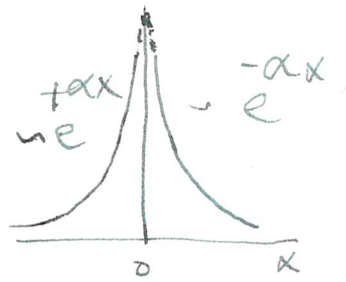
$$\alpha = \frac{maV_0}{\hbar^2}$$

$$\alpha^2 = \frac{m^2a^2V_0^2}{\hbar^4}$$

$$\frac{2m|E|}{\hbar^2} = \frac{m^2a^2V_0^2}{\hbar^4}$$

$$|E| = ma^2V_0^2/\hbar^2$$

or $\boxed{E = -ma^2V_0^2/\hbar^2}$



change of slope

$$\int_{-\epsilon}^{\epsilon} \left(\frac{d^2\psi}{dx^2}\right) dx$$

$$\neq \left|\frac{d\psi_I(x)}{dx}\right| - \left|\frac{d\psi_{II}(x)}{dx}\right|$$

$$= \int_{-\epsilon}^{0} \frac{d^2\psi_I(x)}{dx^2} + \int_{0}^{\epsilon} \frac{d^2\psi_{II}(x)}{dx^2}$$

$$= \frac{d\psi_I}{dx}\Big|_{-\epsilon}^{0} + \frac{d\psi_{II}}{dx}\Big|_{0}^{\epsilon}$$

$$+ \alpha^{3/2} e^{\alpha x}\Big|_{-\epsilon}^{0} - \alpha^{3/2} e^{-\alpha x}\Big|_{0}^{\epsilon}$$

$$\alpha^{3/2}(1 - e^{-\alpha\epsilon})$$

$$- \alpha^{3/2}(e^{-\alpha\epsilon} - 1)$$

$$= 2\alpha^{3/2}(1 - e^{-\alpha\epsilon}) \quad \text{--}(**)$$

$$\frac{d\psi_I}{dx}\Big|_{x=0} = \alpha^{3/2} e^{\alpha x}\Big|_0 = \alpha^{3/2}$$

$$\frac{d\psi_{II}}{dx}\Big|_{x=0} = -\alpha^{3/2} e^{-\alpha x}\Big|_0 = -\alpha^{3/2}$$

change of slope is the sume of the two, i.e.,

$$\alpha^{3/2} - \alpha^{3/2} = 0$$

Also the result in $(**)$ goes to zero as $\epsilon \to 0$ and the two are equal.

5.2.4. For a particle in a box of width L:

$$E_n = \frac{n^2\hbar^2\pi^2}{2mL^2}$$

When the walls move in and if the particle remains in the nth state, the force that acts on it is

$$F = -\frac{\partial E_n}{\partial L} = +\frac{n^2\hbar^2\pi^2}{mL^3}$$

If the classical particle in this box has energy E_n, then

$$\tfrac{1}{2}mv^2 = \frac{n^2\hbar^2\pi^2}{2mL^2}$$

$$v = n\hbar\pi/mL$$

The frequency of collision on a given wall (since it makes one round trip after each collision hence travels a dist. 2L) is

$$\nu = \frac{v}{2L} = \frac{n\hbar\pi}{2mL^2}$$

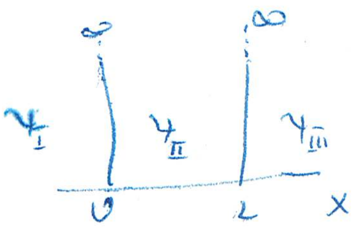
The mom. transfer per collision is = Change in mom./collision

$$2mv = ~~\frac{n^2\hbar^2\pi^2}{2mL^3}~~ = 2n\hbar\pi/L$$

Average force $= 2mv\nu = \dfrac{n^2\hbar^2\pi^2}{mL^3}$

This is actually the average force which acts on the particle. It is equal to that calculated above

5.2.5.



ψ vanishes outside the well as V is infinite. Inside, let $\psi = \psi_{II}$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi$$

$$\frac{d^2\psi}{dx^2} + \frac{2mE}{\hbar^2}\psi = 0$$

$$\frac{d^2\psi}{dx^2} + k^2\psi = 0$$

$$\psi = Ae^{ikx} + Be^{-ikx}$$

ψ should vanish at x = 0, L.

$$A + B = 0 \quad \text{---(1)}$$

$$Ae^{ikL} + Be^{-ikL} = 0 \quad \text{...(2)}$$

or
$$\begin{bmatrix} 1 & 1 \\ e^{ikL} & e^{-ikL} \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

For a nontrivial solution

$$\begin{vmatrix} 1 & 1 \\ e^{ikL} & e^{-ikL} \end{vmatrix} = 0$$

$$e^{-ikL} - e^{ikL} = 0$$

$$-2i\sin kL = 0$$

$\Rightarrow kL = n\pi, \; 0, \pm 1, \pm 1, \ldots$

or $k = n\pi/L$

From (1) $B = -A$

$$\therefore \psi(x) = A\left(e^{i\frac{n\pi}{2}x} - e^{-i\frac{n\pi}{2}x}\right)$$

$$= A\,2iA\sin\frac{n\pi}{2}x$$

$$= C\sin\frac{n\pi}{2}x$$

Now the integer values

5.3.1.

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + (V_r - iV_i)$$

$$H^+ = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_r + iV_i$$

i.e., $H^+ \neq H$ and hence H is not Hermitian

The derivation of the ~~HWVAK PROBABILITY~~ continuity eq.

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + (V_r - iV_i)\psi \quad \ldots (1)$$

Taking the conjugate

$$-i\hbar\frac{\partial\psi^*}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi^* + (V_r + iV_i)\psi^* \quad \ldots (2)$$

Multiplying (1) by ψ^* and (2) by ψ and taking the difference,

$$i\hbar\frac{\partial}{\partial t}(\psi^*\psi) = -\frac{\hbar^2}{2m}(\psi^*\nabla^2\psi - \psi\nabla^2\psi^*) - 2iV_i\psi^*\psi$$

$$\frac{\partial P}{\partial t} = \frac{-\hbar}{2mi}\nabla\cdot(\psi^*\nabla\psi - \psi\nabla\psi^*) - \frac{2}{\hbar}V_i\psi^*\psi$$

$$\frac{\partial P}{\partial t} = \frac{-\hbar^2}{2m_i}\nabla\cdot\bar{J} - \frac{2V_iP}{\hbar}$$

5.3.2.

$$J = \frac{\hbar}{2mi}(\psi^*\nabla\psi - \psi\nabla\psi^*)$$

Now, $\psi = c\tilde{\psi}$

$\psi^* = c^*\tilde{\psi}$, $\tilde{\psi}$ is real.

$$\nabla\psi = c\nabla\tilde{\psi}$$
$$\nabla\psi^* = c^*\nabla\tilde{\psi}$$

$$\Rightarrow \psi^*\nabla\psi = |c|^2\tilde{\psi}\nabla\tilde{\psi}$$
$$\psi\nabla\psi^* = |c|^2\tilde{\psi}\nabla\tilde{\psi}$$

The difference of which vanishes and hence,

$$\bar{J} = 0.$$

5.3.3. $\psi_p = \left(\frac{1}{2\pi\hbar}\right)^{3/2} e^{i(\bar{p}\cdot\bar{r})/\hbar}$

$$\bar{J} = \frac{\hbar}{2mi}(\psi_p^*\nabla\psi_p - \psi_p\nabla\psi_p^*)$$

$$\psi_p^* = \left(\frac{1}{2\pi\hbar}\right)^{3/2} e^{-i(\bar{p}\cdot\bar{r})/\hbar}$$

$$\nabla\psi_p = \frac{i\bar{p}}{\hbar}\left(\frac{1}{2\pi\hbar}\right)^{3/2} e^{\frac{i(\bar{p}\cdot\bar{r})}{\hbar}}$$

$$\nabla\psi_p^* = -\frac{i\bar{p}}{\hbar}\left(\frac{1}{2\pi\hbar}\right)^{3/2} e^{-\frac{i(\bar{p}\cdot\bar{r})}{\hbar}}$$

$$\psi_p^*\nabla\psi_p = \frac{i\bar{p}}{\hbar}\left(\frac{1}{2\pi\hbar}\right)^3$$

$$\psi_p\nabla\psi_p^* = -\frac{i\bar{p}}{\hbar}\left(\frac{1}{2\pi\hbar}\right)^3$$

i.e., $\bar{J} = \frac{\hbar}{2mi}\left\{\frac{i\bar{p}}{\hbar}\left(\frac{1}{2\pi\hbar}\right)^3 + \frac{i\bar{p}}{\hbar}\left(\frac{1}{2\pi\hbar}\right)^3\right\}$

$$= \frac{\bar{p}}{m}\left(\frac{1}{2\pi\hbar}\right)^3$$

or $\bar{J} = \bar{v}\left(\frac{1}{2\pi\hbar}\right)^3 \quad \ldots (1)$

The probability density is $P = \psi^*\psi = \left(\frac{1}{2\pi\hbar}\right)^3$

or $\rho = \left(\frac{1}{2\pi\hbar}\right)^3 \quad \ldots (2)$

Comparison of (1) and (2) gives

$$\bar{J} = \rho \bar{v}$$

$\frac{\partial \rho}{\partial t} = 0$, automatically

$$\nabla \cdot \bar{J} = \nabla \cdot (\rho \bar{v}) = \nabla \rho \cdot \bar{v} + \rho \nabla \cdot \bar{v} = 0$$

i.e., the continuity eq. is satisfied.

S 3.4.

$$\bar{j} = \frac{\hbar}{2mi}(\psi^* \nabla \psi - \psi \nabla \psi^*)$$

$$\psi = A e^{ipx/\hbar} + B e^{-ipx/\hbar}$$

$$\psi^* = A^* e^{-ipx/\hbar} + B^* e^{ipx/\hbar}$$

$$\nabla \psi \equiv \frac{d\psi}{dx}\hat{i}, \quad \nabla \psi^* = \frac{d\psi^*}{dx}\hat{i}$$

$$\frac{d\psi}{dx} = \frac{ip}{\hbar}\left(A e^{ipx/\hbar} - B e^{-ipx/\hbar}\right)$$

$$\frac{d\psi^*}{dx} = \frac{-ip}{\hbar}\left(A^* e^{-ipx/\hbar} - B^* e^{ipx/\hbar}\right)$$

$$\psi^* \frac{d\psi}{dx} = \frac{ip}{\hbar}|A|^2 - A^* B \left(\frac{ip}{\hbar}\right) e^{\frac{-2ipx}{\hbar}} + AB^* \frac{ip}{\hbar} e^{2ipx/\hbar} + |B|^2 \frac{ip}{\hbar}$$

$$\psi \frac{d\psi^*}{dx} = -\frac{ip}{\hbar}|A|^2 + AB^* \frac{ip}{\hbar} e^{2ipx/\hbar} - BA^* \frac{ip}{\hbar} e^{\frac{-2ipx}{\hbar}} + |B|^2 \left(\frac{ip}{\hbar}\right)$$

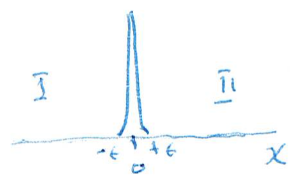
Then,

$$\bar{J} = \frac{\hbar}{2mi}\left\{\frac{2ip}{\hbar}|A|^2 - \frac{2ip}{\hbar}|B|^2\right\}\hat{i}$$

$$= \frac{p}{m}\left\{|A|^2 - |B|^2\right\}\hat{i}$$

or $J = (|A|^2 - |B|^2)\, p/m$

5.4.2 a,



In regions I and II the Schrödinger eq. becomes

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi$$

$$\frac{d^2\psi}{dx^2} + \frac{2mE}{\hbar^2}\psi = 0$$

$$\frac{d^2\psi}{dx^2} + k^2\psi = 0$$

or $\psi = A e^{ikx} + B e^{-ik}$

In region I we have both the incident and reflected waves so that

$$\psi_I = A e^{ikx} + B e^{-ikx}$$

In region II we have only a right-going wave

$$\psi_{II} = C e^{ikx}$$

Where we have the potential

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + aV_0 \delta(x)\psi = E\psi$$

$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}(E - aV_0\delta(x))\psi = 0$$

Let us integrate this eq. from $x = -\epsilon$ to $+\epsilon$.

$$\left.\frac{d\psi}{dx}\right|_{-\epsilon}^{+\epsilon} + \frac{2mE}{\hbar^2}\psi\Big|_{-\epsilon}^{+\epsilon} - \frac{2maV_0}{\hbar^2}\psi(0) = 0$$

$$\Rightarrow \frac{d\psi_{II}(\epsilon)}{dx} - \frac{d\psi_I(-\epsilon)}{dx} + \frac{2mE}{\hbar^2}(\psi(\epsilon) - \psi(-\epsilon)) - \frac{2maV_0}{\hbar^2}\psi(0) = 0$$

In the limit $\epsilon \to 0$ we have

$$\frac{d\psi_{II}(0)}{dx} - \frac{d\psi_I(0)}{dx} - \frac{2maV_0}{\hbar^2}\psi(0) = 0 \quad \text{--- (1)}$$

From the continuity of ψ

$$A + B = C \quad \text{--- 2}$$

Since ψ is written as a linear combination of ψ_I and ψ_{II} (1) would give

$$ikC - ik(A-B) - \frac{2maV_0}{\hbar^2}(A+B+C) = 0$$

and

$$A + B = C$$

Using the expression of C here

$$ik(A+B) - ik(A-B) - \frac{2maV_0}{\hbar^2}(A+B+A+B) = 0$$

$$2ikB - \frac{4maV_0}{\hbar^2}(A+B) = 0$$

$$ikB - \frac{2maV_0}{\hbar^2}(A+B) = 0$$

$$B\left(ik - \frac{2maV_0}{\hbar^2}\right) = \frac{2maV_0}{\hbar^2}A$$

$$B = \frac{\frac{2maV_0}{\hbar^2}}{ik - \frac{2maV_0}{\hbar^2}}A$$

$$= \frac{\frac{2maV_0}{\hbar^2}\left(-ik - \frac{2maV_0}{\hbar^2}\right)}{k^2 + \frac{4m^2a^2V_0^2}{\hbar^4}}A$$

$$|B| = \frac{2maV_0/\hbar^2}{k^2 + 4m^2a^2V_0^2/\hbar^4}\left\{k^2 + \frac{4m^2a^2V_0^2}{\hbar^4}\right\}^{\frac{1}{2}}|A|$$

$$R = \frac{|B|^2}{|A|^2} = \frac{4m^2a^2V_0^2/\hbar^4}{k^2 + 4m^2a^2V_0^2/\hbar^4}$$

$$= \frac{4m^2a^2V_0^2}{\hbar^4k^2 + 4m^2a^2V_0^2}$$

(ii), $T = R - 1$

b)

In regions I & III the Schrödinger eq is the same as that in 8(4)(a) for just the corresponding regions.

$$\psi_I = Ae^{ikx} + Be^{-ikx}$$

$$\psi_{III} = Fe^{ikx} \quad (\text{right-going wave})$$

In region II.

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V_0\psi = E\psi$$

$$\frac{d^2\psi}{dx^2} - \frac{2m}{\hbar^2}(V_0 - E)\psi = 0$$

$$\frac{d^2\psi}{dx^2} - \kappa^2\psi = 0$$

$$\Rightarrow \psi_{II} = Ce^{\kappa x} + De^{-\kappa x}$$

In this cond. both ψ and its derivatives are continuous.

$$\psi_I(-a) = \psi_{II}(-a) \quad \text{--- (1)}$$

$$\frac{d\psi_I(-a)}{dx} = \frac{d\psi_{II}(-a)}{dx} \quad \text{--- 2}$$

$$\psi_{II}(a) = \psi_{III}(a) \quad \text{--- 3}$$

$$\frac{d\psi_{II}(a)}{dx} = \frac{d\psi_{III}(a)}{dx} \quad \text{--- 4}$$

$$\Rightarrow Ae^{-ika} + Be^{ika} = Ce^{-\kappa a} + De^{\kappa a}$$

$$ik(Ae^{-ika} - Be^{ika}) = \kappa(Ce^{-\kappa a} - De^{+\kappa a})$$

$$Ce^{\kappa a} + De^{-\kappa a} = Fe^{ika}$$

$$\kappa(Ce^{\kappa a} - De^{-\kappa a}) = ikFe^{ika}$$

Can be calculated by using Matrix eq.

5.4.3 Consider the eigen value prob. of H.

$$H|E\rangle = E|E\rangle$$

$$\Rightarrow \langle p|H|E\rangle = E\langle p|E\rangle \quad \text{---} (*)$$

But $H = \frac{P^2}{2m} + V$

$$V(x) = -\int f\,dx = -fx$$

i.e., $H = \frac{P^2}{2m} - fX$

In the mom. space (*) becomes

$$\left(\frac{p^2}{2m} - f i\hbar \frac{d}{dp}\right)\psi_E(p) = E\psi_E(p)$$

$$i\hbar f \frac{d\psi_E(p)}{dp} = \left(\frac{p^2}{2m} - E\right)\psi_E(p)$$

$$\frac{d\psi_E(p)}{\psi_E(p)} = \frac{-i}{\hbar f}\left(\frac{p^2}{2m} - E\right)dp$$

$$\ln\psi_E(p) = \frac{-i}{\hbar f}\left(\frac{p^3}{6m} - Ep\right) + c$$

or $\psi_E(p) = A\, e^{\frac{-i}{\hbar f}\left(\frac{p^3}{6m} - Ep\right)}$

Since E is not restricted to be positive, we can normalize $\psi_E(p)$ to the delta function s.t $\langle E|E'\rangle = \delta(E-E')$.

Now,

$$U(t) = \int_{-\infty}^{\infty} |E\rangle\langle E|\, e^{-\frac{iE}{\hbar}t}\, dE$$

$$\Rightarrow \langle p|U(t)|p'\rangle = \int_{-\infty}^{\infty} \langle p|E\rangle\langle E|p'\rangle\, e^{-\frac{iE}{\hbar}t}\, dE$$

$$= \int_{-\infty}^{\infty} \psi_E(p)\,\psi_E^*(p')\, e^{-\frac{iE}{\hbar}t}\, dE$$

$$= \frac{1}{2\pi}\, e^{\frac{i}{6m\hbar f}(p'^3 - p^3)} \int_{-\infty}^{\infty} e^{\frac{i}{\hbar f}(p - p' - ft)E}\, dE$$

$$= e^{\frac{i}{6m\hbar f}(p'^3 - p^3)}\, \delta(p - p' - ft)$$

i.e.,

$$U(p, t; p', 0) = \delta(p - p' - ft)\, e^{\frac{i}{6m\hbar f}(p'^3 - p^3)}$$

7.3.3

$$\int_{-\infty}^{\infty} \psi(x)\Phi(x)dx = \int_{-\infty}^{0} \psi(x)\Phi(x)dx + \int_{0}^{\infty} \psi(x)\Phi(x)dx$$

$$= -\int_{0}^{\infty} \psi(x)\Phi(x)dx + \int_{0}^{\infty} \psi(x)\Phi(x)dx$$

i.e., in the 1st term when $x<0$, $\psi(x)\Phi(x)<0$ and $\psi(x)\Phi(x)dx>0$, so that in place of the minus infinity we can replace the plus infinity.

$$\therefore \int_{-\infty}^{\infty} \psi(x)\Phi(x)dx = 0$$

7.3.4.

$$\langle n'|X|n\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \langle n'|x\rangle\langle x|X|x'\rangle\langle x'|n\rangle\, dx\, dx'$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \psi_{n'}^*(x)\, x\, \delta(x-x')\psi_n(x')\, dx\, dx'$$

$$= \int_{-\infty}^{\infty} \psi_{n'}(x)\, x\, \psi_n(x)\, dx$$

The functions $\psi_n(x)$ are real.

$$x = \left(\frac{\hbar}{m\omega}\right)^{1/2} y$$

$$x\,dx = \frac{\hbar}{m\omega} y\, dy$$

$$\psi_n(x) \to \psi_n(y) = A_n e^{-y^2/2} H_n(y)$$

$$\Rightarrow \langle n'|X|n\rangle = \frac{\hbar A_{n'}A_n}{m\omega}\int_{-\infty}^{\infty} e^{-y^2} H_{n'}(y)\, y\, H_n(y)dy$$

But $H_{n+1}(y) = 2yH_n(y) - 2nH_{n-1}(y)$

or $yH_n(y) = \dfrac{H_{n+1}(y) + 2nH_{n-1}(y)}{2}$

$$\Rightarrow \langle n'|X|n\rangle = \frac{\hbar A_{n'}A_n}{2m\omega}\int_{-\infty}^{\infty} e^{-y^2} H_{n'}(y)[H_{n+1}+2nH_{n-1}]dy$$

$$\langle n'|X|n\rangle = \frac{\hbar A_{n'}A_n}{2m\omega} \cdot \left\{\int_{-\infty}^{\infty} e^{-y^2} H_{n'}(y)H_{n+1}(y)dy + 2n\int_{-\infty}^{\infty} e^{-y^2} H_{n'}(y)H_{n-1}(y)dy\right\}$$

But $\int_{-\infty}^{\infty} H_n(y)H_{n'}(y)e^{-y^2}dy$

$$= \delta_{nn'}(\sqrt{\pi}\, 2^n n!)$$

$$\text{i.e., } \langle n'|X|n\rangle = \frac{\hbar A_{n'}A_n}{2m\omega} \cdot \left\{\delta_{n',n+1}(\sqrt{\pi}\, 2^{n+1}(n+1)!) + 2n(\delta_{n',n-1}(\sqrt{\pi}\, 2^{n-1}(n-1)!))\right\}$$

$$= \frac{\hbar A_{n'}A_n}{2m\omega} 2^n n!\sqrt{\pi} \cdot \left\{\delta_{n',n+1}\, 2(n+1) + \delta_{n',n-1}\right\}$$

$$= \frac{\hbar A_{n'}A_n}{m\omega} 2^n (n+1)!\sqrt{\pi}\, \delta_{n',n+1} + \frac{\hbar A_{n'}A_n}{m\omega} 2^{n-1} n!\sqrt{\pi}\, \delta_{n',n-1}$$

In the first case, $n' = n+1$ for $\delta_{n',n+1}$ not to be zero.

$$A_{n'}A_n = A_{n+1}A_n$$

$$= \left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\left[\frac{1}{2^{2(n+1)}((n+1)!)^2}\cdot\frac{1}{2^{2n}(n!)^2}\right]^{1/4}$$

$$= \left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\left[\frac{1}{2^{4n}\, 2^2 (n+1)^2 (n!)^4}\right]^{1/4}$$

$$= \left(\frac{m\omega}{2\pi\hbar}\right)^{1/2}\frac{1}{\sqrt{n+1}}\frac{1}{2^n n!}$$

In the 2nd term $n' = n-1$

$$A_{n'}A_n = A_{n-1}A_n$$

$$= \left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\left[\frac{1}{2^{2(n-1)}[(n-1)!]^2}\cdot\frac{1}{2^{2n}(n!)^2}\right]^{1/4}$$

$$= \left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\left[\frac{1}{2^{4n-2}}\frac{n^2}{(n!)^2}\cdot\frac{1}{(n!)^2}\right]^{1/4}$$

$$= \left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\left[\frac{2^2 n^2}{2^{4n}(n!)^4}\right]^{1/4}$$

$$= \left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\frac{\sqrt{2n}}{2^n n!} = \left(\frac{m\omega}{2\pi\hbar}\right)^{1/2}\frac{\sqrt{n}}{2^{n-1} n!}$$

Then

$$\langle n'|x|n\rangle$$

$$= \frac{\hbar}{m\omega}\left(\frac{m\omega}{2\pi\hbar}\right)^{1/2}\frac{1}{2^n n!\sqrt{n+1}}\cdot 2^n (n+1)!\sqrt{\pi}\,\delta_{n',n+1}$$

$$+ \frac{\hbar}{m\omega}\left(\frac{m\omega}{2\pi\hbar}\right)^{1/2}\frac{\sqrt{n}\,2^{n-1} n!\sqrt{\pi}}{2^{n-1} n!}\delta_{n',n-1}$$

$$= \left(\frac{\hbar}{2m\omega}\right)^{1/2}\sqrt{n+1}\,\delta_{n',n+1} + \left(\frac{\hbar}{2m\omega}\right)^{1/2}\sqrt{n}\,\delta_{n',n-1}$$

$$= \left(\frac{\hbar}{2m\omega}\right)^{1/2}\left[\delta_{n',n+1}\sqrt{n+1} + \delta_{n',n-1}\sqrt{n}\right]$$

$$-\langle n'|p|n\rangle = \int_{-\infty}^{\infty}\psi_{n'}(x)\left(-i\hbar\frac{d}{dx}\right)\psi_n(x)\,dx$$

$$= -i\hbar\int_{-\infty}^{\infty}\psi_{n'}(x)\frac{d\psi_n(x)}{dx}\,dx$$

$$x = \left(\frac{\hbar}{m\omega}\right)^{1/2} y$$

$$\Rightarrow \frac{d}{dx} = \left(\frac{m\omega}{\hbar}\right)^{1/2}\frac{d}{dy},\quad dx = \left(\frac{\hbar}{m\omega}\right)^{1/2} dy$$

$$\langle n'|p|n\rangle = -i\hbar\left(\frac{m\omega}{\hbar}\right)^{1/2}\left(\frac{\hbar}{m\omega}\right)^{1/2}\int_{-\infty}^{\infty}\psi_{n'}(y)\frac{d\psi_n(y)}{dy}\,dy$$

$$\psi_n(y) = A_n e^{-y^2/2} H_n(y)$$

$$\frac{d\psi_n(y)}{dy} = -y A_n e^{-y^2/2} H_n(y) + A_n e^{-y^2/2} H_n'(y)$$

$$\langle n'|p|n\rangle = -i(\hbar m\omega)^{1/2}\int_{-\infty}^{\infty} A_{n'} e^{-y^2/2} H_{n'}(y)\left[-y A_n e^{-y^2/2} H_n(y) + A_n e^{-y^2/2} H_n'(y)\right] dy$$

$$= +i(\hbar m\omega)^{1/2} A_{n'} A_n\left\{\int_{-\infty}^{\infty} e^{-y^2} H_{n'} y H_n\,dy - \int_{-\infty}^{\infty} e^{-y^2} H_{n'}(y) H_n'(y)\,dy\right\}$$

$$= i(\hbar m\omega)^{1/2} A_{n'} A_n\left\{\int_{-\infty}^{\infty} e^{-y^2} H_{n'}(y)\cdot\left[\frac{H_{n+1}(y) + 2n H_{n-1}(y)}{2}\right]dy - 2n\int_{-\infty}^{\infty} e^{-y^2} H_{n'}(y) H_{n-1}(y)\,dy\right]$$

$$= i(\hbar m\omega)^{1/2} A_{n'} A_n \cdot\left\{\frac{1}{2}\int_{-\infty}^{\infty} e^{-y^2} H_{n'}(y) H_{n+1}(y)\,dy - \int_{-\infty}^{\infty} e^{-y^2} H_{n'}(y) H_{n-1}(y)\,dy\right\}$$

$$= i(\hbar m\omega)^{1/2} A_{n'} A_n \cdot\left\{\frac{1}{2}\delta_{n',n+1}\left(\sqrt{\pi}\,2^{n+1}(n+1)!\right) - \delta_{n',n-1}\left(\sqrt{\pi}\,2^{n-1}(n-1)!\right)\right\}$$

Setting $n' = n+1$ for the first term and $n' = n-1$ for the 2nd term we would obtain

$$\langle n'|p|n\rangle = i\left(\frac{m\omega\hbar}{2}\right)^{1/2}\left[\delta_{n',n+1}\sqrt{n+1} - \delta_{n',n-1}\sqrt{n}\right]$$

7.3.5.

$\langle n|X|n\rangle = \int_{-\infty}^{\infty} \psi_n(x)\, x\, \psi_n(x)\, dx$ (from 7.3.4)

$= \int_{-\infty}^{\infty} x\, \psi_n(x)^2\, dx$

$= \frac{\hbar A_n^2}{m\omega} \int_{-\infty}^{\infty} e^{-y^2}\, y\, H_n(y)^2\, dy$

Using the recursion relation this becomes

$\langle n|X|n\rangle = \frac{\hbar A_n^2}{2m\omega} \int_{-\infty}^{\infty} e^{-y^2} H_n(y) [H_{n+1}(y) + 2n H_{n-1}(y)]\, dy$

$= \frac{\hbar A_n^2}{2m\omega} \left\{ \int_{-\infty}^{\infty} e^{-y^2} H_n H_{n+1}(y)\, dy + 2n \int_{-\infty}^{\infty} e^{-y^2} H_n H_{n-1}(y)\, dy \right\}$

We know, $n = 0, 1, 2, \ldots$ and hence the product $H_n H_{n+1}$ or $H_n H_{n-1}$ is a product of even and odd functions and hence odd for whatever the above values n may take. Further e^{-y^2} is an even function. Thus in either integral we have a product of even and odd functions so that it should vanish in accordance with 7.3.3.

For $\langle n|P|n\rangle$ we goto the X basis in which $P \to -i\hbar \frac{d}{dx}$. This operator when acts on $\psi_n(x)$ it gives $\psi_n'(x) \sim -x\,\psi_n(x) + \psi_n(x) H_n'(x)$ which again is a product of even and odd functions. Thus the integral should vanish by the same reasoning.

$\langle X^2\rangle$?

$-\ (\Delta X)^2 = \langle n|(X - \langle X\rangle)^2|n\rangle$

$= \langle n|(X^2 - 2X\langle X\rangle + \langle X\rangle^2|n\rangle$

$= \langle n|X^2|n\rangle - 2\langle X\rangle^2 + \langle X\rangle^2$

$= \langle n|X^2|n\rangle - \langle X\rangle^2$

$= \langle n|X^2|n\rangle = \langle X^2\rangle$

i.e., $\langle X^2\rangle = (\Delta X)^2$

$-\ \langle P^2\rangle = \langle n|P^2|n\rangle$

$= \langle n|(P^2 - \langle P\rangle)^2|n\rangle$

as, $\langle P\rangle = 0$

or $\langle P^2\rangle = (\Delta P)^2$.

$\langle 1|X^2|1\rangle = \int_{-\infty}^{\infty} \psi_1(x)\, x^2\, \psi_1(x)\, dx$

$= \int_{-\infty}^{\infty} \psi_1(x)^2\, x^2\, dx$

$\psi_1(x) \to \psi_1(y) = A_1 e^{-y^2/2} H_1(y)$

$\psi_1^2(x) \to \psi_1^2(y) = A_1^2 e^{-y^2} H_1^2(y)$

i.e., $\langle 1|X^2|1\rangle = \left(\frac{\hbar}{m\omega}\right)^{3/2} A_1^2 \int_{-\infty}^{\infty} e^{-y^2} H_1(y)^2\, y^2\, dy$

$= \left(\frac{\hbar}{m\omega}\right)^{3/2} A_1^2 \int_{-\infty}^{\infty} e^{-y^2} H_1(y)\, y^2\, H_1(y)\, dy$

From the recursion relation

$y H_1(y) = \frac{H_2(y) + 2H_0(y)}{2}$

~~$y^2 H_1^2(y) = \frac{1}{2}[H_2^2(y) + 4 H_0^2(y) + 4 H_2(y) H_0(y)]$~~

~~or $\langle 1|X^2|1\rangle = \frac{A_1^2}{2} \int_{-\infty}^{\infty}$~~

$[y H_1(y)]^2 = \frac{1}{4}[H_2^2(y) + 4H_0^2(y) + 4H_0(y)H_2(y)]$

$\Rightarrow \langle 1|x^2|1\rangle = \frac{\hbar^{3/2}}{4(m\omega)^{3/2}} A_1^2 \left\{ \int_{-\infty}^{\infty} e^{-y^2} H_2^2(y)\, dy + 4\int_{-\infty}^{\infty} e^{-y^2} H_0^2(y)\, dy + 4\int_{-\infty}^{\infty} e^{-y^2} H_0(y)H_2(y)dy \right\}$

$= \frac{\hbar^{3/2} A_1^2}{4(m\omega)^{3/2}}\{\sqrt{\pi}\, 2^2\, 2! + 4\sqrt{\pi}\}$

$= \frac{12\,\hbar^{3/2} A_1^2}{4(m\omega)^{3/2}}\sqrt{\pi}$

$= 3\left(\frac{\hbar}{m\omega}\right)^{3/2}\left(\frac{m\omega}{\pi\hbar\, 4}\right)^{1/2}\sqrt{\pi}$

$= 3\hbar / 2m\omega$

Similarly with $p^2 \to \hbar^2 \frac{d^2}{dx^2}$ we would have $\langle 1|p^2|1\rangle = \frac{3}{2} m\omega\hbar$.

$\psi_0(x) = A_0 e^{-\frac{m\omega x^2}{2\hbar}} H_0\left(\sqrt{\frac{m\omega}{\hbar}}\, x\right)$

$= A_0 e^{-y^2/2} H_0(y)$

$H_0(y) = 1$

$A_0 = \left[\frac{m\omega}{\pi\hbar}\right]^{1/4}$

or $\psi_0(y) = \left[\frac{m\omega}{\pi\hbar}\right]^{1/4} e^{-y^2/2}$

$(\Delta x)^2 = \langle 0|(X - \langle x\rangle)^2|0\rangle^{1/2}$

$= \langle 0|x^2|0\rangle$

Since $\langle n|x|n\rangle = \langle x\rangle = 0$.

In the X basis with

$x = \left(\frac{\hbar}{m\omega}\right)^{1/2} y$

~~$(\Delta x)^2 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\frac{\hbar}{m\omega}\int_{-\infty}^{\infty} e^{-y^2} H_1^2\, dy$~~

[illegible]

[illegible]

$(\Delta x)^2 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\left(\frac{\hbar}{m\omega}\right)^{3/2}\int_{-\infty}^{\infty} y^2 e^{-y^2} dy$

$= \frac{2}{\sqrt{\pi}}\left(\frac{\hbar}{m\omega}\right)\int_0^{\infty} y^2 e^{-y^2} dy$

$= \frac{1}{\sqrt{\pi}}\left(\frac{\hbar}{m\omega}\right)\Gamma(3/2)$

$= \frac{1}{\sqrt{\pi}}\left(\frac{\hbar}{m\omega}\right)\frac{\sqrt{\pi}}{2} = \frac{\hbar}{2m\omega}$

or $\Delta x = \left(\frac{\hbar}{2m\omega}\right)^{1/2}$

$(\Delta p)^2 = \langle 0|(p - \langle p\rangle)^2|0\rangle$

$= \langle 0|p^2|0\rangle$

$= \int_{-\infty}^{\infty} \psi_0(x)\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\right)\psi_0(x)dx$

$x = \left(\frac{\hbar}{m\omega}\right)^{1/2} y$

$\frac{d^2}{dx^2} = \frac{m\omega}{\hbar}\frac{d^2}{dy^2}$

$\frac{d^2\psi_0(y)}{dy^2} = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left(y^2 e^{-y^2/2} - e^{-y^2/2}\right)$

$(\Delta p)^2 = -\hbar^2\left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\left(\frac{m\omega}{\hbar}\right)\left(\frac{\hbar}{m\omega}\right)^{1/2}\left\{\int_{-\infty}^{\infty} y^2 e^{-y^2} dy - \int_{-\infty}^{\infty} e^{-y^2} dy\right\}$

$= -\frac{2\hbar^2}{\sqrt{\pi}}\left(\frac{m\omega}{\hbar}\right)\left\{\int_0^{\infty} y^2 e^{-y^2} dy - \int_0^{\infty} e^{-y^2} dy\right\}$

$= -\frac{\hbar^2}{\sqrt{\pi}}\left(\frac{m\omega}{\hbar}\right)[\Gamma(3/2) - \Gamma(1/2)]$

$= -\frac{1}{\sqrt{\pi}}(\hbar m\omega)\left(\frac{\sqrt{\pi}}{2} - \sqrt{\pi}\right)$

$= \frac{\hbar m\omega}{2}$

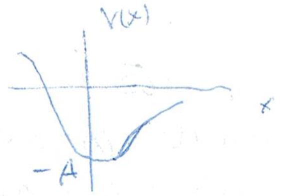
3. Determine the energy levels for a particle moving in a field of potential energy

$$V(x) = A(e^{-2ax} - 2e^{-ax})$$

Solution

The spectrum of positive eigenvalues of the energy is continuous (and the levels are not degenerate), while the spectrum of negative eigen-values is discrete.

~~Schrö~~



Schrödinger's eq reads

$$d^2\psi/dx^2 + (2\mu/\hbar^2)(E - Ae^{-2ax} + 2Ae^{-ax})\psi = 0$$

We introduce a new variable

$$y = \frac{2\sqrt{2\mu A}}{a\hbar} e^{-ax}$$

(taking values from 0 to ∞) and the notation (we consider the discrete spectrum, so that $E<0$)

$$s = \sqrt{-2\mu E}/a\hbar, \quad n = \frac{\sqrt{2\mu A}}{a\hbar} - (s + \tfrac{1}{2}) \quad \text{---(1)}$$

Schrödinger's eq. then takes the form

$$\psi'' + \tfrac{1}{y}\psi' + \left(-\tfrac{1}{4} + \frac{n+s+\tfrac{1}{2}}{y} - \tfrac{s^2}{y^2}\right)\psi = 0$$

As $y \to \infty$, the function ψ behaves asymptotically as $e^{\pm\frac{1}{2}y}$, while as $y \to 0$ it is proportional to $y^{\pm s}$. From considerations of finiteness we must choose the solution which behaves as $e^{-\frac{1}{2}y}$ as $y \to \infty$ and as y^s as $y \to 0$. We make the substitution

$$\psi = e^{-\frac{1}{2}y} y^s w(y)$$

and obtain for w the eq.

$$y w'' + (2s+1-y) w' + n w = 0 \quad \text{--}(2)$$

Which have to be solved with the conditions that w is finite as $y \to 0$, while as $y \to \infty$, w tends to infinity not more rapidly than every finite power of y. Eq. (2) is the eq. for a confluent hypergeometric function.

$$w = F(-n, 2s+1, y)$$

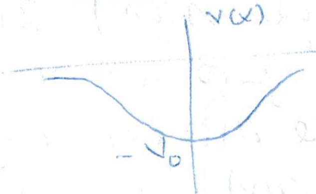
A soln satisfying the required cond. is obtained for non negative integral n (when the function F reduces to a polynomial) According to the def. (1) we thus obtain for the energy levels the values

$$-E_n = A\left[1 - \frac{a\hbar}{\sqrt{2\mu A}}(n+\tfrac{1}{2})\right]^2$$

Where n takes positive integral values from zero to the greatest value for which $\sqrt{2\mu A}/a\hbar > n+\frac{1}{2}$ (so that the parameter s is positive in accordance with its def.). Thus the discrete spectrum contains only a limited no. of levels. If $\frac{\sqrt{2\mu A}}{a\hbar} < \frac{1}{2}$ there is no discrete spectrum at all.

10. The same as prob. 9 but with $V = -V_0/\cosh^2 ax$

Solution



The spectrum of positive eigenvalue of the energy is continuous while that of negative values is discrete; we shall consider the latter. Schrödingers eq. is

$$\frac{d^2\psi}{dx^2} + \frac{2\mu}{\hbar^2}\left(E + \frac{V_0}{\cosh^2 ax}\right)\psi = 0$$

We make the substitution

$$\psi = \frac{w}{\cosh^s ax}, \quad s = \tfrac{1}{2}\left(-1 + \sqrt{1 + \frac{8\mu V_0}{a^2\hbar^2}}\right) \quad \text{--- 1}$$

Obtaining

$$\frac{d^2w}{dx^2} - 2sa\tanh(ax)\frac{dw}{dx} + \left(a^2s^2 + \frac{2\mu E}{\hbar^2}\right)w = 0$$

(s is chosen ~~to be~~ so that the coeff. of w is const.)

This eq. can be reduced to a hypergeometric one by introducing the new variable — 2

$$y = \sinh^2 ax$$

Introducing also the const $\epsilon = \sqrt{-2\mu E}/a\hbar$, we obtain

$$y(1+y)w'' + [(1-s)y + \tfrac{1}{2}]w' + \tfrac{1}{4}(s^2 - \epsilon^2)w = 0 \quad \text{--- (3)}$$

Since the p.E. U(y) is an even function of the coordinate, the wave functions of the stationary states must be either even or odd functions. Since cosh ax is an even function, the parity of ψ is the same as that of w(x). The even and odd (in x) particular integrals of eq. (3) are

$$w_1 = F\left(-\tfrac{1}{2}s + \tfrac{1}{2}\epsilon, -\tfrac{1}{2}s - \tfrac{1}{2}\epsilon, \tfrac{1}{2}, -y\right)$$

$$w_2 = \sqrt{y}\, F\left(-\tfrac{1}{2}s + \tfrac{1}{2}\epsilon + \tfrac{1}{2}, -\tfrac{1}{2}s - \tfrac{1}{2}\epsilon + \tfrac{1}{2}, \tfrac{3}{2}, -y\right)$$

(When x changes sign, y remains the same, while $\sqrt{y} = \sinh ax$ changes sign). In order that $\psi = (1+y)^{-\frac{1}{2}s}\omega$ should reduce to zero as $y \to \infty$, the parameter $\frac{1}{2}\epsilon - \frac{1}{2}s$ must be a negative integer or zero; then F is a polynomial of degree $\frac{1}{2}s - \frac{1}{2}\epsilon$ and ψ tends to zero as $y^{-\frac{1}{2}\epsilon}$ as $y \to \infty$. Similarly, for $\psi = (1+y)^{-\frac{1}{2}s}\omega_2$, this cond. is satisfied if $-\frac{1}{2}s + \frac{1}{2}\epsilon + \frac{1}{2}$ is a negative integer.

Thus the energy levels are determined by $s - \epsilon = n$, or

$$E = -\frac{\hbar^2 a^2}{8\mu}\left[-(1+2n) + \sqrt{1 + \frac{8\mu V_0}{a^2\hbar^2}}\right]^2$$

where n takes positive integral values starting from zero. There is a finite no. of levels, determined by the cond. $\epsilon > 0$, i.e.,

$$2n < \sqrt{1 + [8\mu V_0 / a^2\hbar^2]} - 1$$

1. Check that the familiar scalar product in $V^3(R)$ qualifies as an inner product obeying all the axioms (i) – (iii).

Solution

The inner product of v_i and v_j, denoted by $\langle v_i|v_j\rangle$, is a no. satisfying the following axioms.

(i). $\langle v_i|v_i\rangle \geq 0$ (0 only if $\underline{v_i} = 0$).

(ii) $\langle v_i|v_j\rangle = \langle v_j|v_i\rangle^*$

(iii) $\langle v_i|\alpha v_j + \beta v_k\rangle = \alpha\langle v_i|v_j\rangle + \beta\langle v_i|v_k\rangle$

Consider then two vectors $\underline{v_i}$ & v_j in $V_3(R)$

$\langle v_i|v_j\rangle =$ (length $\underline{v_i}$) · (length of $\underline{v_j}$) · $\cos\theta_{ij}$ ...... (1)

where θ_{ij} is the angle between $\underline{v_i}$ and $\underline{v_j}$

It follows $\langle v_i|v_i\rangle =$ (length of $\underline{v_i}$)$^2 \cos\theta$, $\theta_{ij} = 0$

$=$ (length of $\underline{v_i}$)^2

$= 0$, if only $\underline{v_i} = 0$

i.e., axiom one is satisfied

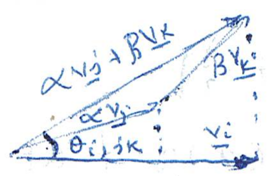
As seen from (1) dot product is real whence

$\langle v_i|v_j\rangle = \langle v_j|v_i\rangle^*$

∴ axiom (ii) is satisfied

$\langle v_i|\alpha v_j + \beta v_k\rangle = |v_i|\,|\alpha v_j + \beta v_k|\cos\theta_{i,jk}$

Consider the figure below.



Clearly,

$|v_i|\,|\alpha v_j + \beta v_k|\cos\theta_{i,jk} = \alpha|v_i|\,|v_j|\cos\theta_{ij} + \beta|v_i|\,|v_k|\cos\theta_{ik}$

i.e., the projection of $\alpha v_j + \beta v_k$ on $\underline{v_i}$ is $\alpha|v_j|\cos\theta_{ij} + \beta|v_k|\cos\theta_{ik}$

$|\alpha v_j + \beta v_k|\cos\theta_{i,jk} = \alpha|v_j|\cos\theta_{ij} + \beta|v_k|\cos\theta_{ik}$

∴ axiom (3) is satisfied and the dot product in $V^3(R)$ qualifies as an inner product.

2. By analyzing the proof of theorem 4. show that the inequality becomes equality if $\underline{v_i} = \lambda \underline{v_j}$, where λ is a real positive scalar.

Solution

The theorem is proved under the conditions

(i) $Re \langle v_i | v_j \rangle \leq |\langle v_i | v_j \rangle|$

(ii) $|\langle v_i | v_j \rangle| \leq |v_i||v_j|$

If for $\underline{v_i} = \lambda \underline{v_j}$, these two inequalitees change to equality, then, obviously, $|v_i + v_j| = |v_i| + |v_j|$

- $|\langle v_i | v_j \rangle| = |\langle \lambda v_j | v_j \rangle| = |\lambda^* \langle v_j | v_j \rangle|$, by axiom (iii)

$= |\lambda^*| |v_j|^2$

$= \lambda |v_j|^2$, by our assumption

$Re \langle v_i | v_j \rangle = Re \langle \lambda v_j | v_j \rangle = Re \{ \lambda^* \langle v_j | v_j \rangle \}$, axiom (iii)

$= Re (\lambda^* |v_j|^2)$

$= \lambda |v_j|^2$, by the assumption.

It follows,

$$Re \langle v_i | v_j \rangle = |\langle v_i | v_j \rangle| \quad \text{--- (1)}$$

- we have that $|\langle v_i | v_j \rangle| = \lambda |v_j|^2$, for $\underline{v_i} = \lambda \underline{v_j}$

$|v_i| |v_j| = |\lambda v_j| |v_j|$

$= |\lambda| |v_j| |v_j|$

$= |\lambda| |v_j|^2$

$= \lambda |v_j|^2$, by the assumption that λ is real and positive.

It follows $|\langle v_i | v_j \rangle| = |v_i| |v_j|$, for $\underline{v_i} = \lambda v_j$ --- (2)

so now from (1) and (2) we conclude that

$$|v_i + v_j| = |v_i| + |v_j|$$

where $\underline{v_i} = \lambda \underline{v_j}$ in the case that λ is real and positive.

3. Now where in the proof of the Gram-Schmidt Theorem did we refer to the LI of the set $\{ \underline{v_1}, \dots, \underline{v_n} \}$. Why was it stipulated in the statement of the theorem?

It was stipulated in the theorem, because the ~~orthogonal vectors, constructed were~~ we constructed orthogonal vectors from this set of vectors only because these vectors are LI even if it was not mentioned in the proof.

From the proof of the theorem we have that

$$|m'\rangle = |V_m\rangle - \sum_{i'=1}^{m-1} \frac{|i'\rangle\langle i'|V_m\rangle}{\langle i'|i'\rangle}$$

If we assume that the set $\{|1'\rangle, |2'\rangle, \dots |m'\rangle\}$ contains the null vector then, it is true that for one of the vectors out of this set, say, $|m'\rangle = 0$.

$$\Rightarrow |V_m\rangle - \sum_{i'=1}^{m-1} |i'\rangle \frac{\langle i'|V_m\rangle}{\langle i'|i'\rangle} = 0$$

$$\text{or } |V_m\rangle = \sum_{i=1}^{m-1} \frac{|i'\rangle\langle i'|V_m\rangle}{\langle i'|i'\rangle}$$

i.e., $|V_m\rangle$ is expressed as in terms of $|V_1\rangle, \dots |V_{m-1}\rangle$.

4. Show that the product of unitary operators is also unitary.

An operator is untary if

$$U U^{+} = I$$

Suppose then U_1 and U_2 are two unitary operators

To show U_1U_2 is unitary.

$$(U_1U_2)(U_1U_2)^{+}$$
$$= (U_1U_2)(U_2^{+}U_1^{+})$$
$$= U_1U_2U_2^{+}U_1^{+}$$
$$= U_1 I U_1^{+}$$
$$= U_1U_1^{+}$$
$$= I$$

i.e., U_1U_2 is unitary.

5. Show that

a, $Tr(\Omega\Lambda) = Tr(\Lambda\Omega)$

b, $Tr(\Omega\Lambda\theta) = Tr(\Lambda\theta\Omega) = Tr(\theta\Omega\Lambda)$

(cyclic permutation

c, The trace of an operator is unaffected by a unitary change of basis. $|i\rangle \to U|i\rangle$.

[Equivalentey, show $Tr\,\Omega = Tr(U^{+}\Omega U)$.]

a, Let $\Gamma = \Omega\Lambda$

$$Tr(\Omega\Lambda) = \sum_i \Gamma_{ii}$$

But from the def. of matrix product

$$\Gamma_{ij} = \sum_k \Omega_{ik} \Lambda_{kj}$$

whence

$$\Gamma_{ii} = \sum_k \Omega_{ik} \Lambda_{ki}$$

$$Tr(\Omega\Lambda) = \sum_i \Gamma_{ii} = \sum_i \sum_k \Omega_{ik} \Lambda_{ki}$$

$$= \sum_i \sum_k \Lambda_{ki} \Omega_{ik}$$

$$= \sum_k \sum_i \Lambda_{ki} \Omega_{ik}$$

$$= Tr(\Lambda\Omega)$$

b, (i) $Tr(\Omega\Lambda\theta) = Tr[\Omega(\Lambda\theta)]$, matrix product is associative

$= Tr[(\Lambda\theta)\Omega]$, by (a) treating $\Lambda\theta$ as one operator

$= Tr(\Lambda\theta\Omega)$

(ii) $Tr(\Omega\Lambda\theta) = Tr[(\Omega\Lambda)\theta]$

$= Tr[\theta(\Omega\Lambda)]$, by (a)

$= Tr(\theta\Omega\Lambda)$

$\therefore$ $Tr(\Omega\Lambda\theta) = Tr(\Lambda\theta\Omega) = Tr(\theta\Omega\lambda)$

c, if $|i\rangle \to U|i\rangle$, then $\langle i|\Omega|i\rangle$

$\to \langle Ui|\Omega|Ui\rangle$

$= \langle i|U^{+}\Omega U|i\rangle$

(i.e.) $\Omega \to U^{+}\Omega U$

$Tr(U^{+}\Omega U) = Tr(\Omega U U^{+})$, by (b)

$= Tr(\Omega I)$

$= Tr\,\Omega$, (thus the trace is unaffected by a unitary change of the basis.

Also $Tr\,\Omega = Tr(I\Omega) = Tr(UU^{+}\Omega)$

$= Tr(U^{+}\Omega U)$, by (b)

Q.E.D.

1. a, Let A be a Hermitian operator

$$\langle A^2 \rangle = \langle \psi | A^2 | \psi \rangle$$
$$= \langle \psi | A^{+} A | \psi \rangle$$
$$= \langle A\psi | A\psi \rangle \geq 0$$

$\therefore \langle A^2 \rangle$ is positive unless $|A\psi\rangle$ is a zero vector.

b, (i) For linear operator Ω

$$\Omega \{ \alpha |V\rangle + \beta |V'\rangle \}$$
$$= \alpha \Omega |V\rangle + \beta \Omega |V'\rangle \quad --(1)$$

If the operator takes the complex conjugate

$$\Omega \{ \alpha |V\rangle + \beta |V'\rangle \}$$
$$= \alpha^* \Omega |V\rangle + \beta^* \Omega |V'\rangle \quad --(2)$$

Compare (1) and (2). Since, in general, α and β are complex $\alpha^* \neq \alpha$, $\beta^* \neq \beta$.
$\therefore$ taking the complex conjugate does not correspond to a linear operator.

(ii) Consider $\langle V | \Omega | V' \rangle$

a, $\langle V | \Omega | V' \rangle$ = ~~[illegible]~~ If Ω is Hermitian.
~~[illegible]~~
~~[illegible]~~
~~[illegible]~~

~~[illegible]~~ $= \langle V | \Omega V' \rangle$
$= \langle V | V'^* \rangle$
$= \sum_i v_i^* v_i'^*$

b, $\langle V | \Omega | V' \rangle = \langle V | \Omega^+ | V' \rangle$. If Ω is Hermitian.
$= \langle \Omega V | V' \rangle$
$= \langle V^* | V' \rangle$
$= \sum_i v_i v_i'$

In general the results in (a) and (b) are diff.
$\therefore$ Taking a complex conjugate does not correspond to a Hermitian operator.

c, $\Omega |V\rangle = |V^*\rangle$

$$\Omega^* |V^*\rangle = |\Omega^* V^*\rangle$$
$$= |(\Omega V)^*\rangle$$
$$= |(V^*)^*\rangle$$
$$= |V\rangle$$

Take $|V'\rangle = |V^*\rangle$

$$\Omega^* |V'\rangle = |V'^*\rangle = \Omega |V'\rangle$$

$$\Rightarrow \Omega^* = \Omega$$

$\therefore$ Taking complex conjugate corresponds to an operator which is its own complex conjugate.
Not necessarily.

2. Yes. Consider a particle (such as the electron in the hydrogen atom which is in central force field.) We know for the angular mom. comp. operators

$$[L_x, L_y] = i\hbar L_z$$
$$[L_y, L_z] = i\hbar L_x$$
$$[L_z, L_x] = i\hbar L_y$$

i.e., they do not commute. But the state $|\psi\rangle = |0\rangle$ is an eigen state of all the three operators with eigenvalue 0. That means in this state all of them could have a well defined value.

(ii) Here you may consider the square of the angular Mom. operator and any of the L_is $(i = x, y, z)$. they commute.

~~So an eigenstate ...~~

If ψ_A is an eigenstate of L^2 it can be of L_x, L_y or L_z so here L^2 and the L_is have well defined values. But not the converse, in general.

3. Let T be the transformation matrix. Now in the new coordinate system

$$H \rightarrow T^{-1} H T$$

If H is invariant

$$H = T^{-1} H T$$

$$[T, H]$$

$$= TH - HT$$

$$= T T^{-1} H T - HT$$

$$= I H T - HT$$

$$= HT - HT$$

$$= 0$$

$$[T, H] = 0$$

and hence the transformation matrix commute with H.

4. a, To show $e^{A} e^{B} = e^{A+B+\frac{1}{2}[A,B]}$

where $[A,[A,B]] = [B,[A,B]] = 0$

Consider a function

$$f(\lambda) = e^{\lambda A} e^{\lambda B}, \quad \lambda\text{- param.}$$

$$\frac{df}{d\lambda} = A e^{\lambda A} e^{\lambda B} + e^{\lambda A} B e^{\lambda B}$$

$$= A e^{\lambda A} e^{\lambda B} + e^{\lambda A} B I e^{\lambda B}$$

Let the identity be $I = e^{\lambda A} e^{-\lambda A}$

$$\frac{df}{d\lambda} = A e^{\lambda A} e^{\lambda B} + e^{\lambda A} B e^{-\lambda A} e^{\lambda A} e^{\lambda B}$$

$$= A f(\lambda) + e^{\lambda A} B e^{-\lambda A} f(A)$$

~~Here we expand $B e^{-\lambda A}$ into Maclaurin series.~~

Consider the following Commutation

$$[B, e^{-\lambda A}] = [B, \sum_n \frac{(-\lambda A)^n}{n!}]$$

By making a series expansion.

$$[B, \sum_n \frac{(-\lambda A)^n}{n!}]$$

$$= [B, I - \lambda A + \frac{\lambda^2 A^2}{2!} - \frac{\lambda^3 A^3}{3!} + \dots]$$

$$= [B, I] - \lambda [B, \lambda A] + [B, \frac{\lambda^2 A^2}{2!}] - [B, \frac{\lambda^3 A^3}{3!}] + \dots$$

$$= 0 - \lambda [B, A] + \frac{\lambda^2}{2!} [B, A^2] - \frac{\lambda^3}{3!} [B, A^3] + \dots$$

$$= -\lambda [B, A] + \frac{\lambda^2}{2!} \{ A[B,A] + [B,A]A \} - \frac{\lambda^3}{3!} \{ [B, A^2] A + A^2 [B, A] \} + \dots$$

$$= -\lambda [B, A] + \frac{\lambda^2}{2!} \{ 2A [B, A] \} - \frac{\lambda^3}{3!} \{ A[B,A]A + [B,A]A^2 + A^2[B,A] \} + \dots$$

$$= -\lambda [B, A] + \lambda^2 A [B, A] - \frac{\lambda^3}{3!} \{ A^2 [B, A] + A^2 [B, A] + A^2 [B, A] \} + \dots$$

$$= -\lambda [B, A] + \lambda^2 A [B, A] - \frac{\lambda^3}{3} \{ 3A^2 [B, A] \} + \dots$$

$$= -\lambda [B, A] + \lambda^2 A [B, A] - \frac{\lambda^3}{2!} A^2 [B, A] + \dots$$

In general ~~$[A^n, B] \neq n A^{n-1}$~~

$$[A^n, B] = nA^{n-1}[A,B]$$

$$[A, B^n] = nB^{n-1}[A,B]$$

$$[B, e^{-\lambda A}] = -\lambda \{ [B,A] - \lambda A [B,A] + \frac{\lambda^2}{2!} A^2 [B,A] + \cdots \}$$

$$= -\lambda e^{-\lambda A} [B,A]$$

$$= \lambda e^{-\lambda A} [A,B]$$

$$B e^{-\lambda A} - e^{-\lambda A} B = \lambda e^{-\lambda A} [A,B]$$

$$B e^{-\lambda A} = \lambda e^{-\lambda A} [A,B] + e^{-\lambda A} B$$

$$\Rightarrow \frac{df}{d\lambda} = A f(\lambda) + \lambda e^{\lambda A} e^{-\lambda A} [A,B] f(\lambda) + e^{\lambda A} e^{-\lambda A} B f(\lambda)$$

$$= A f(\lambda) + \lambda [A,B] f(\lambda) + B f(\lambda)$$

$$= \{A + B + \lambda [A,B]\} f(\lambda)$$

$$\frac{df}{f} = \{A + B + \lambda [A,B]\} d\lambda$$

$$\ln f = (A+B)\lambda + \frac{\lambda^2}{2}[A,B] + C$$

$$f(\lambda) = D e^{(A+B)\lambda + \frac{\lambda^2}{2}[A,B]}$$

Let $D = I$, the identity.

and

$$f(\lambda) = e^{\lambda A} e^{\lambda B} = e^{(A+B)\lambda + \frac{\lambda^2}{2}[A,B]}$$

Let $\lambda = 1$

$$\therefore e^{A} e^{B} = e^{A+B+\frac{1}{2}[A,B]}.$$

b) Let $f(\lambda) = e^{\lambda A} B e^{-\lambda A}$

Applying Taylor's series

$$f(\lambda) = f(0) + \lambda f'(0) + \frac{\lambda^2}{2!} f''(0) + \cdots$$

$$f(0) = B$$

$$f'(\lambda) = A e^{\lambda A} B e^{-\lambda A} - e^{\lambda A} B e^{-\lambda A} A$$

$$= A f(\lambda) - f(\lambda) B$$

At zero $= [A, f(\lambda)]$

~~$f'(0) = AB - BA = [A,B]$~~

~~$f''(\lambda) = A f'(\lambda) - f'(\lambda) B$~~

~~$f''(0) = A f'(0) - f'(0) B$~~

$$f'(0) = [A, f(0)]$$

$$= [A, B]$$

$$f''(\lambda) = [A, f'(\lambda)]$$

$$f''(0) = [A, f'(0)]$$

$$= [A, [A,B]]$$

$$f'''(\lambda) = [A, f''(\lambda)]$$

$$f'''(0) = [A, f''(0)]$$

$$= [A, [A, [A,B]]]$$

And so on.

$$\therefore f(\lambda) = e^{\lambda A} B e^{-\lambda A} = B + \lambda [A,B] + \frac{\lambda^2}{2!}[A,[A,B]] + \frac{\lambda^3}{3!}[A,[A,[A,B]]] + \cdots$$

Choose $\lambda = 1$

$$e^{A} B e^{-A} = B + [A,B] + \frac{1}{2!}[A,[A,B]] + \frac{1}{3!}[A,[A,[A,B]]] + \dots$$

C. Let $[A - B\lambda]^{-1} = \sum_k a_k \lambda^k$

$$[A-B\lambda]^{-1}[A-B\lambda] = \sum_k a_k \lambda^k [A-B\lambda]$$

$$I = \sum_k a_k \lambda^k [A-B\lambda] a_k$$

Equate the coeff. of each power of λ. To the left the power of λ is zero.

$$\Rightarrow \quad I = a_0 A$$

$$I A^{-1} = a_0 A A^{-1}$$

$$A^{-1} = a_0$$

Since to the left we do not have a power > 0 of λ

- First power

$$0 = \lambda A a_1 - \lambda^2 B a_1$$

~~Each coeff~~ so the 2nd term on the right should have a coeft. a_0 (i.e.)

$$0 = A a_1 - B a_0$$

$$A a_1 = B a_0$$

$$A^{-1} A a_1 = A^{-1} B a_0$$

$$a_1 = A^{-1} B A^{-1}$$

- 2nd power

$$0 = a_2 A - B a_1$$

$$A a_2 = B a_1$$

$$A^{-1} A a_2 = A^{-1} B a_1$$

$$a_2 = A^{-1} B A^{-1} B A^{-1}$$

- 3rd power

$$0 = A a_3 - B a_2$$

$$A a_3 = B a_2$$

$$A^{-1} A a_3 = A^{-1} B a_2$$

$$a_3 = A^{-1} B A^{-1} B A^{-1} B A^{-1}$$

In general

$$A a_n = B a_{n-1}$$

$$a_n = A^{-1} B a_{n-1}$$

$$\therefore (A - \lambda B)^{-1} = \sum_{k=0} a_k \lambda^k$$

$$= A^{-1} + A^{-1} B A^{-1} \lambda + A^{-1} B A^{-1} B A^{-1} \lambda^2 + A^{-1} B A^{-1} B A^{-1} B A^{-1} \lambda^3 + \dots$$

5.

$$\bar{v} = \frac{1}{m}\left(\bar{p} - \frac{e\bar{A}}{c}\right)$$

$$A = \tfrac{1}{2}\bar{B}\times\bar{r}$$

$$v_x = \frac{1}{m}\left(p_x - \frac{eA_x}{c}\right)$$

$$v_y = \frac{1}{m}\left(p_y - \frac{eA_y}{c}\right)$$

$$v_z = \frac{1}{m}\left(p_z - \frac{eA_z}{c}\right)$$

$$A_x = \tfrac{1}{2}(\bar{B}\times\bar{r})_x = \tfrac{1}{2}(B_y z - B_z y)$$

$$A_y = \tfrac{1}{2}(\bar{B}\times\bar{r})_y = \tfrac{1}{2}(B_z x - B_x z)$$

$$A_z = \tfrac{1}{2}(\bar{B}\times\bar{r})_z = \tfrac{1}{2}(B_x y - x B_y)$$

To find $[v_x, v_y]$

Let $f = f(x,y,z)$ and in the x,y,z basis

$$v_x = \frac{1}{m}\left[-i\hbar\frac{\partial}{\partial x} - \frac{e}{2c}(B_y z - B_z y)\right]$$

$$v_y = \frac{1}{m}\left[-i\hbar\frac{\partial}{\partial y} - \frac{e}{2c}(B_z x - B_x z)\right]$$

$$v_z = \frac{1}{m}\left[-i\hbar\frac{\partial}{\partial z} - \frac{e}{2c}(B_x y - x B_y)\right]$$

$$[v_x, v_y]f$$

$$= v_x v_y f - v_y v_x f$$

(i) $v_y f = \frac{1}{m}\left[-i\hbar\frac{\partial f}{\partial y} - \frac{e}{2c}(B_z x f - B_x z f)\right]$

$$v_x v_y f = \frac{1}{m^2}\Big[-\hbar^2\frac{\partial^2 f}{\partial x\partial y} + \frac{i\hbar e}{2c}\Big[xf\frac{\partial B_z}{\partial x} + B_z f + B_z x\frac{\partial f}{\partial x} - zf\frac{\partial B_x}{\partial x} - B_x z\frac{\partial f}{\partial x}\Big)$$

$$+ \frac{i\hbar e}{2c}(B_y z - B_z y)\frac{\partial f}{\partial y}$$

$$+ \frac{e^2}{4c^2}(B_y z - B_z y)(B_z x f - B_x z f)\Big]$$

(ii) $v_x f$

$$= \frac{1}{m}\left[-i\hbar\frac{\partial f}{\partial x} - \frac{e}{2c}(B_y z f - B_z y f)\right]$$

$$v_y v_x f$$

$$= \frac{1}{m^2}\Big[-\hbar^2\frac{\partial^2 f}{\partial y\partial x}$$

$$+ \frac{i\hbar e}{2c}\Big(zf\frac{\partial B_y}{\partial y} + zB_y\frac{\partial f}{\partial y} - yf\frac{\partial B_z}{\partial y} - B_z f - B_z y\frac{\partial f}{\partial y}\Big)$$

$$+ \frac{i\hbar e}{2c}(B_z x - B_x z)\frac{\partial f}{\partial x}$$

$$+ \frac{e^2}{4c^2}(B_z x - B_x z)(B_y z f - B_z y f)\Big]$$

$$\nabla\cdot B = 0 \qquad \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0$$

$$[v_x, v_y]f$$

$$= \frac{1}{m^2}\Big[\frac{i\hbar e}{2c}\Big(xf\frac{\partial B_z}{\partial x} - zf\frac{\partial B_x}{\partial x} + 2B_z f - zf\frac{\partial B_y}{\partial y} + yf\frac{\partial B_z}{\partial y}\Big]$$

$$x\frac{\partial B_z}{\partial x} + z\frac{\partial B_z}{\partial z} + y\frac{\partial B_z}{\partial y} = (\bar{r}\cdot\nabla)B_z$$

$$\frac{i\hbar f}{2m^2 c}\Big(x\frac{\partial B_z}{\partial x} - z\frac{\partial B_x}{\partial x} - z\frac{\partial B_y}{\partial y} + y\frac{\partial B_z}{\partial y}$$

For a given field the derivatives vanish and $[v_x, v_y]f$

$$= \frac{i\hbar e}{2m^2 c}2B_z f$$

$$= \frac{i\hbar e}{m^2 c}B_z f$$

or $[v_x, v_y] = \frac{i\hbar e}{m^2 c}B_z$

Phys. 642

Problems and Solutions

1. Consider two angular momenta with $j_1 = j_2 = 1$. Obtain the complete set of coupled representation states in terms of the uncoupled states. Hence write down the values of all the C.G. coeff. relevant to this prob. (State the phase convention.)

Solution

The coupled states in terms of the uncoupled states are written as follows:

$$|jm\rangle = \sum_{m_1, m_2} |j_1 m_1 j_2 m_2\rangle \cdot \langle j_1 m_1 j_2 m_2 | jm\rangle$$

Since $m = m_1 + m_2$

$$|jm\rangle = \sum_{m_1} |j_1 m_1 j_2\, m - m_1\rangle \cdot \langle j_1 m_1 j_2\, m - m_1 | jm\rangle$$

With $j_1 = j_2 = 1$, the possible values of j are $2, 1, 0$.

1. $j = 2$: $m = 2, 1, 0, -1, -2$.

$$\Rightarrow |2m\rangle = \sum_{m_1=-1}^{1} |1 m_1 1\, m - m_1\rangle \langle 1 m_1 1\, m - m_1 | 2m\rangle$$

– $|22\rangle = |1111\rangle\langle 1111|22\rangle$

only $m_1 = 1$ contributes since $m_2 \neq (1, 0, -1)$ in the other cases.

By the Condon-Shortley phase convention the coeff. of the top state $|1111\rangle$ is chosen to be positive (+1).

i.e.) $\langle 1111|22\rangle = 1$

$\Rightarrow |22\rangle = |1111\rangle$

– $|21\rangle = |1011\rangle\langle 1011|21\rangle + |1110\rangle\langle 1110|21\rangle$

$$J_-|22\rangle = \sqrt{(2+2)(2-2+1)}\,|21\rangle = 2|21\rangle$$

$$|21\rangle = \frac{1}{2} J_-|22\rangle$$

$$= \frac{1}{2}(J_{1-} + J_{2-})|1111\rangle$$

$$= \frac{\sqrt{2}}{2}|1011\rangle + \frac{\sqrt{2}}{2}|1110\rangle$$

$$= \frac{1}{\sqrt{2}}(|1011\rangle + |1110\rangle)$$

$\Rightarrow \langle 1011|21\rangle = \frac{1}{\sqrt{2}}$, $\langle 1110|21\rangle = \frac{1}{\sqrt{2}}$ which are the C.G. coefficients.

– $|20\rangle = |111-1\rangle\langle 111-1|20\rangle + |1010\rangle\langle 1010|20\rangle + |1-111\rangle\langle 1-111|20\rangle$

$$J_-|21\rangle = \sqrt{6}\,|20\rangle$$

$$|20\rangle = \frac{1}{\sqrt{6}} J_-|21\rangle$$

$$= \frac{1}{\sqrt{6}}\frac{1}{\sqrt{2}}(J_{1-} + J_{2-})(|1011\rangle + |1110\rangle)$$

$$= \frac{1}{\sqrt{12}}(\sqrt{2}|1-111\rangle + \sqrt{2}|1010\rangle + \sqrt{2}|1010\rangle + \sqrt{2}|111-1\rangle)$$

$$= \frac{1}{\sqrt{6}}|1-111\rangle + \sqrt{\frac{2}{3}}|1010\rangle + \frac{1}{\sqrt{6}}|111-1\rangle$$

$$\Rightarrow \langle 111-1|20\rangle = \frac{1}{\sqrt{6}}$$

$$\langle 1010|20\rangle = \sqrt{\frac{2}{3}}$$

$$\langle 1-111|20\rangle = \frac{1}{\sqrt{6}}$$

$- \ |2\,-1\rangle = |1\,0\,1\,-1\rangle\langle 1\,0\,1\,-1|2\,-1\rangle$

$+|1\,-1\,1\,0\rangle\langle 1\,-1\,1\,0|2\,-1\rangle$

$J_-|2\,0\rangle = \sqrt{6}\,|2\,-1\rangle$

$|2\,-1\rangle = \frac{1}{\sqrt{6}} J_-|2\,0\rangle$

$= \frac{1}{\sqrt{6}}(J_{1-} + J_{2-})$

$\left\{\frac{1}{\sqrt{6}}|1\,-1\,1\,1\rangle + \sqrt{\frac{2}{3}}|1\,0\,1\,0\rangle + \frac{1}{\sqrt{6}}|1\,1\,1\,-1\rangle\right\}$

$= \frac{\sqrt{2}}{3}|1\,-1\,1\,0\rangle + \frac{\sqrt{2}}{6}|1\,0\,1\,-1\rangle + \frac{\sqrt{2}}{6}|1\,-1\,1\,0\rangle + \frac{\sqrt{2}}{3}|1\,0\,1\,-1\rangle$

$= \frac{1}{\sqrt{2}}(|1\,-1\,1\,0\rangle + |1\,0\,1\,-1\rangle)$

$\Rightarrow \langle 1\,0\,1\,-1|2\,-1\rangle = \frac{1}{\sqrt{2}}$

$\langle 1\,-1\,1\,0|2\,-1\rangle = \frac{1}{\sqrt{2}}$

$- \ |2\,-2\rangle = |1\,-1\,1\,-1\rangle\langle 1\,-1\,1\,-1|2\,-2\rangle$

$J_-|2\,-1\rangle = 2\,|2\,-2\rangle$

$|2\,-2\rangle = \frac{1}{2} J_-|2\,-1\rangle$

$= \frac{1}{2}(J_{1-} + J_{2-})$

$\frac{1}{\sqrt{2}}\{|1\,-1\,1\,0\rangle + |1\,0\,1\,-1\rangle\}$

$= \frac{1}{2\sqrt{2}}(\sqrt{2}\,|1\,-1\,1\,-1\rangle + \sqrt{2}\,|1\,-1\,1\,-1\rangle)$

$= |1\,-1\,1\,-1\rangle$

$\Rightarrow \langle 1\,-1\,1\,-1|2\,-2\rangle = 1.$

2. $j = 1, \ m = 1, 0, -1$

$$|1\,m\rangle = \sum_{m_1=-1}^{1} |1\,m_1\,1\,m-m_1\rangle\langle 1\,m_1\,1\,m-m_1|1\,m\rangle$$

$- \ |1\,1\rangle = |1\,1\,1\,0\rangle\langle 1\,1\,1\,0|1\,1\rangle + |1\,0\,1\,1\rangle\langle 1\,0\,1\,1|1\,1\rangle$

$\langle 1\,1|1\,1\rangle = 1$

$\Rightarrow \langle 1\,1\,1\,0|1\,1\rangle^2 + \langle 1\,0\,1\,1|1\,1\rangle^2 = 1$

(since the C.G. coeffs are real)

$\langle 1\,1|2\,1\rangle = 0$

$\Rightarrow \langle 1\,1\,1\,0|1\,1\rangle + \langle 1\,0\,1\,1|1\,1\rangle = 0$

$\langle 1\,0\,1\,1|1\,1\rangle = -\langle 1\,1\,1\,0|1\,1\rangle$

i.e., $\langle 1\,1\,1\,0|1\,1\rangle^2 + \langle 1\,1\,1\,0|1\,1\rangle^2 = 1$

$2\langle 1\,1\,1\,0|1\,1\rangle^2 = 1$

$\langle 1\,1\,1\,0|1\,1\rangle = \frac{1}{\sqrt{2}}$

We have chosen the positive value in view of the Condon-Shortley phase convention in which the top state for each j should have a positive phase for $m_1 = j_1$.

$\langle 1\,0\,1\,1|1\,1\rangle = -\frac{1}{\sqrt{2}}$

$|1\,1\rangle = \frac{1}{\sqrt{2}}(|1\,1\,1\,0\rangle - |1\,0\,1\,1\rangle)$

$- \ |1\,0\rangle = |1\,1\,1\,-1\rangle\langle 1\,1\,1\,-1|1\,0\rangle + |1\,0\,1\,0\rangle\langle 1\,0\,1\,0|1\,0\rangle + |1\,-1\,1\,1\rangle\langle 1\,-1\,1\,1|1\,0\rangle$

$J_-|1\,1\rangle = \sqrt{2}\,|1\,0\rangle$

$|1\,0\rangle = \frac{1}{\sqrt{2}} J_-|1\,1\rangle$

$= \frac{1}{2}(J_{1-} + J_{2-})(|1\,1\,1\,0\rangle - |1\,0\,1\,1\rangle)$

$= \frac{1}{2}\left(\sqrt{2}\,|1010\rangle - \sqrt{2}\,|1-111\rangle\right.$

$\left. + \sqrt{2}\,|111-1\rangle - \sqrt{2}\,|1010\rangle\right)$

$= \frac{1}{\sqrt{2}}\left(|111-1\rangle - |1-111\rangle\right)$

$\Rightarrow \langle 111-1|10\rangle = \frac{1}{\sqrt{2}}$

$\langle 1010|10\rangle = 0$

$\langle 1-111|10\rangle = -1/\sqrt{2}$

— $|1-1\rangle = |101-1\rangle\langle 101-1|1-1\rangle$

$+ |1-110\rangle\langle 1-110|1-1\rangle$

$J_-|10\rangle = \sqrt{2}\,|1-1\rangle$

$|1-1\rangle = \frac{1}{\sqrt{2}} J_-|10\rangle$

$= \frac{1}{\sqrt{2}}(J_{1-} + J_{2-})$

$\frac{1}{\sqrt{2}}\left(|111-1\rangle - |1-111\rangle\right)$

$= \frac{1}{2}\left(\sqrt{2}\,|101-1\rangle\right.$

$\left. - \sqrt{2}\,|1-110\rangle\right)$

$= \frac{1}{\sqrt{2}}\left(|101-1\rangle - |1-110\rangle\right)$

$\Rightarrow \langle 101-1|1-1\rangle = \frac{1}{\sqrt{2}}$

$\langle 1-110|1-1\rangle = -1/\sqrt{2}$

— $j=0,\ m=0$

$|0m\rangle = \sum_{m_1=-1}^{1} |1m_1 1 m-m_1\rangle$

$\cdot \langle 1m_1 1 m-m_1|0m\rangle$

$\Rightarrow |00\rangle = |111-1\rangle\langle 111-1|00\rangle$

$+ |1010\rangle\langle 1010|00\rangle$

$+ |1-111\rangle\langle 1-111|00\rangle$

$\langle 00|00\rangle = 1$

$\Rightarrow \langle 111-1|00\rangle^2 + \langle 1010|00\rangle^2$

$+ \langle 1-111|00\rangle^2 = 1 \quad \cdots (*)$

$\langle 00|20\rangle = 0$

$\Rightarrow \frac{1}{\sqrt{6}}\langle 111-1|00\rangle$

$+ \sqrt{2/3}\,\langle 1010|00\rangle + \frac{1}{\sqrt{6}}\langle 1-111|00\rangle = 0$

$\langle 111-1|00\rangle + 2\langle 1010|00\rangle$

$+ \langle 1-111|00\rangle = 0 \quad \cdots (**)$

$\langle 00|10\rangle = 0$

$\Rightarrow \frac{1}{\sqrt{2}}\langle 111-1|00\rangle - \frac{1}{\sqrt{2}}\langle 1-111|00\rangle = 0$

or $\langle 1-111|00\rangle = \langle 111-1|00\rangle$

Using in $(**)$

$2\langle 111-1|00\rangle + 2\langle 1010|00\rangle = 0$

or $\langle 1010|00\rangle = -\langle 111-1|00\rangle$

Using in $(*)$

$\langle 111-1|00\rangle^2 + \langle 111-1|00\rangle^2$

$+ \langle 111-1|00\rangle^2 = 1$

$3\langle 111-1|00\rangle^2 = 1$

or $\langle 111-1|00\rangle = \frac{1}{\sqrt{3}}$

Where we have used the Condon-Shortley phase convention.

$\therefore |00\rangle = \frac{1}{\sqrt{3}}\left(|111-1\rangle - |1010\rangle\right.$

$\left. + |1-111\rangle\right)$

2. By expanding $e^{-i\theta M_y}$ in a power series and using

$$M_y = \frac{1}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

derive the eq.

$$d^{1/2}_{mm'}(\theta) = \begin{bmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix}$$

Solution

$$e^{-i\theta M_y} = I - i\theta M_y - \frac{\theta^2}{2!}M_y^2 + i\frac{\theta^3}{3!}M_y^3 + \frac{\theta^4}{4!}M_y^4 - i\frac{\theta^5}{5!}M_y^5 - \frac{\theta^6}{6!}M_y^6 + \dots$$

$$M_y = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$M_y^2 = \frac{1}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{2^2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$M_y^3 = \frac{1}{8}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \frac{1}{2^3}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$M_y^4 = \frac{1}{16}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{2^4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$M_y^5 = \frac{1}{32}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \frac{1}{2^5}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\Rightarrow e^{-i\theta M_y} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - i\frac{\theta}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - \frac{(\frac{\theta}{2})^2}{2!}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + i\frac{(\frac{\theta}{2})^3}{3!}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{(\frac{\theta}{2})^4}{4!}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - i\frac{(\frac{\theta}{2})^5}{5!}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \dots$$

$$= \begin{bmatrix} \left(1 - \frac{(\frac{\theta}{2})^2}{2!} + \frac{(\frac{\theta}{2})^4}{4!} + \dots\right) & -\frac{\theta}{2} + \frac{(\frac{\theta}{2})^3}{3!} - \frac{(\theta/2)^5}{5!} + \dots \\ \frac{\theta}{2} - \frac{(\theta/2)^3}{3!} + \frac{(\theta/2)^5}{5!} + \dots & 1 - \frac{(\frac{\theta}{2})^2}{2!} + \frac{(\theta/2)^4}{4!} + \dots \end{bmatrix}$$

or $$e^{-i\theta M_y} = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\theta/2 \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} = d^{1/2}_{mm'}(\theta)$$

3. Derive the equation

$$d^1_{mm'}(\theta) = \begin{pmatrix} \frac{1+\cos\theta}{2} & \frac{\sin\theta}{\sqrt{2}} & \frac{1-\cos\theta}{2} \\ -\frac{\sin\theta}{\sqrt{2}} & \cos\theta & \frac{\sin\theta}{\sqrt{2}} \\ \frac{1-\cos\theta}{2} & -\frac{\sin\theta}{\sqrt{2}} & \frac{1+\cos\theta}{2} \end{pmatrix}$$

Solution

Let $$M_y = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix}$$

$$e^{-i\theta M_y} = I - i\theta M_y - \frac{\theta^2}{2!}M_y^3 + i\frac{\theta^3}{3!}M_y^3 + \frac{\theta^4}{4!}M_y^4 - i\frac{\theta^5}{5!}M_y^5 - \frac{\theta^6}{6!}M_y^6 + \dots$$

$$M_y^2 = \frac{1}{2}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$M_y^3 = \frac{1}{2\sqrt{2}}\begin{bmatrix} 0 & -2i & 0 \\ 2i & 0 & -2i \\ 0 & 2i & 0 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix}$$

$$M_y^4 = \frac{1}{2}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$M_y^5 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix}$$

$$M_y^6 = \frac{1}{2}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow e^{-i\theta M_y} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{i\theta}{\sqrt{2}} \begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix} - \frac{\theta^2}{2!}\frac{1}{2}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix} + \frac{i\theta^3}{3!}\frac{1}{\sqrt{2}}\begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix}$$

$$+ \frac{\theta^4}{4!}\frac{1}{2}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix} - \frac{i\theta^5}{5!}\frac{1}{\sqrt{2}}\begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix} - \frac{\theta^6}{6!}\frac{1}{2}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix} + \cdots$$

$$= \begin{bmatrix} 1 - \frac{\theta^2/2!}{2} + \frac{\theta^4/4!}{2} \cdots & -\frac{\theta}{\sqrt{2}} + \frac{\theta^3/3!}{\sqrt{2}} - \frac{\theta^5/5!}{\sqrt{2}} \cdots & \frac{\theta^2/2!}{2} - \frac{\theta^4/4!}{2} + \frac{\theta^6/6!}{2} \cdots \\ \frac{\theta}{\sqrt{2}} - \frac{\theta^3/3!}{\sqrt{2}} + \frac{\theta^5/5!}{\sqrt{2}} \cdots & 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} \cdots & -\frac{\theta}{\sqrt{2}} + \frac{\theta^3/3!}{\sqrt{2}} - \frac{\theta^5/5!}{\sqrt{2}} \cdots \\ +\frac{\theta^2/2!}{2} - \frac{\theta^4/4!}{2} + \frac{\theta^6/6!}{2} \cdots & \frac{\theta}{\sqrt{2}} - \frac{\theta^3/3!}{\sqrt{2}} + \frac{\theta^5/5!}{\sqrt{2}} \cdots & 1 - \frac{\theta^2/2!}{2} + \frac{\theta^4/4!}{2} - \frac{\theta^6/6!}{2} \cdots \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1 + 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots}{2} & \frac{-(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots)}{\sqrt{2}} & \frac{1 - (1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots)}{2} \\ \frac{\theta - \theta^3/3! + \theta^5/5! + \cdots}{\sqrt{2}} & 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots & \frac{-(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots)}{\sqrt{2}} \\ \frac{1 - (1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots)}{2} & \frac{\theta - \frac{\theta^3}{3!} + \theta^5/5! + \cdots}{\sqrt{2}} & \frac{1 + 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1+\cos\theta}{2} & -\frac{\sin\theta}{\sqrt{2}} & \frac{1-\cos\theta}{2} \\ \frac{\sin\theta}{\sqrt{2}} & \cos\theta & -\frac{\sin\theta}{\sqrt{2}} \\ \frac{1-\cos\theta}{2} & \frac{\sin\theta}{\sqrt{2}} & \frac{1+\cos\theta}{2} \end{bmatrix}$$

$= d^1_{mm'}(\theta)$.

This same result can be obtained by using the eq

$$d^j_{mm'}(\theta) = (-)^{j-m'} \sqrt{\frac{(j+m')!}{(j-m')!\,(j+m)!\,(j-m)!}} \left(\sin\frac{\theta}{2}\right)^{m'-m} \left(\cos\frac{\theta}{2}\right)^{m+m'} \cdot \left[(1-t)^{m-m'} t^{-m-m'} \cdot \left(\frac{d}{dt}\right)^{(j-m')} \left\{ t^{j+m} (1-t)^{j-m} \right\}\right]_{t=\cos^2\frac{\theta}{2}}$$

4. The angular momentum operators associated with spin 1/2 are

$$\vec{J} = \frac{\hbar}{2}\vec{\sigma}$$

with $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

J_z is diagonal with eigenvalue $\pm\hbar/2$.

i. Write down the components of $\vec{J}$ in the direction of the unit vector $\hat{n} = (\theta, \phi)$ with θ & ϕ being the polar angles of $\hat{n}$.

ii. What are the possible results of a measurement of the component of angular momentum in the direction $\hat{n}$? (or find the eigenvalues of $J_n = \hat{n}\cdot\vec{J}$.) Find the corresponding normalized eigenvectors.

Deduce the results for the case when $\hat{n}$ is along the z-axis.

iii. Suppose the spin measurement along $\hat{n}$ gives $+\hbar/2$, what are the probabilities of finding $\pm\hbar/2$ in an immediately subsequent measurement along z-direction?

iv. Let the spin ½ particle have a magnetic moment

$\vec{\mu} = \frac{e}{mc}\vec{\sigma}$. Apply a uniform magnetic field $\vec{B}$ in the z-direction. Let J_x be measured at $t=0$ and the result be $\hbar/2$.

a) What is the state vector of the system at time t?

b) What are the probabilities of the outcomes of various measurements made on the system at time t?

Let us apply an additional field

$$\vec{B}_1 = B_1(\hat{i}\cos\omega t + \hat{j}\sin\omega t)$$

let $|\psi(t=0)\rangle = \begin{pmatrix}1\\0\end{pmatrix}$

Discuss the time evolution of the system.

Solution

(i) $\hat{n} = n_x\hat{i} + n_y\hat{j} + n_z\hat{k}$

Since it is a polar vector of unit magnitude,

$$n_x = \sin\theta\cos\varphi$$

$$n_y = \sin\theta\sin\varphi$$

$$n_z = \cos\theta$$

$$\vec{J} = \frac{\hbar}{2}\vec{\sigma} = \frac{\hbar}{2}(\sigma_x\hat{i} + \sigma_y\hat{j} + \sigma_z\hat{k})$$

So the comp. of $\vec{J}$ along $\hat{n}$ is

$$J_n = \hat{n}\cdot\vec{J}$$

$$= \frac{\hbar}{2}(n_x\sigma_x + n_y\sigma_y + n_z\sigma_z)$$

$$= \frac{\hbar}{2}\left\{\sin\theta\cos\varphi\begin{pmatrix}0&1\\1&0\end{pmatrix} + \sin\theta\sin\varphi\begin{pmatrix}0&-i\\i&0\end{pmatrix} + \cos\theta\begin{pmatrix}1&0\\0&-1\end{pmatrix}\right\}$$

$$= \frac{\hbar}{2}\begin{bmatrix}\cos\theta & \sin\theta(\cos\varphi - i\sin\varphi)\\ \sin\theta(\cos\varphi + i\sin\varphi) & -\cos\theta\end{bmatrix}$$

or $J_n = \frac{\hbar}{2}\begin{bmatrix}\cos\theta & \sin\theta\, e^{-i\varphi}\\ \sin\theta\, e^{i\varphi} & -\cos\theta\end{bmatrix}$

(ii) The possible results of a measurement of the comp. of the angular momentum in the direction of $\vec{n}$ are the eigenvalues of J_n, i.e.,

$$J_n|J_n\rangle = J_n|J_n\rangle \quad \cdots (*)$$

Let $|J_n\rangle = \begin{bmatrix}A_1\\A_2\end{bmatrix}$

(*) gives

$$\frac{\hbar}{2}\begin{bmatrix}\cos\theta & \sin\theta\, e^{-i\varphi}\\ \sin\theta\, e^{i\varphi} & -\cos\theta\end{bmatrix}\begin{bmatrix}A_1\\A_2\end{bmatrix} = J_n\begin{bmatrix}A_1\\A_2\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}\frac{\hbar}{2}\cos\theta - J_n & \frac{\hbar}{2}\sin\theta\, e^{-i\varphi}\\ \frac{\hbar}{2}\sin\theta\, e^{i\varphi} & -\frac{\hbar}{2}\cos\theta - J_n\end{bmatrix}\begin{bmatrix}A_1\\A_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \cdots (**)$$

For this to hold true for nontrivial soln. the determinant of the left matrix should vanish

$$\begin{vmatrix}\frac{\hbar}{2}\cos\theta - J_n & \frac{\hbar}{2}\sin\theta\, e^{-i\varphi}\\ \frac{\hbar}{2}\sin\theta\, e^{i\varphi} & -\frac{\hbar}{2}\cos\theta - J_n\end{vmatrix} = 0$$

$$-\left(\frac{\hbar^2}{4}\cos^2\theta - J_n^2\right) - \frac{\hbar^2}{4}\sin^2\theta = 0$$

$$J_n^2 - \frac{\hbar^2}{4}(\cos^2\theta + \sin^2\theta) = 0$$

$$J_n^2 - \frac{\hbar^2}{4} = 0$$

or $J_n^2 = \frac{\hbar^2}{4}$, $J_n = \pm\frac{\hbar}{2}$,

∴ The results of the measurement are

$J_n = \hbar/2$ or $J_n = -\hbar/2$

Eigenvectors

- $J_n = \frac{\hbar}{2}$

From $**$

$$\begin{bmatrix} \frac{\hbar}{2}(\cos\theta - 1) & \frac{\hbar}{2}\sin\theta\, e^{-i\varphi} \\ \frac{\hbar}{2}\sin\theta\, e^{i\varphi} & -\frac{\hbar}{2}(\cos\theta + 1) \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(\cos\theta - 1) A_1 + \sin\theta\, e^{-i\varphi} A_2 = 0$$

$$\sin\theta\, e^{i\varphi} A_1 - (\cos\theta + 1) A_1 = 0$$

The 1st eq. gives

$$A_2 = \frac{1-\cos\theta}{\sin\theta} e^{i\varphi} \cdot A_1$$

$$= \frac{2\sin^2\frac{\theta}{2}}{\sin\theta} e^{i\varphi} A_1$$

$$= \tan\frac{\theta}{2} e^{i\varphi} A_1$$

From the requirement of normalization:

$$(A_1^*, A_2^*) \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = |A_1|^2 + |A_2|^2 = 1$$

$$|A_1|^2 + \tan^2\frac{\theta}{2} |A_1|^2 = 1$$

$$|A_1|^2 (1 + \tan^2\frac{\theta}{2}) = 1$$

$$|A_1|^2 \sec^2\frac{\theta}{2} = 1$$

Choosing the positive value

$$A_1 = \cos\theta/2$$

$$\Rightarrow A_2 = \tan\frac{\theta}{2} \cdot \cos\frac{\theta}{2} e^{i\varphi}$$

$$= \sin\frac{\theta}{2} e^{i\varphi}$$

$$\therefore |J_n = +\tfrac{\hbar}{2}\rangle = \begin{bmatrix} \cos\theta/2 \\ \sin\frac{\theta}{2} e^{i\varphi} \end{bmatrix}$$

or $|+\hat{n}\rangle = \begin{bmatrix} \cos\theta/2 \\ \sin\frac{\theta}{2} e^{i\varphi} \end{bmatrix}$

- $J_n = -\hbar/2$

Again from $(**)$

$$\begin{bmatrix} \frac{\hbar}{2}(\cos\theta + 1) & \frac{\hbar}{2}\sin\theta\, e^{-i\varphi} \\ \frac{\hbar}{2}\sin\theta\, e^{i\varphi} & -\frac{\hbar}{2}(\cos\theta - 1) \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(\cos\theta + 1) A_1 + \sin\theta\, e^{-i\varphi} A_2 = 0$$

$$\sin\theta\, e^{i\varphi} A_1 - (\cos\theta - 1) A_2 = 0$$

The 1st eq. gives

$$A_1 = -\frac{\sin\theta}{\cos\theta + 1} e^{-i\varphi} A_2$$

$$= -\tan\theta/2\, e^{-i\varphi} A_2$$

From the requirement of normalization

$$|A_1|^2 + |A_2|^2 = 1$$

$$\tan^2\theta/2 |A_2|^2 + |A_2|^2 = 1$$

$$|A_2|^2 (\tan^2\theta/2 + 1) = 1$$

$$|A_2|^2 \sec^2\theta/2 = 1$$

Choosing the positive value

$$A_2 = \cos\theta/2$$

$$\Rightarrow A_1 = -\tan\theta/2 \cdot \cos\theta/2\, e^{-i\varphi}$$

$$= -\sin\theta/2\, e^{-i\varphi}$$

$$|J_n = -\tfrac{\hbar}{2}\rangle = \begin{bmatrix} -\sin\theta/2\, e^{-i\varphi} \\ \cos\theta/2 \end{bmatrix}$$

or $|-\hat{n}\rangle = \begin{bmatrix} -\sin\theta/2\, e^{-i\varphi} \\ +\cos\theta/2 \end{bmatrix}$

If $\vec{n}$ is along the z-axis ($\vec{n}=\vec{k}$) $\theta=0$. Still now the eigenvalues remain to be the same and

$$|+,\vec{k}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$|-,\vec{k}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

iii. In the $\hat{n}$ basis

$$|+,\vec{k}\rangle = |+,\hat{n}\rangle\langle +,\vec{n}|+,\vec{k}\rangle + |-,\vec{n}\rangle\langle -,\vec{n}|+,\vec{k}\rangle$$

But for $J_n = +\hbar/2$,

$$|+,\hat{k}\rangle = |+,\hat{n}\rangle\langle +,\vec{n}|+,\vec{k}\rangle$$

So, the ~~result~~ probability of obtaining $J_z = \hbar/2$ will be

$$|\langle +,\vec{n}|+,\vec{k}\rangle|^2$$

$$= |(\cos\theta/2,\ \sin\theta/2\, e^{-i\varphi})\begin{pmatrix} 1 \\ 0 \end{pmatrix}|^2$$

$$= \cos^2\theta/2$$

Similarly, the probability of obtaining $J_z = -\hbar/2$ will be

$$|\langle +,\hat{n}|-,\vec{k}\rangle|^2$$

$$= |(\cos\theta/2,\ \sin\theta/2\, e^{-i\varphi})\cdot\begin{pmatrix} 0 \\ 1 \end{pmatrix}|^2$$

$$= \sin^2\theta/2$$

Which are valid in the limiting cases of $\theta=0$ & $\theta=\pi$.

iv. a) In general the state vector of a spin 1/2 particle is given by

$$|\psi(t)\rangle = \begin{bmatrix} \psi_{1/2}(t) \\ \psi_{-1/2}(t) \end{bmatrix}$$

This state vector evolves in time according to Schrödinger's eq., i.e.,

$$i\hbar \frac{d|\psi\rangle}{dt} = H|\psi\rangle$$

for our case $H = -\vec{\mu}\cdot\vec{B}$

$$= -\frac{e}{mc}\vec{s}\cdot B\vec{k} = -\frac{eB}{mc}s_z$$

$$\Rightarrow i\hbar\frac{d|\psi(t)\rangle}{dt} = H|\psi\rangle = -\frac{eB}{mc}s_z|\psi\rangle$$

$$\frac{d|\psi\rangle}{dt} = i\frac{eBs_z}{mc\hbar}|\psi\rangle$$

$$= i\frac{eB}{mc\hbar}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\cdot\begin{bmatrix} \psi_{1/2} \\ \psi_{-1/2} \end{bmatrix}$$

$$= i\frac{eB}{mc\hbar}\begin{bmatrix} \psi_{1/2} \\ -\psi_{-1/2} \end{bmatrix}$$

$$\Rightarrow \frac{d}{dt}\begin{bmatrix} \psi_{1/2} \\ \psi_{-1/2} \end{bmatrix} = \frac{ieB}{mc\hbar}\begin{bmatrix} \psi_{1/2} \\ -\psi_{-1/2} \end{bmatrix}$$

or $$\frac{d\psi_{1/2}}{dt} = \frac{ieB}{mc\hbar}\psi_{1/2} \quad \cdots *$$

$$\frac{d\psi_{-1/2}}{dt} = -\frac{ieB}{mc\hbar}\psi_{-1/2} \quad \cdots **$$

These are eigenvalue eqs since we may write

$$i\hbar\frac{d\psi_{1/2}}{dt} = H\psi_{1/2} = -\frac{eB}{mc}\psi_{+1/2} \quad \cdots 1$$

$$i\hbar\frac{d\psi_{-1/2}}{dt} = H\psi_{-1/2} = \frac{eB}{mc}\psi_{-1/2} \quad \cdots 2$$

And the eigenvalues are $\pm\frac{eB}{mc}$

We know solve (*) & (**)

$$\psi_{\frac{1}{2}}(t) = c_1 e^{i\frac{eB}{mc\hbar}t} = c_1 e^{i\omega t}$$

$$\psi_{-\frac{1}{2}}(t) = c_2 e^{-i\frac{eB}{mc\hbar}t} = c_2 e^{-i\omega t}$$

If at time $t=0$, a measurement of J_x gives a value of $+\hbar/2$, our state is $|+,\hat{i}\rangle$. This means $\varphi = 0$, $\theta = \pi/2$ and the normalized state vector of J_n is

$$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$$

$$\Rightarrow \psi_{1/2}(0) = c_1 = \frac{1}{\sqrt{2}}$$

$$\psi_{-1/2}(0) = c_2 = \frac{1}{\sqrt{2}}$$

∴ at time t the state vector of the system is

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}e^{i\omega t}\\ e^{-i\omega t}\end{bmatrix}$$

b. In this state we can measure the angular momenta J_x, J_y, J_z.

- $J_x = +\hbar/2$

$\varphi = 0$, $\theta = \pi/2 \Rightarrow \hat{n} = \hat{i}$

corresponding state, $|+,\hat{i}\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$

$$\Rightarrow P(J_x = +\tfrac{\hbar}{2}) = |\langle\psi|+,\hat{i}\rangle|^2$$

$$= \left|\frac{1}{\sqrt{2}}(e^{-i\omega t}, e^{i\omega t})\cdot\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}\right|^2$$

$$= \frac{1}{4}\left|e^{-i\omega t} + e^{i\omega t}\right|^2$$

$$= \frac{1}{4}\cdot(2\cos\omega t)^2$$

$$= \cos^2\omega t$$

- $J_x = -\hbar/2$

$\varphi = \pi$, $\theta = \pi/2 \Rightarrow \hat{n} = -\hat{i}$

$$|-,\hat{i}\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$$

$$\Rightarrow P(J_x = -\tfrac{\hbar}{2}) = |\langle\psi|-,\hat{i}\rangle|^2$$

$$= \left|\frac{1}{\sqrt{2}}(e^{-i\omega t}, e^{i\omega t})\cdot\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}\right|^2$$

$$= \frac{1}{4}\left|e^{-i\omega t} - e^{i\omega t}\right|^2$$

$$= \sin^2\omega t$$

- $J_y = +\frac{\hbar}{2}$

$\varphi = \pi/2$, $\theta = \pi/2 \Rightarrow \hat{n} = \hat{j}$

$$|+,\hat{j}\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\i\end{bmatrix}$$

$$\Rightarrow P(J_y = +\tfrac{\hbar}{2})$$

$$= |\langle\psi|+,\hat{i}\rangle|^2$$

$$= \left|\frac{1}{\sqrt{2}}(e^{-i\omega t}, e^{i\omega t})\frac{1}{\sqrt{2}}\begin{bmatrix}1\\i\end{bmatrix}\right|^2$$

$$= \frac{1}{4}\left|e^{-i\omega t} + i e^{i\omega t}\right|^2$$

$$= \frac{1}{4}\left|\cos\omega t(1+i) - \sin\omega t(1+i)\right|$$

$$= \frac{1}{4}\left|(\cos\omega t - \sin\omega t)(1+i)\right|^2$$

$$= \frac{1}{4}\left(\cos^2\omega t + \sin^2\omega t - 2\sin\omega t\cos\omega t\right)\cdot(2)$$

$$= \frac{1}{2}(1 - \sin 2\omega t)$$

$$= \cos^2(\omega t + \pi/4)$$

- $P_y = -\hbar/2$

$\varphi = 3\pi/2, \ \theta = \pi/2$

$$|-,\hat{j}\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \end{bmatrix}$$

$$\Rightarrow P(S_y = -\hbar/2) = |\langle\psi|-,\hat{j}\rangle|^2$$

$$= \left|\frac{1}{\sqrt{2}}(e^{-i\omega t}, e^{i\omega t})\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \end{bmatrix}\right|^2$$

$$= \frac{1}{4}\left|e^{-i\omega t} + i e^{i\omega t}\right|^2$$

$$= \frac{1}{4}\left|\cos\omega t(1-i) + \sin\omega t(1-i)\right|^2$$

$$= \frac{1}{4}\left|(\cos\omega t + \sin\omega t)(1-i)\right|^2$$

$$= \frac{1}{2}(\cos\omega t + \sin\omega t)^2$$

$$= \cos^2(\omega t - \pi/4)$$

- $S_z = +\hbar/2$

$\theta = 0, \ \hat{n} = \hat{k}$

$$|+,\hat{k}\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow P(S_z = +\hbar/2) = |\langle\psi|+,\hat{k}\rangle|^2$$

$$= \left|\frac{1}{\sqrt{2}}(e^{-i\omega t}, e^{i\omega t})\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right|^2$$

$$= \frac{1}{2}\left|e^{i\omega t}\right|$$

$$= 1/2$$

- $S_z = -\hbar/2$

$\theta = \pi$

$$|-,\hat{k}\rangle = \begin{bmatrix} 0 \\ e^{i\varphi} \end{bmatrix}$$

$$\Rightarrow P(S_z = -\hbar/2) = |\langle\psi|-,\hat{k}\rangle|^2$$

$$= \left|\frac{1}{\sqrt{2}}(e^{-i\omega t}, e^{i\omega t})\begin{bmatrix} 0 \\ e^{i\varphi} \end{bmatrix}\right|^2$$

$$= \frac{1}{2}\left|e^{i(\omega t + \varphi)}\right|^2$$

$$= 1/2$$

- If we apply a time dependent field B_1 as given, the total field will be

$$\bar{B}_T = B\hat{k} + B_1(\hat{i}\cos\omega t + \hat{j}\sin\omega t)$$

$$\Rightarrow H_{mag} = -\bar{\mu}\cdot\bar{B}_T$$

$$= -\frac{eB}{mc}S_z - \frac{eB_1}{mc}(S_x\cos\omega t + S_y\sin\omega t)$$

$$= -\hbar\omega_0 S_z - \hbar\omega_1(S_x\cos\omega t + S_y\sin\omega t)$$

Once again the time evolution of the state of the particle is governed by Schrödinger's eq -

$$i\hbar\frac{d|\psi(t)\rangle}{dt} = H_{mag}|\psi(t)\rangle$$

$$|\psi(t)\rangle = \begin{bmatrix} \psi_{1/2}(t) \\ \psi_{-1/2}(t) \end{bmatrix}$$

$$H_{mag}|\psi(t)\rangle = -\hbar\omega_0\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{bmatrix} \psi_{1/2} \\ \psi_{-1/2} \end{bmatrix}$$

$$-\hbar\omega_1\cos\omega t\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{bmatrix} \psi_{1/2} \\ \psi_{-1/2} \end{bmatrix}$$

$$-\hbar\omega_1\sin\omega t\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} \psi_{1/2} \\ \psi_{-1/2} \end{pmatrix}$$

$$H_{mag}|\psi(t)\rangle = i\hbar \frac{d|\psi(t)\rangle}{dt}$$

$$= -\hbar\omega_0 \begin{bmatrix} \psi_{1/2} \\ -\psi_{-1/2} \end{bmatrix}$$

$$-\hbar\omega_1 \cos\omega t \begin{bmatrix} \psi_{-1/2} \\ \psi_{1/2} \end{bmatrix}$$

$$-i\hbar\omega_1 \sin\omega t \begin{bmatrix} -\psi_{-1/2} \\ \psi_{1/2} \end{bmatrix}$$

$$= \begin{bmatrix} -\hbar\omega_0 \psi_{1/2} - \hbar\omega_1 e^{-i\omega t}\psi_{-1/2} \\ \hbar\omega_0 \psi_{-1/2} - \hbar\omega_1 e^{i\omega t}\psi_{1/2} \end{bmatrix}$$

or $$i\hbar \begin{bmatrix} \frac{d\psi_{1/2}}{dt} \\ \frac{d\psi_{-1/2}}{dt} \end{bmatrix} = \begin{bmatrix} -\hbar\omega_0\psi_{1/2} - \hbar\omega_1 e^{-i\omega t}\psi_{-1/2} \\ \hbar\omega_0\psi_{-1/2} - \hbar\omega_1 e^{i\omega t}\psi_{1/2} \end{bmatrix}$$

which gives

$$\frac{d\psi_{1/2}}{dt} = i\omega_0\psi_{1/2} + i\omega_1 e^{-i\omega t}\psi_{-1/2}$$

$$\frac{d\psi_{-1/2}}{dt} = -i\omega_0\psi_{-1/2} + i\omega_1 e^{i\omega t}\psi_{1/2}$$

As a trial solution consider

$$\psi_{1/2} = c_1 e^{i\omega_0 t}$$

$$\psi_{-1/2} = c_2 e^{-i\omega_0 t}$$

where $c_1 \equiv c_1(t)$, $c_2 \equiv c_2(t)$

$$\Rightarrow \dot{c}_1 = i\omega_1 e^{-i(\omega+2\omega_0)t} \cdot c_2$$

$$\dot{c}_2 = i\omega_1 e^{i(\omega+2\omega_0)t} \cdot c_1$$

Once again as a trial solution (which actually satisfies the above two eqs) we may take

$$c_1 = B_1 e^{-i(\omega+2\omega_0+\alpha)t}$$

$$c_2 = B_2 e^{-i\alpha t}$$

$B_1 \equiv$ constant, $B_2 \equiv$ constant which when substituted in the D.E's for c_1 & c_2 will give

$$-(\omega+2\omega_0+\alpha)B_1 = \omega_1 B_2$$

$$-\alpha B_2 = \omega_1 B_1$$

$$\Rightarrow \frac{\omega+2\omega_0+\alpha}{\omega_1} = \frac{\omega_1}{\alpha}$$

or $$\alpha^2 + (\omega+2\omega_0)\alpha - \omega_1^2 = 0$$

$$\Rightarrow \alpha_\pm = -\left(\omega_0+\frac{\omega}{2}\right) \pm \sqrt{\left(\omega_0+\frac{\omega}{2}\right)^2+\omega_1^2}$$

So c_2 may be given as

$$c_2 = K^- e^{i\Omega_- t} + K^+ e^{i\Omega_+ t}$$

Now, $$c_1 = \frac{-i}{\omega_1} e^{-i(\omega+2\omega_0)t} \cdot \dot{c}_2$$

$$= \frac{e^{-i(\omega+2\omega_0)t}}{\omega_1}\left(\Omega_- K^- e^{i\Omega_- t} + \Omega_+ K^+ e^{i\Omega_+ t}\right)$$

A/4 / H/7 p/6

$$\Rightarrow \psi_{1/2}(t) = \frac{e^{-i(\omega+\omega_0)t}}{\omega_1}\left(\Omega_- K^- e^{i\Omega_- t} + \Omega_+ K^+ e^{i\Omega_+ t}\right)$$

$$\psi_{-1/2}(t) = K^- e^{i(\Omega_- - \omega_0)t} + K^+ e^{i(\Omega_+ - \omega_0)t}$$

At $t=0$,

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{bmatrix} \frac{\Omega_- K^- + \Omega_+ K^+}{\omega_1} \\ K^- + K^+ \end{bmatrix}$$

$$K^- = -K^+ = \frac{\omega_1}{\Omega_- - \Omega_+}$$

$$= \frac{\omega_1}{-2\sqrt{\left(\omega_0+\frac{\omega}{2}\right)^2+\omega_1^2}}$$

$$\Rightarrow c_1 = \frac{e^{-i(\omega+2\omega_0)t}}{\omega_1} \cdot e^{-i(\omega_0+\frac{\omega}{2})t}$$

$$\cdot \Big[\Omega_- K^- \big(\cos\sqrt{(\omega_0+\omega/2)^2+\omega_1^2}\,t - i\sin\sqrt{(\omega_0+\omega/2)^2+\omega_1^2}\,t\big)$$

$$+ \Omega_+ K_+\big(\cos\sqrt{(\omega_0+\tfrac{\omega}{2})^2+\omega_1^2}\,t + i\sin\sqrt{(\omega_0+\tfrac{\omega}{2})^2+\omega_1^2}\,t\big)$$

$$= \frac{e^{-\frac{i}{2}(3/2\,\omega+3\omega_0)t}}{\omega_1}$$

$$\cdot\Big\{\frac{\omega_1(\omega_0+\frac{\omega}{2})+\sqrt{\ }}{2\sqrt{\ }}(\cos\Gamma - i\sin\Gamma)$$

$$+\left(\frac{\omega_1}{2}\right)\frac{-(\omega_0+\frac{\omega}{2})+\Gamma}{\sqrt{\ }}(\cos\Gamma+i\sin\Gamma)\Big\}$$

$$= \frac{e^{-i3(\omega_0+\frac{\omega}{2})t}}{2\Gamma}\Big\{2\sqrt{\ }\cos\Gamma - i2(\omega_0+\tfrac{\omega}{2})\sin\Gamma\Big\}$$

$$= \frac{e^{-i3(\omega_0+\frac{\omega}{2})t}}{\sqrt{(\omega_0+\frac{\omega}{2})^2+\omega_1^2}}\Big\{\sqrt{(\omega_0+\tfrac{\omega}{2})^2+\omega_1^2}\cos\Gamma - i(\omega_0+\tfrac{\omega}{2})\sin\Gamma\Big\}$$

or $\psi_{1/2} = c_1 e^{i\omega_0 t}$

$$= \frac{e^{-i(2\omega_0+\frac{3\omega}{2})t}}{\sqrt{(\omega_0+\frac{\omega}{2})^2+\omega_1^2}}$$

$$\Big\{\sqrt{(\omega_0+\tfrac{\omega}{2})^2+\omega_1^2}\cos\sqrt{(\omega_0+\tfrac{\omega}{2})^2+\omega_1^2}\,t$$

$$- i(\omega_0+\tfrac{\omega}{2})\sin\sqrt{(\omega_0+\tfrac{\omega}{2})^2+\omega_1^2}\,t\Big\}$$

Similarly,

$$\psi_{-1/2} = i\,e^{-i(2\omega_0+\frac{\omega}{2})t}\cdot\frac{\omega_1\sin\sqrt{(\omega_0+\frac{\omega}{2})^2+\omega_1^2}}{\sqrt{(\omega_0+\frac{\omega}{2})^2+\omega_1^2}}$$

The probability that the spin is ~~down~~ up at time t is

$$|\langle\psi(t)|+,\vec{k}\rangle|^2$$

$$= |(\psi_{1/2}^*, \psi_{-1/2}^*)[\,]|^2$$

$$= |\psi_{1/2}^*|^2$$

$$= \frac{((\omega_0+\frac{\omega}{2})^2+\omega_1^2)\cos^2\Gamma + (\omega_0+\frac{\omega}{2})^2\sin^2\Gamma}{(\omega_0+\frac{\omega}{2})^2+\omega_1^2}$$

$$= \frac{(\omega_0+\frac{\omega}{2})^2+\omega_1^2\cos\sqrt{(\omega_0+\frac{\omega}{2})^2+\omega_1^2}\,t}{(\omega_0+\omega/2)^2+\omega_1^2}$$

That it is down at t is

$$1 - |\langle\psi(t)|+,\vec{k}\rangle|^2$$

$$= \frac{\omega_1^2\sin\sqrt{(\omega_0+\frac{\omega}{2})^2+\omega_1^2}\,t}{(\omega_0+\frac{\omega}{2})^2+\omega_1^2}$$

5. Consider a hydrogen atom in a state $|n\,\ell\,m_s\rangle$. Use first order perturbation theory to calculate the corrections due to relativistic effect and the spin-orbit interaction.

Solution

Suppose the electron in the hydrogen is moving at a velocity

$\bar{v}$. But we consider a fram in which the electron is at rest so that the proton will be moving at a velocity $-\bar{v}$ and will produce a magnetic field

$$\bar{B} = -\frac{e}{c}\,\frac{\bar{v}\times\bar{r}}{r^3}$$

At the site of the electron the interaction of the magnetic moment of the electron with this field leads to the spin-orbit energy

$$H_{SO} = -\bar{\mu}\cdot\bar{B}$$

$$= \frac{e}{mcr^3}\,\bar{\mu}\cdot(\bar{p}\times\bar{r})$$

$$= -\frac{e}{mc}\,\frac{\bar{\mu}\cdot(\bar{r}\times\bar{p})}{r^3}$$

$$= -\frac{e}{mc}\,\frac{\bar{\mu}\cdot\bar{L}}{r^3}$$

$$= \left(-\frac{e}{mc}\right)\left(-\frac{e}{mc}\right)\frac{\bar{S}\cdot\bar{L}}{r^3}$$

$$= \frac{e^2}{m^2c^2r^3}\,\bar{S}\cdot\bar{L}$$

The correct expression as found by Thomas has a factor ½ so that

$$H_{S.O.} = \frac{e^2}{2m^2c^2r^3}\,\vec{S}\cdot\vec{L}$$

Since $\bar{V}/\!/\bar{L}\times\bar{B}$ the states at a given n are n^2-fold degenerate, we must state with a basis that diagonalizes $H_{S.O.}$.

$$\bar{J} = \bar{L}+\bar{S}$$

$$\Rightarrow J^2 = L^2+S^2+2\bar{L}\cdot\bar{S}$$

$$\bar{L}\cdot\bar{S} = \frac{1}{2}(J^2-L^2-S^2)$$

$$\text{or } H_{SO} = \frac{e^2}{4m^2c^2r^3}(J^2-L^2-S^2)$$

$$E^1_{nlm} = \langle nlm|H'|nlm\rangle$$

In this basis

$$\langle j'm'\,l'\tfrac{1}{2}|H_{SO}|jm\,l\tfrac{1}{2}\rangle$$

$$\text{or } \langle l'\tfrac{1}{2}\,j'm'|H_{SO}|l\tfrac{1}{2}\,jm\rangle$$

$$= \delta_{jj'}\,\delta_{mm'}\,\delta_{ll'}\,\frac{e^2}{4m^2c^2}\left\langle\frac{1}{r^3}\right\rangle_{nl}\hbar^2\left[j(j+1)-l(l+1)-\tfrac{1}{2}(\tfrac{1}{2}+1)\right]$$

$$= \frac{e^2}{4m^2c^2}\left\langle\frac{1}{r^3}\right\rangle_{nl}\cdot\hbar^2\left[j(j+1)-l(l+1)-\tfrac{3}{4}\right]$$

(Note that two states with the same total jm, but built from diff. ls are orthogonal because of the orthogonality of the spherical harmonics)

But at this value of s
$j = l\pm\frac{1}{2}$.

i. $j = l+\frac{1}{2}$

$$E^1_{SO} = \frac{\hbar^2e^2}{4m^2c^2}\cdot l\left\langle\frac{1}{r^3}\right\rangle_{nl}$$

$$\text{But } \left\langle\frac{1}{r^3}\right\rangle_{nl} = \frac{1}{a_0^3}\,\frac{1}{n^3 l(l+\frac{1}{2})(l+1)}$$

$$\Rightarrow E^1_{SO} = \frac{\hbar^2e^2}{4m^2c^2a_0^3}\,\frac{1}{n^3(l+\frac{1}{2})(l+1)}$$

ii. $j = l-\frac{1}{2}$

$$E^1_{SO} = -\frac{\hbar^2e^2}{4m^2c^2}(l+1)\left\langle\frac{1}{r^3}\right\rangle_{nl}$$

$$= \frac{-\hbar^2e^2}{4m^2c^2a_0^3}\,\frac{1}{n^3 l(l+\frac{1}{2})}$$

or in general for $j = l\pm\frac{1}{2}$

$$E^1_{SO} = \frac{\hbar^2e^2}{4m^2c^2a_0^3}\,\frac{\left\{\begin{matrix} l \\ -(l+1)\end{matrix}\right\}}{n^3 l(l+\frac{1}{2})(l+1)}$$

This formula holds for $l\neq 0$. If $l=0$, $\langle 1/r^3\rangle$ diverges and $\langle\bar{L}\cdot\bar{S}\rangle$ vanishes.

In terms of the fine structure constant $\alpha = e^2/\hbar c$ (with $a_0 = \frac{\hbar^2}{me^2}$)

$$E^1_{so} = \frac{1}{4} mc^2\alpha^4 \frac{\left\{ \begin{array}{c} l \\ -(l+1) \end{array} \right\}}{n^3 l(l+\frac{1}{2})(l+1)}$$

To order $(\frac{v}{c})^4$ the K.E. of the electron is

$$T = (c^2p^2 + m^2c^4)^{1/2} - mc^2$$

$$= \frac{p^2}{2m} - \frac{p^4}{8m^3c^2} + O(p^6)$$

$$H_T = -p^4/8m^3c^2$$

The 1st order correction corresponding to which will be

$$E^1_T = -\frac{1}{8m^3c^2}\langle nlm|p^4|nlm\rangle$$

$$p^4 = 4m^2\left(\frac{p^2}{2m}\right)^2 = 4m^2\left(H_0 + \frac{e^2}{r}\right)^2$$

$$\Rightarrow E^1_T = -\frac{1}{2mc^2}\left[(E_n^0)^2 + 2E_n^0 e^2 \langle \tfrac{1}{r} \rangle_{nlm} + e^4 \langle \tfrac{1}{r^2} \rangle_{nlm}\right]$$

From the virial theorem

$$-\langle \tfrac{e^2}{r} \rangle_{nlm} = 2E_n^0$$

and by calculation

$$\langle \tfrac{e^4}{r^2} \rangle_{nlm} = \frac{e^4}{a_0^2 n^3 (l+\frac{1}{2})} = \frac{4(E_n^0)^2 n}{l+\frac{1}{2}}$$

$$\Rightarrow E^1_T = -\frac{(E_n^0)^2}{2mc^2}\left(-3 + \frac{4n}{l+\frac{1}{2}}\right)$$

$$= -\frac{1}{2}mc^2\alpha^4\left[-\frac{3}{4n^4} + \frac{1}{n^3(l+\frac{1}{2})}\right]$$

So now $E^1_{so} + E^1_T$, the total fine structure energy shift is

$$E^1_{fs} = -\frac{mc^2\alpha^2}{2n^2}\cdot\frac{\alpha^2}{n}\left(\frac{1}{j+\frac{1}{2}} - \frac{3}{4n}\right)$$

for both $j = l \pm \frac{1}{2}$.

6. Obtain an approximate expression for the energy shift of the ground state of the hydrogen atom due to the finite size of the proton assuming that the proton is a uniformly charged sphere of radius $R = 10^{-13}$ cm.

Let us find the interaction energy first by finding the p.d.

$$\epsilon_0 \oint \vec{E}\cdot d\vec{s} = e' = e(r/R)^3$$

$$\epsilon_0 E\, 4\pi r^2 = e(r/R)^3$$

or $$E = \frac{1}{4\pi\epsilon_0}\frac{er}{R^3}$$

$$= \frac{er}{R^3} \quad \text{(in Gaussian system of units)}$$

$$\text{p.d.} = V(r) - V(R) = -\int_R^r E\,dr$$

$$= \frac{-e}{R^3}\int_R^r r\,dr$$

$$= \frac{-er^2}{2R^3} + \frac{e}{2R}$$

But $V(R) = \frac{e}{R}$

$$\Rightarrow V(r) = -\frac{er^2}{2R^3} + \frac{3e}{2R}$$

or the P.E. of interaction with the electron is

$$U(r) = \frac{e^2r^2}{2R^3} - \frac{3e^2}{2R}, \quad r \le R$$

$$= -\frac{e^2}{r}, \quad r > R$$

So now as a correction to the Hamiltonian we have

$$H_1 = \frac{e^2}{r} + \frac{e^2 r^2}{2R^3} - \frac{3e^2}{2R}, \quad r \le R$$

$$= 0, \quad r \gg R$$

The corresponding first order energy shift is

$$E^1_{100} = \langle 100 | H_1 | 100 \rangle$$

$$= e^2 \langle 100 | \tfrac{1}{r} | 100 \rangle$$

$$+ \frac{e^2}{2R^3} \langle 100 | r^2 | 100 \rangle$$

$$- \frac{3e^2}{2R} \underbrace{\langle 100 | 100 \rangle}_{1}$$

$$\psi_{100} = \left(\frac{1}{\pi a_0^3}\right)^{1/2} e^{-r/a_0}$$

$$\Rightarrow E^1_{100} = \frac{e^2}{\pi a_0^3} \int_0^R e^{-r/a_0} \left(\frac{1}{r} + \frac{r^2}{2R^3} - \frac{3}{2R}\right) \cdot e^{-r/a_0} \cdot r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= \frac{4e^2}{a_0^3} \int_0^R e^{-2r/a_0} \left(\frac{1}{r} + \frac{r^2}{2R^3} - \frac{3}{2R}\right) \cdot r^2 dr$$

Since $r < R_0 \ll a_0$, we may set $e^{-2r/a_0} \sim 1 \; (r/a_0 \sim 0)$

$$\Rightarrow E^1_{100} = \frac{4e^2}{a_0^3} \left(\frac{r^2}{2} + \frac{r^5}{10R^3} - \frac{Br^3}{2R} \right)_0^R$$

$$= \frac{4e^2}{a_0^3} \cdot \frac{R^2}{10}$$

or $$E^1_{100} = \frac{2e^2 R^2}{5a_0^3}$$

$$a_0 = 0.5 \times 10^{-8} \text{ cm}$$

$$R = 10^{-13} \text{ cm}$$

$$\Rightarrow E^1_{100} = 4.35 \times 10^{-9} \text{ eV}$$

7. A hydrogen atom is in the ground state. Assume the K.E. to be given by the relativistic formula. Estimate the correction to the energy using first order perturbation theory.

Solution

$$T = (c^2p^2 + m^2c^4)^{1/2} - mc^2$$

$$= mc^2\left(1 + \frac{p^2}{m^2c^2}\right)^{1/2} - mc^2$$

$$= mc^2\left(1 + \frac{p^2}{2m^2c^2} - \frac{p^4}{8m^4c^4} + \dots\right) - mc^2$$

$$= \frac{p^2}{2m} - \frac{p^4}{8m^3c^2} + \dots$$

If we want the correction to order p^4 (or v^4) we have

$$T = \frac{p^2}{2m} - \frac{p^4}{8m^3c^2}$$

$$\Rightarrow H = \frac{p^2}{2m} - \frac{p^4}{8m^3c^2} - \frac{e^2}{r} = H_0 - \frac{p^4}{8m^3c^2}$$

or $H_1 = -\dfrac{p^4}{8m^3c^2}$ may be regarded as a perturbation due to relativistic effect.

So now the first order energy shift is given by

$$E^1_{nlm} = \langle nlm | H_1 | nlm \rangle$$

But since we are considering the ground state

$$E^1_{T,100} = \langle 100 | H^1 | 100 \rangle$$

$$= -\frac{1}{8m^3c^2} \langle 100 | p^4 | 100 \rangle$$

$$H_0 = \frac{p^2}{2m} - \frac{e^2}{r}$$

$$\Rightarrow \frac{p^2}{2m} = H_0 + \frac{e^2}{r}$$

or $$p^4 = 4m^2 \left(H_0 + \frac{e^2}{r}\right)^2$$

i.e., $$E^1_{T,100} = -\frac{1}{2mc^2} \langle 100 | (H_0 + \frac{e^2}{r})^2 | 100 \rangle$$

$$= \frac{-1}{2mc^2} \Big\{ \langle 100 | H_0^2 | 100 \rangle + \langle 100 | 2H_0 \frac{e^2}{r} | 100 \rangle + \langle 100 | \frac{e^4}{r^2} | 100 \rangle \Big\}$$

$$= \frac{-1}{2mc^2} \Big\{ (E^0_{100})^2 + 2e^2 E^0_{100} \langle 100 | \frac{1}{r} | 100 \rangle + e^4 \langle 100 | \frac{1}{r^2} | 100 \rangle \Big\}$$

$$\langle 100 | \frac{e^2}{r} | 100 \rangle = -2E^0_{100}, \text{ virial theorem}$$

$$(\langle V \rangle = 2E)$$

$$\langle nlm | \frac{e^4}{r^2} | nlm \rangle = \frac{e^4}{a_0^2 n^3 (l+\frac{1}{2})}$$

$$= \frac{4n\, E^{0\,2}_{n}}{l+\frac{1}{2}}$$

Since $n=1 \Rightarrow l=0$, in our case

$$\langle 100 | \frac{e^4}{r^2} | 100 \rangle = 8(E^0_{100})^2$$

$$\Rightarrow E^1_{T,100} = -\frac{(E^0_{100})^2}{2mc^2} \{1 - 4 + 8\}$$

$$= -\frac{5(E^0_{100})^2}{2mc^2}$$

$$E_n = -\frac{me^4}{2\hbar^2 n^2} = \frac{-e^2}{2a_0 n^2}$$

i.e., $$E^0_{100} = \frac{-e^2}{2a_0}$$

$$\therefore E^1_{T,100} = \frac{-5e^4}{8mc^2 a_0^2}$$

8. A nucleus A of spin 1 is excited in an even state. Energetically the emission of an alpha particle is possible

$$A \rightarrow B + \alpha$$

The nucleus B produced in this reaction is stable, has zero spin, and is also in an even state. Use the principle of conservation of energy angular momentum and of parity to show that this reaction is forbidden.

solution

The original nucleus A has spin 1 and its parity is even. The product nucleus B has even parity (given) and its spin is 0. Also, the spin of the α-particle is zero. Then the requirement of the conservation of angular mom. will imply that the orbital angular mom. corresponding to the relative motion of the α-particle and the product nucleus B should be 1. But since parity is $\pi = (-1)^l$, after the reaction we get $\pi = (-1)^1 = -1$ (the system is in a state of odd parity). Thus parity is not conserved and the reaction is forbidden.

9. In a hydrogen like atom the 2S and 2p levels are separated by a small energy difference Δ due to a small effect which practically does not affect the wave functions of these states. The atom is placed in an electric field $E\hat{k}$. Neglecting the effect of the more distant levels obtain a general expression for the energy shifts of the n=2 levels as a function of E [Neglect spin]. Do not evaluate explicitly any non zero integrals you may come across.]

Solution

$$P.d. = -\int \vec{E}\cdot d\vec{l} = -\int_0^z E\,dl = -Ez$$

The corresponding energy of the electron in this field will be

$$V = eEz$$

Now z is something like a component T_{10} of a vector operator and for n=2, the possible states are

$l = 1,\ m_l = 1, 0, -1$

$|211\rangle, |210\rangle, |21-1\rangle$

$l = 0,\ m_l = 0$

$|200\rangle$

(i.e.) we have four states. Now we can calculate the 16 matrix elements of V in these states (basis). But

$$\langle 2l'm'|T_{10}|2lm\rangle$$

$$= \langle lm, 10|l'm'\rangle\langle 2l'\|T_1\|2l\rangle$$

$= 0$ unless $m' = m$.

$l' = l+1, l, l-1$

Therefore, the only non zero matrix elements are

$$\langle 200|eEz|210\rangle$$

$$\langle 210|eEz|200\rangle$$

where we have neglected

$\langle 210|eEz|210\rangle$ $\quad (-1)^l = (-1)^1 = -1$

$\langle 200|eEz|200\rangle$ $\quad (-1)^0 = 1$

on parity arguments i.e., z connects states of diff. parity.

$$\psi_{200} = \frac{1}{\sqrt{32\pi a_0^3}}\left(2 - \frac{r}{a_0}\right)e^{-r/2a_0}$$

$$\psi_{210} = \frac{1}{\sqrt{32\pi a_0^3}}\, e^{-r/2a_0}\left(\frac{r}{a}\right)\cos\theta$$

$\Rightarrow \langle 210|eEz|200\rangle$

$$= \frac{eE}{32\pi a_0^3}\int_0^\infty\int_0^\pi\int_0^{2\pi}[e^{-r/2a}\left(\frac{r}{a}\right)\cos\theta$$

$$(r\cos\theta)\left(2-\frac{r}{a_0}\right)e^{-\frac{r}{2a_0}} r^2\sin\theta$$

$$\cdot dr\, d\theta\, d\varphi]$$

$$= \frac{eE}{16a_0^4}\int_0^\infty\int_0^\pi[e^{-r/a_0}\left(2-\frac{r}{a_0}\right)r^4\cos^2\theta\sin\theta$$

$$dr\, d\theta]$$

$$= \frac{eE}{24a_0^4}\int_0^\infty e^{-r/a_0}\left(2r^4-\frac{r^5}{a_0}\right)dr$$

This can be integrated by parts repeatedly or we can obtain from tables: the first gives $48a_0^5$ and the second gives $-120a_0^5$.

$eE\langle 210|z|200\rangle$

$$= +\frac{eE}{24a_0^4}(48-120)a_0^5$$

$$= -\frac{72}{24}\,\frac{eEa_0^5}{a_0^4}$$

$$= -3eEa_0$$

i.e., $V \to \begin{bmatrix} 0 & -3eEa_0 \\ -3eEa_0 & 0 \end{bmatrix}$ (rows/columns labelled $l=1$, 0)

Taking into account the energy shift between the 2S & 2P states, the Hamiltonian in the applied field goes over to

$$\begin{pmatrix} 0 & -3eEa_0 \\ -3eEa_0 & \Delta \end{pmatrix}$$

$$= \begin{pmatrix} 0 & K \\ K & \Delta \end{pmatrix}, \quad K = -3eEa_0$$

To diagonalize this matrix of H we solve the eigenvalue prob.

$$\begin{vmatrix} 0-\Lambda & K \\ K & \Delta-\Lambda \end{vmatrix} = 0$$

$$\Lambda(\Lambda-\Delta)-K^2=0$$

$$\Lambda^2-\Delta\Lambda-K^2=0$$

or $\Lambda_\pm = \dfrac{\Delta\pm\sqrt{\Delta^2+4K^2}}{2}$

$$\Rightarrow H = \begin{bmatrix} \frac{\Delta-\sqrt{\Delta^2+36e^2E^2a_0^2}}{2} & 0 \\ 0 & \frac{\Delta+\sqrt{\Delta^2+36e^2E^2a_0^2}}{2} \end{bmatrix}$$

i.e., the energy of the state $|210\rangle$ (2P state) is shifted by $\frac{\Delta-\sqrt{\Delta^2+36e^2E^2a_0^2}}{2}$ and that of $|200\rangle$ (2S state) is shifted by $\frac{\Delta+\sqrt{\Delta^2+36e^2E^2a_0^2}}{2}-\Delta = \frac{-\Delta+\sqrt{\Delta^2+36e^2E^2a_0^2}}{2}$.

Of course $|211\rangle, |21{-1}\rangle$ are not shifted as the matrix elements of V in these states are zero. (Note for strong fields $|e a_0 E|(6) \gg \Delta$, we obtain a linear Stark effect i.e., shifts proportional to E.)

10. e^+ and e^- have the same absolute magnetic moment but opposite g factors. Show that the ground state of the e^+e^- atom (positronium) a $^1S_0$, $^3S_1$ doublet

Cannot have a linear Zeeman effect if this is true. Argue in terms of the total magnetic moment operator.

Solution

The Zeeman effect takes place for an atom placed in an external magnetic field. Suppose we choose $\bar{B}$ along the z-axis. Now, the interaction Hamiltonian will be

$$H' = -\bar{\mu}\cdot\bar{B} = -\mu_z B_0$$

Where $\bar{\mu} = \bar{\mu}^+ - \bar{\mu}_-$ is the magnetic moment of the positronium.

$$\bar{\mu} = \frac{e\hbar}{2mc}(\bar{\sigma}^+ - \bar{\sigma}^-)$$

$$= \frac{e}{mc}\left(\frac{\hbar}{2}\bar{\sigma}^+ - \frac{\hbar}{2}\bar{\sigma}^-\right)$$

$$= \frac{e}{mc}(\bar{S}^+ - \bar{S}^-)$$

$$H = -\frac{eB_0}{mc}(S_z^+ - S_z^-)$$

Since we are adding two angular momenta the possible states of the positronium in terms of the uncoupled ones are

$$|S M_s\rangle = \sum_{m_s^+ m_s^-} |s^+ m_s^+ s^- m_s^-\rangle \langle s^+ m_s^+ s^- m_s^- | S M_s\rangle$$

$$= \sum_{m_s^+ = -1/2}^{1/2} |s^+ m_s^+ s^- M_s - m_s^+\rangle \langle s^+ m_s^+ \, M_s - m_s^+ | S M_s\rangle$$

Since $M_s = m_s^+ + m_s^-$

$S = 1, \quad M_s = 1, 0, -1$

$S = 0, \quad M_s = 0$

i.e.,

$$|11\rangle = |\tfrac{1}{2}\tfrac{1}{2}\rangle$$

$$S_-|11\rangle = \sqrt{2}\,|10\rangle$$

$$= (S_-^+ + S_-^-)\,|\tfrac{1}{2}\tfrac{1}{2}\rangle$$

$$= |-\tfrac{1}{2},\tfrac{1}{2}\rangle + |\tfrac{1}{2},-\tfrac{1}{2}\rangle$$

$$|10\rangle = \frac{1}{\sqrt{2}}\left(|-\tfrac{1}{2},\tfrac{1}{2}\rangle + |\tfrac{1}{2},-\tfrac{1}{2}\rangle\right)$$

$$S_-|10\rangle = \sqrt{2}\,|1,-1\rangle$$

$$= (S_-^+ + S_-^-)\frac{1}{\sqrt{2}}\left(|-\tfrac{1}{2},\tfrac{1}{2}\rangle + |\tfrac{1}{2},-\tfrac{1}{2}\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(|-\tfrac{1}{2},-\tfrac{1}{2}\rangle + |-\tfrac{1}{2},-\tfrac{1}{2}\rangle\right)$$

$$= \frac{2}{\sqrt{2}}|-\tfrac{1}{2},-\tfrac{1}{2}\rangle$$

$$|1,-1\rangle = |-\tfrac{1}{2},-\tfrac{1}{2}\rangle$$

$|00\rangle$ may be obtained from $|\tfrac{1}{2},-\tfrac{1}{2}\rangle$ or $|-\tfrac{1}{2},\tfrac{1}{2}\rangle$ so that

$$|00\rangle = \alpha|\tfrac{1}{2},-\tfrac{1}{2}\rangle + \beta|-\tfrac{1}{2},\tfrac{1}{2}\rangle$$

Normalization

$$\langle 00|00\rangle = 1$$

$$\alpha^2 + \beta^2 = 1$$

Orthogonality with $|10\rangle$

$$\alpha + \beta = 0$$

$$\beta = -\alpha$$

$$\alpha^2 + \alpha^2 = 1$$

$$2\alpha^2 = 1$$

or $\alpha = \pm 1/\sqrt{2}$

Using the Condon Shortley phase convention

$|00\rangle = \frac{1}{\sqrt{2}}(|\tfrac{1}{2},-\tfrac{1}{2}\rangle - |-\tfrac{1}{2},\tfrac{1}{2}\rangle)$

these are the unperturbed ground states of the positronium.

$$H' = -\frac{eB_0}{mc}(S_z^+ - S_z^-)$$

Next we find the energy shift due to the linear Zeeman effect to first order.

$$E^1_{sm_s} = \langle s m_s | H' | s m_s \rangle$$

$$- H'|11\rangle = -\frac{eB_0}{mc}(S_z^+ - S_z^-)|\tfrac{1}{2}\tfrac{1}{2}\rangle$$

$$= -\frac{eB_0}{mc}(\tfrac{1}{2}-\tfrac{1}{2})|\tfrac{1}{2}\tfrac{1}{2}\rangle$$

$$= 0$$

$$\Rightarrow E^1_{11} = \langle 11|H'|11\rangle = 0$$

$$- H^1|10\rangle = -\frac{eB_0}{mc}(S_z^+ - S_z^-)$$

$$\frac{1}{\sqrt{2}}(|-\tfrac{1}{2},\tfrac{1}{2}\rangle + |\tfrac{1}{2},-\tfrac{1}{2}\rangle)$$

$$= -\frac{eB_0}{\sqrt{2}mc}(-\tfrac{1}{2}|-\tfrac{1}{2},\tfrac{1}{2}\rangle + \tfrac{1}{2}|\tfrac{1}{2},-\tfrac{1}{2}\rangle$$

$$- \tfrac{1}{2}|-\tfrac{1}{2},\tfrac{1}{2}\rangle + \tfrac{1}{2}|\tfrac{1}{2},-\tfrac{1}{2}\rangle)$$

$$= -\frac{eB_0}{mc}\frac{1}{\sqrt{2}}(|\tfrac{1}{2},-\tfrac{1}{2}\rangle$$

$$- |-\tfrac{1}{2},\tfrac{1}{2}\rangle)$$

$$\Rightarrow E^1_{10} = \langle 10|H'|10\rangle$$

$$= -\frac{eB_0}{\sqrt{2}mc}(0 - 1 + 1 - 0)$$

$$= 0$$

$$- H^1|1\,-1\rangle = -\frac{eB_0}{mc}(S_z^+ - S_z^-)|-\tfrac{1}{2},-\tfrac{1}{2}\rangle$$

$$= -\frac{eB_0}{mc}(-\tfrac{1}{2}+\tfrac{1}{2})|-\tfrac{1}{2},-\tfrac{1}{2}\rangle$$

$$= 0$$

$$\Rightarrow E^1_{1-1} = 0$$

$$H^1|00\rangle = -\frac{eB_0}{mc}(S_z^+ - S_z^-)$$

$$\frac{1}{\sqrt{2}}(|\tfrac{1}{2},-\tfrac{1}{2}\rangle - |-\tfrac{1}{2},\tfrac{1}{2}\rangle)$$

$$= -\frac{eB_0}{\sqrt{2}mc}(\tfrac{1}{2}|\tfrac{1}{2},-\tfrac{1}{2}\rangle + \tfrac{1}{2}|-\tfrac{1}{2},\tfrac{1}{2}\rangle$$

$$+ \tfrac{1}{2}|\tfrac{1}{2},-\tfrac{1}{2}\rangle + \tfrac{1}{2}|-\tfrac{1}{2},\tfrac{1}{2}\rangle)$$

$$= -\frac{eB_0}{\sqrt{2}mc}(|\tfrac{1}{2},-\tfrac{1}{2}\rangle + |-\tfrac{1}{2},\tfrac{1}{2}\rangle)$$

$$\Rightarrow E^1_{00} = \langle 00|H^1|00\rangle$$

$$= -\frac{eB_0}{\sqrt{2}mc}(1 + 0 - 0 - 1)$$

$$= 0$$

∴ The ground state ${}^1S_0, {}^3S_1$ of the positronium cannot have a linear Zeeman effect.

${}^{2S+1}D_J$ — $2S+1$ - is multiplicity, J is total angular mom.

11. An atom with no permanent magnetic moment is said to be diamagnetic. If one neglects the spin of the electron, what is the induced diamagnetic moment

for a hydrogen atom in its ground state when a weak magnetic field is applied?

Solution

Here we have to find the energy shift due to the applied field. If there is a shift, then this should be due to the fact that a dipole moment is induced in the atom — since we have neglected the spin magnetic moment.

$$H = \frac{1}{2m}\left(\bar{P} - \frac{e\bar{A}}{c}\right)^2 - \frac{e^2}{r}$$

$$= \frac{1}{2m}\left(P^2 - \frac{e}{c}(\bar{P}\cdot\bar{A} + \bar{A}\cdot\bar{P}) + \frac{e^2\bar{A}^2}{c^2}\right) - \frac{e^2}{r}$$

~~For a weak field~~

$$H = \frac{1}{2m}\left(\bar{P}^2 - \frac{e}{c}(\bar{P}\cdot\bar{A} + \bar{A}\cdot\bar{P})\right) - \frac{e^2}{r} + \frac{e^2A^2}{2mc^2}$$

Consider ~~a homogeneous~~ the Coulomb Gauge

$$(\bar{P}\cdot\bar{A} + \bar{A}\cdot\bar{P})\psi$$

$$\to (-i\hbar\,\nabla\cdot\bar{A} - i\hbar\,\bar{A}\cdot\nabla)\psi$$

$$= -i\hbar\,\nabla\cdot(\bar{A}\psi) - i\hbar\,\bar{A}\cdot\nabla\psi$$

$$= -i\hbar\underbrace{\nabla\cdot\bar{A}}_{0} - i\hbar\bar{A}\cdot\nabla\psi - i\hbar\bar{A}\cdot\nabla\psi$$

$$= -2i\hbar\,\bar{A}\cdot\nabla\psi$$

$$\Rightarrow 2\bar{A}\cdot(-i\hbar\nabla\psi)$$

$$\to 2\bar{A}\cdot\bar{P}\,\psi$$

or $\bar{P}\cdot\bar{A} + \bar{A}\cdot\bar{P} = 2\bar{A}\cdot\bar{P}$

$\Longleftrightarrow$ $P\cdot\bar{A} \equiv \bar{A}\cdot\bar{P}$ where $\nabla\cdot\bar{A} = 0$.

$$H = \frac{P^2}{2m} - \frac{e}{mc}\bar{A}\cdot\bar{P} - \frac{e^2}{r} + \frac{e^2A^2}{2mc^2}$$

For a uniform magnetic field, $\bar{A} = \frac{1}{2}\bar{B}\times\bar{r}$

$$H = \frac{P^2}{2m} - \frac{e}{2mc}\bar{B}\times\bar{r}\cdot\bar{P} - \frac{e^2}{r} + \frac{e^2A^2}{2mc^2}$$

$$= \frac{P^2}{2m} - \frac{e}{2mc}\bar{r}\times\bar{P}\cdot\bar{B} - \frac{e^2}{r} + \frac{e^2A^2}{2mc^2}$$

$$= \frac{P^2}{2m} - \frac{e}{2mc}\bar{L}\cdot\bar{B} - \frac{e^2}{r} + \frac{e^2A^2}{2mc^2}$$

$$= \left(\frac{P^2}{2m} - \frac{e^2}{r}\right) - \frac{e}{2mc}\bar{L}\cdot\bar{B} + \frac{e^2A^2}{2mc^2}$$

$$= H_0 - \frac{e}{2mc}\bar{L}\cdot\bar{B} + \frac{e^2A^2}{2mc^2}$$

$$H' = -\frac{e}{2mc}\bar{L}\cdot\bar{B} + \frac{e^2A^2}{2mc^2}$$

For a weak field, $H' = -\frac{e}{2mc}\bar{L}\cdot\bar{B}$

$$\Rightarrow E'_{100} = \langle 100|H'|100\rangle$$

$$-\frac{e}{2mc}\langle 100|\bar{L}|100\rangle\cdot\bar{B}$$

$$= 0$$

As $L_x|100\rangle = 0$

$L_y|100\rangle = 0$

$L_z|100\rangle = 0$

Correspondingly no magnetic moment will be induced. However, suppose we include $\frac{e^2\bar{A}^2}{2mc^2}$ and for a homogeneous field $\bar{A} = \frac{1}{2}\bar{B}\times\bar{r}$ i.e.,

$$H' = -\frac{e}{2mc}\bar{L}\cdot\bar{B} + \frac{e^2(\bar{B}\times\bar{r})^2}{8mc^2}$$

$$E_{10}^{1} = \langle 100|H'|100\rangle$$

$$= \frac{e^2}{8mc^2}\langle 100|(\bar{B}\times\bar{r})^2|100\rangle$$

for $\bar{B}$ along the z-axis

$$|\bar{B}\times\bar{r}| = Br\sin\theta$$

$$E_1^1 = \frac{e^2B^2}{8mc^2}\langle 100|r^2\sin^2\theta|100\rangle$$

$$\psi_{00} = \frac{1}{\sqrt{\pi a_0^3}}e^{-r/a_0}$$

$$E_1^1 = \frac{e^2B^2}{8mc^2(\pi a_0^3)}\int_0^\infty\int_0^\pi\int_0^{2\pi}\left(e^{-r/a_0}(r\sin^2\theta)e^{-r/a}\right.$$

$$\left.\cdot r^2\sin\theta\, dr\, d\theta\, d\varphi\right)$$

$$= \frac{e^2B^2}{4mc^2a_0^3}\int_0^\infty\int_0^\pi e^{-2r/a_0}r^4(1-\cos^2\theta)\, d\cos\theta\, dr$$

$$= \frac{2e^2B^2}{3mc^2a_0^3}\int_0^\infty e^{-2r/a_0}r^4\, dr$$

$$= \frac{2e^2B^2}{3mc^2a_0^3}\cdot\frac{4a_0}{2}\cdot\frac{3a_0}{2}\frac{2a_0}{2}\frac{a_0}{2}\cdot\frac{a_0}{2}$$

$$= \frac{e^2B^2a_0^2}{4mc^2}$$

$\therefore$ the diamagnetic moment $\bar{\mu}$ is

$$\bar{\mu} = -\frac{\partial E_1^1}{\partial \bar{B}} = -\frac{e^2a_0^2\bar{B}}{2mc^2}$$

12. Derive the dipole-dipole magnetic interaction energy of a proton and an antiproton at a fixed distance a in an eigen state of total spin in terms of the proton magnetic moment μ_0. Two magnetic dipoles have the interaction energy

$$V = \frac{1}{r^3}\left\{(\bar{\mu}_1\cdot\bar{\mu}_2) - \frac{3(\bar{\mu}_1\cdot\bar{r})(\bar{\mu}_2\cdot\bar{r})}{r^2}\right\}$$

Solution

Now we would like to find the expectation values of V in the coupled states of the two particles proton-p and anti proton $\bar{p}$.

We know that

$$\bar{\mu}_p = 2\mu_0\bar{S}_p$$

$$\bar{\mu}_{\bar{p}} = -2\mu_0\bar{S}_{\bar{p}}$$

suppose these two particles lie along the z-axis being separated by a distance $r = a$.

$$\bar{\mu}_p\cdot\bar{\mu}_{\bar{p}} = -4\mu_0^2\,\bar{S}_p\cdot\bar{S}_{\bar{p}}$$

the total spin is $\bar{S} = \bar{S}_p + \bar{S}_{\bar{p}}$

$$\Rightarrow S^2 = S_p^2 + S_{\bar{p}}^2 + 2\bar{S}_p\cdot\bar{S}_{\bar{p}}$$

$$\bar{S}_p\cdot\bar{S}_{\bar{p}} = \frac{1}{2}(S^2 - S_p^2 - S_{\bar{p}}^2)$$

or $\bar{\mu}_p\cdot\bar{\mu}_{\bar{p}} = -2\mu_0^2(S^2 - S_p^2 - S_{\bar{p}}^2)$

$$\bar{\mu}_p\cdot\bar{r} = 2\mu_0\bar{S}_p\cdot a\bar{k} = 2\mu_0 a S_{pz}$$

$$\bar{\mu}_{\bar{p}}\cdot\bar{r} = -2\mu_0\bar{S}_{\bar{p}}\cdot a\bar{k} = -2\mu_0 a S_{\bar{p}z}$$

i.e.,

$$V = \frac{1}{a^3}\left\{-2\mu_0^2(S^2 - S_p^2 - S_{\bar{p}}^2) + 12\mu_0^2 S_{pz}S_{\bar{p}z}\right\}$$

Since we are considering two spin-½ particles the states are

$$|\tfrac{1}{2}\tfrac{1}{2}Sm_s\rangle \equiv |Sm_s\rangle$$

$$\Rightarrow (S^2 - S_p^2 - S_{\bar{p}}^2)|\tfrac{1}{2}\tfrac{1}{2}Sm_s\rangle$$

$$= S^2 - \tfrac{1}{2}(\tfrac{1}{2}+1) - \tfrac{1}{2}(\tfrac{1}{2}+1) = S^2 - \tfrac{3}{2}|\tfrac{1}{2}\tfrac{1}{2}Sm_s\rangle$$

or $S^2 - S_p^2 - S_{\bar{p}}^2 \to S^2 - 3/2$

$S_{p_z} S_{\bar{p}_z} = \frac{1}{2}(S_z^2 - S_{p_z}^2 - S_{\bar{p}_z}^2)$

If we go to the uncoupled states $|\tfrac{1}{2} m_{s_p} \tfrac{1}{2} m_{s_{\bar{p}}}\rangle \equiv |m_{s_p} m_{s_{\bar{p}}}\rangle$

$S_{p_z} S_{\bar{p}_z} |m_{s_p} m_{s_{\bar{p}}}\rangle$

$= \frac{1}{2}(S_z^2 - S_{p_z}^2 - S_{\bar{p}_z}^2) |m_{s_p} m_{s_{\bar{p}}}\rangle$

$= \frac{1}{2}(S_z^2 - \frac{1}{4} - \frac{1}{4}) |m_{s_p} m_{s_{\bar{p}}}\rangle$

(since $m_{s_p}, m_{s_{\bar{p}}} = \pm \frac{1}{2}$)

$= \frac{1}{2}(S_z^2 - \frac{1}{2}) |m_{s_p} m_{s_{\bar{p}}}\rangle$

i.e., $S_{p_z} S_{\bar{p}_z} \longrightarrow \frac{1}{2}(S_z^2 - \frac{1}{2})$

$$\therefore V = \frac{1}{a^3}\{-2\mu_0^2 (S^2 - \tfrac{3}{2}) + 6\mu_0^2 (S_z^2 - \tfrac{1}{2})\}$$

$$= \frac{\mu_0^2}{a^3}(6 S_z^2 - 2S^2)$$

$$= -\frac{\mu_0^2}{a^3}(2S^2 - 6S_z^2)$$

Now the coupled states are $|11\rangle, |10\rangle, |1{-1}\rangle, |00\rangle$

$\Rightarrow V_{11} = \langle 11|V|11\rangle$

$$= -\frac{\mu_0^2}{a^3}\langle 11|(2S^2 - 6S_z^2)|11\rangle$$

$$= -\frac{\mu_0^2}{a^3}\{2\cdot 1(1+1) - 6\cdot 1\}, \quad \hbar = 1$$

$$= \frac{2\mu_0^2}{a^3}$$

$V_{10} = \langle 10|V|10\rangle$

$$= -\frac{\mu_0^2}{a^3}\langle 10|(2S^2 - 6S_z^2)|10\rangle$$

$$= -\frac{\mu_0^2}{a^3}\{2\cdot 1(1+1) - 6\cdot 0\}$$

$$= -\frac{4\mu_0^2}{a^3}$$

$V_{1,-1} = \langle 1,-1|V|1,-1\rangle$

$$= -\frac{\mu_0^2}{a^3}\langle 1,-1|(2S^2 - 6S_z^2)|1,-1\rangle$$

$$= -\frac{\mu_0^2}{a^3}\{2\cdot 1(1+1) - 6\cdot 1\}$$

$$= \frac{2\mu_0^2}{a^3}$$

$V_{00} = \langle 00|V|00\rangle$

$$= -\frac{\mu_0^2}{a^3}\langle 00|(2S^2 - 6S_z^2)|00\rangle$$

$$= -\frac{\mu_0^2}{a^3}\cdot 0$$

$$= 0.$$

13. (i) Show that for a free zero rest mass spin ½ particle the wave eq. can be written in the form

$$c(\hat{\sigma}\cdot\hat{p})\psi = i\hbar\frac{\partial\psi}{\partial t}$$

where the comps of $\hat{\sigma}$ are the Pauli matrices and $\hat{p}$ is the linear mom. operator.

(ii). Discuss the conservation of angular mom. in this case.

(iii) Show that the spin of the particle in a positive (negative) energy state is parallel (antiparallel) to its momentum.

Solution

i. In relativity theory the energy of a particle is given by

$$E = c\sqrt{p^2 + m_0^2 c^2}$$

Now we linearize this eq. by demanding that it be given by

$$E = c\sqrt{p^2 + m_0^2 c^2} = c\sum_{\mu=0}^{3} \beta_\mu P_\mu \quad \cdots (*)$$

where

$P_0 = m_0 c$, $P_1 = P_x$, $P_2 = P_y$, $P_3 = P_z$

or $E = c\sqrt{p^2+m_0^2c^2}$

$c(\beta_0 P_0 + \beta_1 P_1 + \beta_2 P_2 + \beta_3 P_3)$

$= c(\beta_0 m_0 c + \beta_1 P_x + \beta_2 P_y + \beta_3 P_z)$

For a particle of rest mass $m_0 = 0$,

$E = cp = c(\beta_1 P_x + \beta_2 P_y + \beta_3 P_z)$

$\Rightarrow c^2p^2 = c^2[\beta_1^2 P_x^2 + \beta_2^2 P_y^2 + \beta_3^2 P_z^2 + (\beta_1\beta_2 + \beta_2\beta_1)P_xP_y + (\beta_2\beta_3 + \beta_3\beta_2)P_yP_z + (\beta_1\beta_3 + \beta_3\beta_1)P_xP_z]$

(Assuming the P_i-s commute.)

or $c^2(P_x^2 + P_y^2 + P_y^2)$

$= c^2(\beta_1^2 P_x^2 + \beta_2 P_y^2 + \beta_3^2 P_z^2 + (\beta_1\beta_2 + \beta_2\beta_1)P_xP_y + (\beta_2\beta_3 + \beta_3\beta_2)P_yP_z + (\beta_1\beta_3 + \beta_3\beta_1)P_xP_z]$

$\Leftrightarrow \beta_1^2 = \beta_2^2 = \beta_3^2 = 1$

$\beta_1\beta_2 + \beta_2\beta_1 = \beta_2\beta_3 + \beta_3\beta_2 = \beta_1\beta_3 + \beta_3\beta_1 = 0$

or $\beta_i^2 = 1, \ i = 1,2,3$

$\beta_i\beta_j + \beta_j\beta_i = 2\delta_{ij}$

which is the property of the pauli spin ½ matrices

$\sigma_1 = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

$\sigma_3 = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\therefore \beta_1 = \sigma_1, \ \beta_2 = \sigma_2, \ \beta_3 = \sigma_3$

$\Rightarrow E = c(\vec{\beta}\cdot\vec{p}) = c(\vec{\sigma}\cdot\vec{p})$

Since $E = i\hbar\frac{\partial}{\partial t}$, the W.E. of spin ½ particle of rest mass zero is

$c(\vec{\sigma}\cdot\vec{p})\psi = i\hbar\frac{\partial\psi}{\partial t}$

(ii).

As can be seen, the Hamiltonian of the particle is $c(\vec{\sigma}\cdot\vec{p})$ and considering orbital motion in addition to the spin, the total angular mom. of the particle is $\vec{J} = \vec{L} + \vec{S}$.

From Ehrenfest's theorem we recall that

$\dot{\Omega} = -\frac{i}{\hbar}[\Omega, H]$

If Ω is conserved, $\dot{\Omega} = 0$ i.e., $[\Omega, H] = 0$.

$J_x = L_x + S_x$

$[J_x, c(\vec{\sigma}\cdot\vec{p})]$

$= [L_x + S_x, c(\vec{\sigma}\cdot\vec{p})]$

$= c[L_x, \vec{\sigma}\cdot\vec{p}] + c[S_x, \vec{\sigma}\cdot\vec{p}]$

$= c[L_x, \sigma_x P_x + \sigma_y P_y + \sigma_z P_z]$

$+ \frac{\hbar c}{2}[\sigma_x, \sigma_x P_x + \sigma_y P_y + \sigma_z P_z], \ \hbar \to 1$

$= ic\sigma_y P_z - ic\sigma_z P_y + \frac{c}{2}(2i\sigma_z P_y) - \frac{c}{2}(2i\sigma_y P_z)$

$= 0$

Similarly

$[J_y, c(\vec{\sigma}\cdot\vec{p})] = 0$

$[J_z, c(\vec{\sigma}\cdot\vec{p})] = 0$

or $\dot{J}_x = \dot{J}_y = \dot{J}_z = 0$.

That means total angular mom. is conserved.

N.B. We have used

$[L_i, \sigma_j] = 0$. Any L_i with any σ_j commutes.

$[L_i, P_i] = 0$

$[L_i, P_j] = i\hbar\, \epsilon_{ijk} P_k$

$[\sigma_i, \sigma_j]_+ = 2\delta_{ij}$

$\sigma_i \sigma_j = i\sigma_k$

($i, j, k = 1, 2, 3$ and cyclic permutation is considered in the last case.)

(iii) $c(\vec{\sigma}\cdot\vec{p})\psi = i\hbar \frac{\partial \psi}{\partial t} = E\psi$

or $c(\vec{\sigma}\cdot\vec{p})\psi = E\psi$

$= \pm pc\psi$, since $E^2 = p^2c^2$

$\vec{\sigma}\cdot\vec{p}\,\psi = \pm p\psi$

$\sigma p \cos\theta = \pm p$

$\sigma \cos\theta = \pm 1$

$\sigma = |\vec{\sigma}| = [\sigma_x^2 + \sigma_y^2 + \sigma_z^2]^{1/2}$

$= \sqrt{3}$

$\cos\theta = \pm 1/\sqrt{3}$

$\therefore \theta = -125.3°$

or $\theta = 54.7°$

On the other hand if $|\sigma| = 1$ (we are considering the z-axis), $\cos\theta = \pm 1$

If $E = pc$, $\cos\theta = 1 \Rightarrow \hat{\sigma} \parallel \vec{p}$

If $E = -pc$, $\cos\theta = -1 \Rightarrow$ spin and mom. are antiparallel.

14. An atom with $j = 1/2$, $m_j = 1/2$ is in a uniform magnetic field. Suddenly the field is rotated by $\theta = 60°$. Find the probability that the atom is in the sublevels $m_j = 1/2$ or $m_j = -1/2$, relative to the new field, immediately after the change in field.

Solution

Initially suppose $\vec{B}$ lies along the z-axis and consider a new coordinate x_r, y_r, z_r rotated by an angle 60° relative to the former coordinate system x, y, z. We claim after the rotation the W.Fs remain the same since the rotation occurs suddenly. In the original frame

$$\vec{J} = \frac{1}{2}\vec{\sigma} = \frac{1}{2}(\sigma_x, \sigma_y, \sigma_z)$$

since we are considering spin ½ particle (as $j = 1/2$)

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Next we find the comp. of $\vec{J}$ along the new orientation ($\hat{n}$) of the field.

$$J_n = \hat{n}\cdot\vec{J}$$

$$= n_x J_x + n_y J_y + n_z J_z$$

$$= \frac{\hbar}{2}\begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\varphi} \\ \sin\theta\, e^{i\varphi} & -\cos\theta \end{pmatrix}$$

(∵ prob. 4.)

Again prob. 4 gives the result that the eigenstates of J_n are

$$|J_n = +\tfrac{\hbar}{2}\rangle = \begin{bmatrix} \cos\theta/2 \\ \sin\theta/2\, e^{i\varphi} \end{bmatrix}$$

$$|J_n = -\tfrac{\hbar}{2}\rangle = \begin{bmatrix} -\sin\frac{\theta}{2}\, e^{-i\varphi} \\ \cos\frac{\theta}{2} \end{bmatrix}$$

Now, the probability that

i. $|+,\hat{k}\rangle \to |+,\hat{n}\rangle$ is (since $m_j = \frac{1}{2}$ our initial state is $|+,\hat{k}\rangle$):

$$|\langle +,\hat{n}|+,\hat{k}\rangle|^2$$

$$= \left|\left(\cos\tfrac{\theta}{2},\ \sin\tfrac{\theta}{2}\, e^{-i\varphi}\right)\begin{bmatrix}1\\0\end{bmatrix}\right|^2$$

$$= \cos^2\tfrac{\theta}{2}$$

$$= \cos^2\tfrac{\pi}{6} = \tfrac{3}{4}$$

ii. $|+,\hat{k}\rangle \to |-,\hat{n}\rangle$ is

$$|\langle -,\hat{n}|+,\hat{k}\rangle|^2$$

$$= \left|\left(-\sin\tfrac{\theta}{2}\, e^{-i\varphi},\ \cos\tfrac{\theta}{2}\right)\begin{bmatrix}1\\0\end{bmatrix}\right|^2$$

$$= \sin^2\tfrac{\theta}{2}$$

$$= \sin^2\tfrac{\pi}{6}$$

$$= \tfrac{1}{4}$$

15. A system consists of two particles, one of spin ½ and one of spin 0. Show that the orbital angular mom. is an integral of motion for any law of interaction between these particles.

Solution (Also, at the back)

For the system since we can find the simultaneous eigenstates of H & J^2, J^2 is well defined with eigenvalue $j(j+1)$.

$$j = l+\tfrac{1}{2},\ l-\tfrac{1}{2}$$

$$\Rightarrow\ l = j-\tfrac{1}{2},\ j+\tfrac{1}{2}$$

But for this system the parity $(-1)^l$ is conserved which means the orbital angular mom. in these states of J^2 & H is well defined.

$$\Leftrightarrow\ [L^2, H] = 0$$

$\therefore L^2$ is conserved.

16. Show that the orbital angular mom. of the motion of two α-particles is always even. ($l = 0, 2, 4, \ldots$)

Solution

A system of two α-particles is bosonic as each of them is a boson ($s=0$). Thus the W.F. of this system is symmetric. Moreover, the parity of the states is even since the state has to be symmetric $(-1)^l = 1$

$\therefore l = 0, 2, 4, \ldots$ (always even)

17. Is it possible for ${}^{8}_{4}Be$ nucleus in an excited state with spin 1 to decay into two α particles.

Solution

$${}^{8}_{4}Be \to \alpha + \alpha$$

$J({}^{8}_{4}Be) = 1$, $l = 0$, at rest

$J(\alpha+\alpha) = l = 0, 2, 4, \ldots$ prob. 16.

i.e., angular mom. is not conserved and hence ${}^{8}_{4}Be$ of $S=1$ cannot decay in this manner.

18. Using the transformation properties of a scalar field ψ under infinitesimal rotation about $\hat{x}$ find the expressions of L_x, L_y, L_z in spherical coordinates.

Solution

Under such a rotation about the x-axis,

$x \to x,\ y \to y - \alpha z,\ z \to z + \alpha y$

to first order in α.

$\delta z = \alpha y = \alpha r \sin\theta \sin\varphi$

Also, $\delta z = \delta(r\cos\theta) = -r\sin\theta\,\delta\theta$

comparison gives

$\delta\theta = -\alpha\sin\varphi$

$\therefore\ \theta \to \theta + \delta\theta = \theta - \alpha\sin\varphi$

Since x is unchanged

$\delta x = \delta(r\sin\theta\cos\varphi) = 0$

$\cos\theta\cos\varphi\,\delta\theta$

$-\sin\theta\sin\varphi\,\delta\varphi = 0$

$\delta\varphi = \cot\theta\cot\varphi\,\delta\theta$

$\therefore\ \varphi \to \varphi + \delta\varphi = \varphi + \cot\theta\cot\varphi\,\delta\theta$

$= \varphi + \cot\theta\cot\varphi\cdot(-\alpha\sin\varphi)$

$= \varphi - \alpha\cot\theta\cos\varphi$

$\Rightarrow\ e^{i\alpha L_x}\psi(r,\theta,\varphi)$

$\approx (1 + i\alpha L_x)\psi(r,\theta,\varphi)$

$= \psi(r,\theta - \alpha\sin\varphi, \varphi - \alpha\cot\theta\cos\varphi)$

$\approx \psi(r,\theta,\varphi) - \alpha\sin\varphi\dfrac{\partial\psi}{\partial\theta} - \alpha\cot\theta\cos\varphi\dfrac{\partial\psi}{\partial\varphi}$

(Again to first order in α)

or $(1 + i\alpha L_x)\psi$

$= \left[1 - \alpha\left(\sin\varphi\dfrac{\partial}{\partial\theta} + \cot\theta\cos\varphi\dfrac{\partial}{\partial\varphi}\right)\right]\psi$

or $L_x = i\left(\sin\varphi\dfrac{\partial}{\partial\theta} + \cot\theta\cos\varphi\dfrac{\partial}{\partial\varphi}\right)$

For infinitesimal rot. along y-axis

Solution

$x \to x + \alpha z$

$y \to y$

$z \to z - \alpha x$

$\delta z = -\alpha x = -\alpha r\sin\theta\cos\varphi$

$\delta z = \delta(r\cos\theta) = -r\sin\theta\,\delta\theta$

$\Rightarrow\ \delta\theta = \alpha\cos\varphi$

$\therefore\ \theta \to \theta + \delta\theta = \theta + \alpha\cos\varphi$

$\delta y = \delta(r\sin\theta\sin\varphi) = 0$

$\cos\theta\sin\varphi\,\delta\theta + \sin\theta\cos\varphi\,\delta\varphi = 0$

$\Rightarrow \delta\varphi = -\cot\theta\tan\varphi\,\delta\theta$

$= -\cot\theta\tan\varphi\,\alpha\cos\varphi$

$= -\alpha\cot\theta\sin\varphi$

$\therefore\ \varphi \to \varphi + \delta\varphi = \varphi - \alpha\cot\theta\sin\varphi$

$\Rightarrow\ e^{i\alpha L_y}\psi(r,\theta,\varphi) = (1 + i\alpha L_y)\psi(r,\theta,\varphi)$

$= \psi(r,\theta + \delta\theta, \varphi + \delta\varphi)$

$= \psi(r,\theta + \alpha\cos\varphi, \varphi - \alpha\cot\theta\sin\varphi)$

$\approx \psi(r,\theta,\varphi) + \alpha\cos\varphi\dfrac{\partial\psi}{\partial\theta} - \alpha\cot\theta\sin\varphi\dfrac{\partial\psi}{\partial\varphi}$

$\Rightarrow\ L_y = -i\left(\cos\varphi\dfrac{\partial}{\partial\theta} - \cot\theta\sin\varphi\dfrac{\partial}{\partial\varphi}\right)$

For rotation about z by infinitesimal α

$x \to x + \alpha y,\ y \to y - \alpha x,\ z \to z$

$\delta x = \alpha y = \alpha r\sin\theta\sin\varphi$

$\delta x = \delta(r \sin\theta \cos\varphi)$

$= r\cos\theta\cos\varphi\,\delta\theta$

$- r\sin\theta\sin\varphi\,\delta\varphi$

but $\delta z = \delta(r\cos\theta) = 0$

$= -r\sin\theta\,\delta\theta = 0$

$\Rightarrow \delta\theta = 0$

(i.e.) $\delta x = \alpha r \sin\theta \sin\varphi$

$= -r\sin\theta\sin\varphi\,\delta\varphi$

or $\delta\varphi = -\alpha$

Now, $\theta + \delta\theta = \theta$

$\varphi + \delta\varphi = \varphi - \alpha$

$e^{i\alpha L_z}\psi(r,\theta,\varphi)$

$= \psi(r,\theta,\varphi-\alpha)$

$(1 + i\alpha L_z)\psi(r,\theta,\varphi)$

$= \psi(r,\theta,\varphi) - \alpha\frac{\partial\psi}{\partial\varphi}$

$\Rightarrow L_z = i\frac{\partial}{\partial\varphi}$

19. Prove the following commutation relations

a, $[\hat{L}_j, \hat{x}_k] = i\epsilon_{jk\ell}\hat{x}_\ell$

b, $[\hat{L}_j, \hat{P}_k] = i\epsilon_{jk\ell}\hat{P}_\ell$

1,2,3 — x,y,z and $\epsilon_{jk\ell}$ is the Levi-civita symbol.

Solution

a, — $[L_1, x_1]$

$= [(x_2P_3 - x_3P_2), x_1]$

$= [x_2P_3, x_1] - [x_3P_2, x_1]$

$= x_2[P_3, x_1] + [x_2, x_1]P_3$

$- x_3[P_2, x_1] - [x_3, x_1]P_2$

Going to the coordinate basis and making these operators commutators act on a function we can show

$[x_2, x_1] = 0, [x_3, x_1] = 0$

$[P_2, x_2] = -i, \hbar \to 1$

$[P_3, x_1]\psi$

$= (P_3x_1 - x_1P_3)\psi$

$\to -i\frac{\partial}{\partial x_3}(x_1\psi) + ix_1\frac{\partial\psi}{\partial x_3}$

$= -i\frac{\partial x_1}{\partial x_3} - ix_1\frac{\partial\psi}{\partial x_3} + ix_1\frac{\partial\psi}{\partial x_3}$ (with $\frac{\partial x_1}{\partial x_3} = 0$)

$= 0$

$\Rightarrow [P_3, x_1] = 0$, Also $[P_2, x_1] = 0$

(i.e.) $[L_1, x_1] = 0$

— $[L_1, x_2]$

$= [x_2P_3 - x_3P_2, x_2]$

$= [x_2P_3, x_2] + [x_3P_2, x_2]$

$= x_2[P_3, x_2] + [x_2, x_2]P_3$

$- x_3[P_2, x_2] - [x_3, x_2]P_2$

$= 0 + 0 - x_3(-i) - 0$

$= ix_3$

$[L_1, x_3] = [x_2P_3 - x_3P_2, x_3]$

$= [x_2P_3, x_3] - [x_3P_2, x_3]$

$= x_2[P_3, x_3] + [x_2, x_3]P_3$

$- x_3[P_2, x_3] - [x_3, x_3]P_2$

$[L_1, X_3]$

$= X_2(-i) + 0 - 0 - 0$

$= -i X_2$

$- [L_2, X_2] = 0$, similarly

$[L_2, X_1]$

$= [X_3 P_1 - X_1 P_3, X_1]$

$= [X_3 P_1, X_1] - [X_1 P_3, X_1]$

$= X_3 [P_1, X_1] + [X_3, X_1] P_1$
$- X_1 [P_3, X_1] - [X_1, X_1] P_3$

$= -i X_3 + 0 - 0 - 0$

$= -i X_3$

Also, we can find

$[L_2, X_3] = i X_1$

$[L_3, X_1]$

$= [X_1 P_2 - X_2 P_1, X_1]$

$= [X_1 P_2, X_1] - [X_2 P_1, X_1]$

$= X_1 [P_2, X_1] + [X_1, X_1] P_2$
$- X_2 [P_1, X_1] - [X_2, X_1] P_1$

$= 0 + 0 - X_2(-i) - 0$

$= i X_2$

It is also possible to show

$[L_3, X_2] = -i X_1$

$[L_3, X_3] = 0$

In general

$[L_j, X_k] = i \epsilon_{jkl} X_l$

b, $[L_1, P_1]$

$= [X_2 P_3 - X_3 P_2, P_1]$

$= [X_2 P_3, P_1] - [X_3 P_2, P_1]$

$= X_2 [P_3, P_1] + [X_2, P_1] P_3$
$- X_3 [P_2, P_1] - [X_3, P_1] P_2$

$= 0 + 0 - 0 - 0$

$= 0$

$[L_1, P_2]$

$= [X_2 P_3 - X_3 P_2, P_2]$

$= X_2 [P_3, P_2] + [X_2, P_2] P_3$
$- X_3 [P_2, P_2] - [X_3, P_2] P_2$

$= 0 + i P_3 - 0 - 0$

$= i P_3$

$[L_1, P_3]$

$= [X_2 P_3 - X_3 P_2, P_3]$

$= X_2 [P_3, P_3] + [X_2, P_3] P_3$
$- X_3 [P_2, P_3] - [X_3, P_3] P_2$

$= 0 + 0 - 0 - i P_2$

$= -i P_2$

$[L_2, P_1]$

$= [X_3 P_1 - X_1 P_3, P_1]$

$= X_3 [P_1, P_1] + [X_3, P_1] P_1$
$- X_1 [P_3, P_1] - [X_1, P_1] P_3$

$= 0 + 0 - 0 - i P_3$

$= -i P_3$

Similarly we can show

$[L_2, P_2] = 0$, $[L_2, P_3] = iP_1$

$[L_3, P_1] = iP_2$, $[L_3, P_2] = -iP_1$

$[L_3, P_3] = 0$

And in general

$$[L_j, P_k] = i\epsilon_{jkl} P_l$$

20. Prove

a, $[\vec{L}, P_x^2 + P_y^2 + P_z^2] = 0$

b, $[\vec{L}, x^2 + y^2 + z^2] = 0$

Solution

a. $[\vec{L}, P_x^2]$

$= [\hat{i} L_x + \hat{j} L_y + \hat{k} L_z, P_x^2]$

$= \hat{i}[L_x, P_x^2] + \hat{j}[L_y, P_x^2] + \hat{k}[L_z, P_x^2]$

$= \hat{i}(P_x[L_x, P_x] + [L_x, P_x]P_x)$

$+ \hat{j}(P_x[L_y, P_x] + [L_y, P_x]P_x)$

$+ \hat{k}(P_x[L_z, P_x] + [L_z, P_x]P_x)$

$= 2(-\hat{j} P_x P_z + \hat{k} P_x P_y)$ - from 19 b

$[\vec{L}, P_y^2]$

$= [\hat{i} L_x + \hat{j} L_y + \hat{k} L_z, P_y^2]$

$= \hat{i}[L_x, P_y^2] + \hat{j}[L_y, P_y^2]$

$+ \hat{k}[L_z, P_y^2]$

$= \hat{i}(P_y[L_x, P_y] + [L_x, P_y]P_y)$

$+ \hat{j}(P_y[L_y, P_y] + [L_y, P_y]P_y)$

$+ \hat{k}(P_y[L_z, P_y] + [L_z, P_y]P_y)$

$= 2(\hat{i} P_y P_z - \hat{k} P_x P_y)$ - from 19 b.

We may also find

$$[\vec{L}, P_z^2] = 2(\hat{j} P_x P_z - \hat{i} P_y P_z)$$

$\Rightarrow [\vec{L}, P_x^2 + P_y^2 + P_z^2] = 0$

b, $[\vec{L}, x^2 + y^2 + z^2]$

$= [\vec{L}, x^2] + [\vec{L}, y^2] + [\vec{L}, z^2]$

$[\vec{L}, x^2] = x[\vec{L}, x] + [\vec{L}, x]x$

$= \hat{j}(x[L_y, x] + [L_y, x]x)$

$+ \hat{k}(x[L_z, x] + [L_z, x]x)$

$= 2(\hat{k} xy - \hat{j} xz)$

$[\vec{L}, y^2] = y[\vec{L}, y] + [\vec{L}, y]y$

$= \hat{i}(y[L_x, y] + [L_x, y]y)$

$+ \hat{k}(y[L_z, y] + [L_z, y]y)$

$= 2(\hat{i} yz - \hat{k} xy)$

Also, $[\vec{L}, z^2] = 2(\hat{j} xz - \hat{i} yz)$

$\Rightarrow [\vec{L}, x^2 + y^2 + z^2] = 0$

21. Find the transformation of the spherical harmonics Y_{11}, Y_{10} & Y_{1-1} under a rotation of the coordinate system through the Euler angles θ, ψ, Φ.

Solution

Since $j = 1$, the W.Fs are two Y_{1m}s and the corresponding state vectors are $|1m\rangle$s, $m = 1, 0, -1$. So now we find the transformation matrix $d^1(\vec{\alpha})$ in this basis.

$$d^1(\vec{\alpha}) \rightarrow \begin{pmatrix} \langle 11|d^1(\vec{\alpha})|11\rangle & \langle 11|d^1(\vec{\alpha})|10\rangle & \langle 11|d^1(\vec{\alpha})|1-1\rangle \\ \langle 10|d^1(\vec{\alpha})|11\rangle & \langle 10|d^1(\vec{\alpha})|10\rangle & \langle 10|d^1(\vec{\alpha})|1-1\rangle \\ \langle 1-1|d^1(\vec{\alpha})|11\rangle & \langle 1-1|d^1(\vec{\alpha})|10\rangle & \langle 1-1|d^1(\vec{\alpha})|1-1\rangle \end{pmatrix}$$

In general $d^1_{mm'}(\Phi, \theta, \psi) = e^{-im\Phi} e^{-im'\psi} d^1_{mm'}(\theta)$

But $d^1_{m m'}(\theta)$

$$= (-)^{1-m'} \sqrt{\frac{(1+m')!}{(1-m')!(1+m)!(1-m)!}}$$

$$\cdot \left(\sin\frac{\theta}{2}\right)^{m-m'} \left(\cos\frac{\theta}{2}\right)^{-m-m'}$$

$$\left[\left(\frac{d}{dt}\right)^{1-m'} \left[t^{1+m} (1-t)^{1-m} \right] \right]_{t=\cos^2\frac{\theta}{2}}$$

$m, m' = 1, 0, -1$

$$d^1_{11}(\theta) = (-)^0 \sqrt{\frac{(1+1)!}{0!(1+1)!\,0!}} \cdot \left(\sin\frac{\theta}{2}\right)^0 \left(\cos\frac{\theta}{2}\right)^0$$

$$\cdot \left(\frac{d}{dt}\right)^0 \left[t^2 (1-t)^0 \right]\Big|_{t=\cos^2\frac{\theta}{2}}$$

$$= \cos^2\frac{\theta}{2}$$

$$\Rightarrow d^1_{11}(\Phi,\theta,\Psi) = e^{-i(\Phi+\Psi)} \cos^2\frac{\theta}{2}$$

$$d^1_{01}(\theta) = (-)^0 \sqrt{\frac{2}{0!\,1!\,1!}} \frac{1}{\sin\frac{\theta}{2}} \cdot \frac{1}{\cos\theta/2} \left(\cos^2\frac{\theta}{2} \sin^2\frac{\theta}{2} \right)$$

$$= \frac{\sqrt{2}\sin\theta}{2} = \frac{\sin\theta}{\sqrt{2}}$$

$$\Rightarrow d^1_{01} = e^{-i\Psi} \frac{\sin\theta}{\sqrt{2}}$$

$$d^1_{-11}(\theta) = (-)^0 \sqrt{\frac{2!}{0!\,0!\,2!}} \frac{1}{\sin^2\frac{\theta}{2}} \left(\cos\theta/2\right)^0 \sin^4\theta/2$$

$$= \sin^2\theta/2$$

$$\Rightarrow d^1_{-11}(\Phi,\theta,\Psi) = e^{i(\Phi-\Psi)} \sin^2\frac{\theta}{2}$$

$$d^1_{10}(\theta) = -\sqrt{\frac{1!}{1!\,2!\,0!}} \sin\frac{\theta}{2} \cdot \frac{1}{\cos\frac{\theta}{2}}$$

$$\left[2\cos^2\theta/2 \right]$$

$$= -\frac{\sin\theta}{\sqrt{2}}$$

$$\Rightarrow d^1_{10}(\Phi,\theta,\Psi) = -e^{-i\Phi} \frac{\sin\theta}{\sqrt{2}}$$

$$d^1_{1,-1}(\theta) = (-)^2 \sqrt{\frac{0!}{2!\,2!\,0!}} \sin^2\frac{\theta}{2} \left(\cos\frac{\theta}{2}\right)$$

$$\frac{d^2}{dt^2}\left[t^2 \right]_{t=\cos^2\frac{\theta}{2}}$$

$$= \frac{2\sin^2\frac{\theta}{2}}{2} = \sin^2\frac{\theta}{2}$$

$$\Rightarrow d^1_{1,-1}(\Phi,\theta,\Psi) = e^{-i(\Phi-\Psi)} \sin^2\frac{\theta}{2}$$

$$d^1_{00}(\theta) = -\sqrt{\frac{1!}{1!\,1!\,1!}} \left(\sin\frac{\theta}{2}\right)^0 \left(\cos\frac{\theta}{2}\right)^0$$

$$\left(1 - 2\cos^2\theta/2\right)$$

$$= \cos\theta$$

$$\Rightarrow d^1_{00}(\Phi,\theta,\Psi) = \cos\theta$$

$$d^1_{-10} = -\sqrt{\frac{1!}{1!\,0!\,2!}} \frac{1}{\sin\theta/2} \cos\theta/2$$

$$\left(-2(1-\cos^2\theta/2)\right)$$

$$= \frac{\sin\theta}{\sqrt{2}}$$

$$\Rightarrow d^1_{-10}(\Phi,\theta,\Psi) = e^{i\Phi} \frac{\sin\theta}{\sqrt{2}}$$

$$d^1_{1,-1}(\theta) = (-)^2 \sqrt{\frac{0!}{2!\,2!\,0!}} \sin^2\frac{\theta}{2}$$

$$\cdot 2$$

$$= \sin^2\theta/2$$

$$\Rightarrow d^1_{1,-1} = e^{-i(\Phi-\Psi)} \sin^2\frac{\theta}{2}$$ (Already done)

$$d^1_{0,-1} = (-)^2 \sqrt{\frac{0!}{2!\,1!\,1!}} \sin^2\theta/2 \, \cos^m\frac{\theta}{2}$$

$$(-2)$$

$$= -\frac{\sin\theta}{\sqrt{2}}$$

$$\Rightarrow d^1_{0,-1}(\Phi,\theta,\Psi) = -e^{i\Psi} \frac{\sin\theta}{\sqrt{2}}$$

$$d^1_{-1,-1}(\theta) = (-1)^2 \sqrt{\frac{0!}{2!\,0!\,2!}}\ \cos^2\frac{\theta}{2}$$

$$(-2) = \cos^2\theta/2$$

$$\Rightarrow d^1_{-1,-1}(\Phi,\theta,\psi) = e^{i(\Phi+\psi)} \cos^2\frac{\theta}{2}$$

i.e., the transformation matrix becomes

$$d^1(\Phi,\theta,\psi) = \begin{bmatrix} e^{-i(\varphi+\psi)}\cos^2\frac{\theta}{2} & -\frac{e^{i\varphi}\sin\theta}{\sqrt2} & e^{-i(\varphi-\psi)}\sin^2\frac{\theta}{2} \\ \frac{e^{-i\psi}\sin\theta}{\sqrt2} & \cos\theta & -\frac{e^{i\psi}\sin\theta}{\sqrt2} \\ e^{i(\varphi-\psi)}\sin^2\frac{\theta}{2} & \frac{e^{i\varphi}\sin\theta}{\sqrt2} & e^{i(\varphi+\psi)}\cos^2\frac{\theta}{2} \end{bmatrix}$$

And the spherical harmonics transform as

$$\begin{bmatrix} e^{-i(\varphi+\psi)}\cos^2\frac{\theta}{2} & -\frac{e^{i\varphi}\sin\theta}{\sqrt2} & e^{-i(\varphi-\psi)}\sin^2\frac{\theta}{2} \\ \frac{e^{-i\psi}\sin\theta}{\sqrt2} & \cos\theta & -\frac{e^{i\psi}\sin\theta}{\sqrt2} \\ e^{+i(\varphi-\psi)}\sin^2\frac{\theta}{2} & \frac{e^{i\varphi}\sin\theta}{\sqrt2} & \cos^2\frac{\theta}{2} \end{bmatrix}\begin{bmatrix} Y_{11} \\ Y_{10} \\ Y_{1,-1} \end{bmatrix} = \begin{bmatrix} Y'_{11} \\ Y'_{10} \\ Y'_{1,-1} \end{bmatrix}$$

22. Solve and classify the eigenvalues of the Hamiltonian

$$H = A(\sigma_z^1 + \sigma_z^2) + B\,\vec\sigma^1\cdot\vec\sigma^2$$

Where σ_1 and σ_2 are the pauli spin matrices for the particles (1) & (2) resp. (The pauli principle is not considered.)

Solution

We now consider the coupling of two angular momenta with $\hbar \to 1$

$$\sigma_z^1 + \sigma_z^2 = 2(S_z^1 + S_z^2) = 2S_z$$

$$\vec\sigma^1\cdot\vec\sigma^2 = \frac{1}{2}(\sigma^2 - \sigma_1^2 - \sigma_2^2)$$

$$= \frac{1}{2}(4S^2 - 4S_1^2 - 4S_2^2)$$

$$= 2(S^2 - S_1^2 - S_2^2)$$

$$\Rightarrow H = 2AS_z + 2B(S^2 - S_1^2 - S_2^2)$$

$$= 2AS_z + 2B(S^2 - 3/2)$$

The possible states of this system are $|11\rangle, |10\rangle, |1,-1\rangle, |00\rangle$

$$H|11\rangle = 2A + 2B[1(1+1) - \tfrac{1}{2}(\tfrac{1}{2}+1) - \tfrac{1}{2}(\tfrac{1}{2}+1)]$$

$$= 2A + B$$

$$H|10\rangle = 2B[1(1+1) - \tfrac{1}{2}(\tfrac{1}{2}+1) - \tfrac{1}{2}(\tfrac{1}{2}+1)]$$

$$= B$$

$$H|1,-1\rangle = -2A + 2B[1(1+1) - \tfrac{1}{2}(\tfrac{1}{2}+1) - \tfrac{1}{2}(\tfrac{1}{2}+1)]$$

$$= -2A + B$$

$$H|00\rangle = 2B(0 - \tfrac{1}{2}(\tfrac{1}{2}+1) - \tfrac{1}{2}(\tfrac{1}{2}+1))$$

$$= -3B$$

23. If one considers systems capable of emitting particles of half integral spin, one encounters operators v obeying the commutation relations

1. $[v, J_z] = \frac{1}{2}v$

2. $[[v, J^2], J^2] = \frac{1}{2}(vJ^2 + J^2 v) + \frac{3v}{16}$

Where J is the angular mom. of the emitting system. Find selection rules following from (1) and (2), in a matrix representation which makes J_z and J^2 diagonal

(eigenvalues m & $j(j+1)$ resp.)
In other words, what matrix elements $\langle m'j'|v|mj\rangle$ can be nonzero.

[Hint: let $X_j = j(j+1)$]

Solution

1. Let us find the matrix elements of v

$\langle j'm'|v|jm\rangle$

$= \langle j'm'|2[v, J_z]|m_j\rangle$

$= 2\langle j'm'|(vJ_z - J_z v)|jm\rangle$

$= 2(m - m')\langle j'm'|v|jm\rangle$

$\Rightarrow [2(m-m') - 1]\langle j'm'|v|jm\rangle = 0$

$\langle j'm'|v|jm\rangle = 0$

unless $2(m - m') - 1 = 0$

or $m' = m - \frac{1}{2} \Leftarrow m = m' + \frac{1}{2}$

So this is the selection rule in the first case.

2. Let us take the matrix element of the commutator in (2)

$\langle j'm'|[[v, J^2], J^2]|jm\rangle$

$= \frac{1}{2}\langle j'm'|(\frac{1}{2}(vJ^2 + J^2 v) + \frac{3}{16}v)|jm\rangle$

$\Rightarrow \langle j'm'|(vJ^2J^2 - J^2vJ^2)|jm\rangle$

$- \langle j'm'|(J^2vJ^2 - J^2J^2v)|jm\rangle$

$= \frac{1}{2}j(j+1)\langle j'm'|v|jm\rangle$

$+ \frac{1}{2}j'(j'+1)\langle j'm'|v|jm\rangle$

$+ \frac{3}{16}\langle j'm'|v|jm\rangle$

or $j(j+1)j(j+1)\langle j'm'|v|jm\rangle$

$- j'(j'+1)j(j+1)\langle j'm'|v|jm\rangle$

$- j(j+1)j'(j'+1)\langle j'm'|v|jm\rangle$

$+ j'(j'+1)j'(j'+1)\langle j'm'|v|jm\rangle$

$= [\frac{1}{2}j(j+1) + \frac{1}{2}j'(j'+1) + \frac{3}{16}]\langle j'm'|v|jm\rangle$

with $X_j = j(j+1)$, $X_{j'} = j'(j'+1)$

$(X_j^2 - 2X_jX_{j'} + X_{j'}^2)\langle j'm'|v|jm\rangle$

$= (\frac{1}{2}(X_j + X_{j'}) + \frac{3}{16})\langle j'm'|v|jm\rangle$

$\Rightarrow \langle j'm'|v|jm\rangle = 0$

unless $2(X_j^2 - 2X_jX_{j'} + X_{j'}^2) - X_j - X_{j'} - \frac{3}{8} = 0$

$2(X_{j'} - X_j)^2 - X_j - X_{j'} - \frac{3}{8} = 0$

$2X_{j'}^2 - (4X_j + 1)X_{j'} + 2X_j^2 - X_j - \frac{3}{8} = 0$

$X_{j'} = \frac{4X_j + 1 \pm \sqrt{(4X_j+1)^2 - 8(2X_j^2 - X_j - \frac{3}{8})}}{4}$

$= \frac{4j(j+1) + 1 \pm \sqrt{16j^2(j+1)^2 + 8j(j+1) + 1 - 16j^2(j+1)^2 + 8j(j+1) + 3}}{4}$

$= \frac{4j(j+1) + 1 \pm \sqrt{16j(j+1) + 4}}{4}$

or $j'(j'+1) = \frac{4j(j+1) + 1 \pm 2\sqrt{4j(j+1) + 1}}{4}$

So this is the selection rule to be fulfilled.

$j'(j'+1) = j(j+1) \pm \frac{1}{2}\sqrt{4j(j+1) + 1}$

24. A system consists of two distinguishable particles

each with intrinsic spin $\frac{1}{2}$. The spin-spin interaction of the particles is $J\vec{\sigma}_1\cdot\vec{\sigma}_2$, where J is a constant. An external magnetic field $\vec{H}$ is applied. The magnetic moment of the two particles are $\alpha\vec{\sigma}_1$ & $\beta\vec{\sigma}_2$. Find the exact energy eigenvalues of this system.

Solution

The Hamiltonian of this system is

$$H = -(\alpha\vec{\sigma}_1 + \beta\vec{\sigma}_2)\cdot\vec{H} + J\,\vec{\sigma}_1\cdot\vec{\sigma}_2$$

Suppose $\vec{H} = B_0\vec{k}$

$$\Rightarrow H = -(\alpha\sigma_{1z} + \beta\sigma_{2z})H_0 + J\,\vec{\sigma}_1\cdot\vec{\sigma}_2$$

$$= -2(\alpha S_{1z} + \beta S_{2z})B_0 + 4J\,\vec{S}_1\cdot\vec{S}_2$$

$$= -2B_0(\alpha S_{1z} + \beta S_{2z}) + 2J(S^2 - S_1^2 - S_2^2)$$

$\alpha S_{1z} + \beta S_{2z}$

$$= \frac{\alpha S_{1z} + \beta S_{2z}}{2} + \frac{\alpha S_{1z} + \beta S_{2z}}{2} + \frac{\beta S_{1z} - \beta S_{1z}}{2} + \frac{\alpha S_{2z} - \alpha S_{2z}}{2}$$

$$= \frac{\alpha}{2}(S_{1z} + S_{2z}) + \frac{\beta}{2}(S_{1z} + S_{2z}) + \frac{\alpha}{2}(S_{1z} - S_{2z}) - \frac{\beta}{2}(S_{1z} - S_{2z})$$

$$= \frac{\alpha+\beta}{2}\underbrace{(S_{1z} + S_{2z})}_{S_z} + \frac{\alpha-\beta}{2}(S_{1z} - S_{2z})$$

$$\Rightarrow H = -H_0(\alpha+\beta)(S_{1z} + S_{2z}) - H_0(\alpha-\beta)(S_{1z} - S_{2z}) + 2J(S^2 - S_1^2 - S_2^2)$$

Going to the eigenbasis of S^2, S_1^2, S_2^2 & S_z we obtain

$S_1^2 = S_2^2 = \frac{1}{2}(\frac{1}{2}+1)$

$$H = 2J(S^2 - \tfrac{3}{2}) - B_0(\alpha+\beta)S_z - B_0(\alpha-\beta)(S_{1z} - S_{2z})$$

The coupled states of the system are the triplets and the singlet which have got even & odd parity resp. Thus the matrix elements of the last term (which is odd under exchange) vanish in these states. We may find non zero elements only between the triplet and singlet states.

$$(S_{1z} - S_{2z})|00\rangle$$

$$= (S_{1z} - S_{2z})\frac{1}{\sqrt{2}}(|\tfrac{1}{2},-\tfrac{1}{2}\rangle - |-\tfrac{1}{2},\tfrac{1}{2}\rangle)$$

$$= \frac{1}{\sqrt{2}}\cdot\frac{1}{2}(|\tfrac{1}{2},-\tfrac{1}{2}\rangle + |-\tfrac{1}{2},\tfrac{1}{2}\rangle) + \frac{1}{\sqrt{2}}\cdot\frac{1}{2}(|\tfrac{1}{2},-\tfrac{1}{2}\rangle + |-\tfrac{1}{2},\tfrac{1}{2}\rangle)$$

$$= \frac{1}{\sqrt{2}}(|\tfrac{1}{2},-\tfrac{1}{2}\rangle + |-\tfrac{1}{2},\tfrac{1}{2}\rangle)$$

$$= |10\rangle$$

$\Rightarrow \langle 10|H|00\rangle = -H_0(\alpha-\beta)$

(Which is the only nonzero matrix element of the last term in H.)

$$\langle 11|H|11\rangle = J - B_0(\alpha+\beta)$$

$$\langle 10|H|10\rangle = J$$

$$\langle 1,-1|H|1,-1\rangle = J + B_0(\alpha+\beta)$$

$$\langle 00|H|00\rangle = -3J$$

(As seen the last term in H does not contribute to the eigenvalues of the states with $S=1$ and $S_z = \pm 1$ and they are eigenstates with eigenvalues $E_\pm = \mp(\alpha+\beta)B_0 + J$.) For those states with $S_z = 0$ we have

$$\langle 00|H|00\rangle = -3J$$

$$\langle 10|H|00\rangle = \langle 00|H|10\rangle = -B_0(\alpha-\beta)$$

$$\langle 10|H|10\rangle = J$$

Since $|10\rangle$ & $|00\rangle$ are not eigenstates of the last term of H we'll diagonalize H in these states.

$$\Rightarrow H(S_z=0)$$

$$= \begin{array}{c} \\ S=0 \\ S=1 \end{array}\begin{array}{c} S=0 \quad\quad\quad S=1 \\ \begin{pmatrix} -3J & -B_0(\alpha-\beta) \\ -B_0(\alpha-\beta) & J \end{pmatrix} \end{array}$$

To find the eigenvalues

$$\begin{vmatrix} -3J-\lambda & -B_0(\alpha-\beta) \\ -B_0(\alpha-\beta) & J-\lambda \end{vmatrix} = 0$$

$$(3J+\lambda)(J-\lambda) + B_0^2(\alpha-\beta)^2 = 0$$

$$3J^2 - 2J\lambda - \lambda^2 + B_0^2(\alpha-\beta)^2 = 0$$

$$-\lambda^2 - 2J\lambda + 3J^2 + B_0^2(\alpha-\beta)^2 = 0$$

i.e. $$\lambda = \frac{2J \pm \sqrt{4J^2 + 4[3J^2 + B_0^2(\alpha-\beta)^2]}}{-2}$$

or The eigenvalue of the Hamiltonian becomes

$$E_{\pm}(S_z=0) = -J \pm \sqrt{4J^2 + B_0^2(\alpha-\beta)^2}$$

25. A particle of mass m moves in a 2-D square potential well

$$V(x) = 0 \quad \text{for } |x|, |y| < a$$
$$= \infty \quad \text{otherwise}$$

Determine the expectation values of the coordinate operators x, y for the ground state when a small perturbation $V = F_1 x + F_2 y$ (F_1, F_2 constant) is applied. Consider terms only to first order in F_1 and F_2. You need not compute the final matrix elements. It is sufficient to express them in integral form and state explicitly which ones are not zero.

Solution

From a time independent perturbation theory we have to first order

$$|n\rangle = |n^0\rangle + |n^1\rangle$$

$$\text{where } |n^1\rangle = \sum_{m\neq n} \frac{|m^0\rangle\langle m^0|V|n^0\rangle}{E_n^0 - E_m^0}$$

To the ground state

$$|0\rangle = |0^0\rangle + \sum_{m\neq 0} \frac{|m^0\rangle\langle m^0|V|0^0\rangle}{E_0^0 - E_m^0}$$

$$= |0^0\rangle + \sum_{m\neq 0} \frac{|m^0\rangle V_{m0}}{E_0^0 - E_m^0} \quad \cdots (1)$$

$$\text{where } V_{m0} = \langle m^0|(F_1 X + F_2 Y)|0^0\rangle$$
$$= F_1\langle m^0|X|0^0\rangle + F_2\langle m^0|Y|0^0\rangle$$
$$= F_1 X_{m0} + F_2 Y_{m0} \quad \cdots (2)$$

from (1)

$$\langle 0|X|0\rangle = \langle 0^0|X|0^0\rangle + \sum_{m\neq 0} \frac{\langle 0^0|X|m^0\rangle V_{m0}}{E_0^0 - E_m^0}$$

We have used the fact that $\langle m^0|X|0^0\rangle$ are real since we have chosen real eigen functions.

Treating X as a vector operator and since the ground state has a definite (even) parity, the first term is zero. Thus to first order in F's.

$$\text{or } X_{00} = 2\sum_{m\neq 0} \frac{X_{0m} V_{m0}}{E_0^0 - E_m^0}$$

$$= 2\sum_{m\neq 0} \frac{F_1 X_{m0} X_{0m} + F_2 Y_{m0} X_{0m}}{E_0^0 - E_m^0} \quad \cdots (3)$$

$$\psi_0 = \frac{1}{a}\cos\left(\frac{\pi x}{2a}\right)\cos\left(\frac{\pi y}{2a}\right)$$

The 2nd term in (3) is always zero $Y_{m0} X_{0m} = 0$ since if $X_{m0} \neq 0$

then $y_{mo} = 0$ or vice versa.

Finally choosing real wave functions $X_{mo} = \langle m|X|0\rangle = \langle 0|X|m\rangle = X_{om}$

$$\langle 0|X|0\rangle = 2F_1 \sum_{m\neq 0} \frac{X_{om}^2}{E_0^0 - E_m^0}$$

and $$\langle 0|Y|0\rangle = 2F_2 \sum \frac{Y_{om}^2}{E_0^0 - E_m^0}$$

X_{om} is nonvanishing for m even; i.e., $m = 2n$ – n integer so that

$$X_{om} = X_{0,2n}$$

$$= \frac{1}{a}\int_{-a}^{a} x \sin\frac{\pi 2nx}{2a} \cos\frac{\pi x}{2a}\,dx$$

$$= \frac{1}{a}\int_{-a}^{a} x \sin\frac{\pi n x}{a} \cos\frac{\pi x}{2a}\,dx$$

and $$E_m^0 = E_{2n} = \frac{\pi^2(2n)^2}{8ma^2}$$

$$= \frac{4\pi n^2}{8ma^2}$$

$$E_0^0 = \frac{\pi^2}{8ma^2}$$

$$\therefore E_0 - E_m = \frac{\pi^2}{8ma^2}(1-4n^2)$$

26. Tritium (the isotope H^3 of hydrogen) undergoes spontaneous beta decay, emitting an electron of maximum energy about 17 KeV. The nucleus remaining is He^3. Calculate the probability that an electron of this energy ion is left in a quantum state of principal q. no. 2. Neglect nuclear recoil, and assume that the tritium atom was initially in its ground state.

Solution

Since we have spontaneous decay, the sudden perturbation approximation holds true. That is the state of the atomic (bound) electron is the same just before and just after the beta decay.

Thus if the initial state is $n=1, Z=1$, the amplitude for finding the electron in the state $n=2, Z=2$ immediately after the decay is

$$\langle n=2, Z=2 | n=1, Z=1\rangle$$

And the normalized states are

$$|n=1, Z=1\rangle = \frac{1}{\sqrt{\pi a^3}}\, e^{-r/a}$$

$$|n=2, Z=2\rangle = \frac{1}{\sqrt{\pi a^3}}\, e^{-r/a}(1 - r/a)$$

$$\Rightarrow \langle n=2, Z=2 | n=1, Z=1\rangle$$

$$= \frac{1}{\pi a^3}\int_0^\infty (1 - r/a)\, e^{-2r/a}\, 4\pi r^2\,dr$$

$$= \frac{4}{a^3}\left\{\int_0^\infty r^2 e^{-\frac{2r}{a}} - \frac{1}{a}\int_0^\infty r^3 e^{-\frac{2r}{a}}\,dr\right\}$$

$$= \frac{4}{a^3}\left\{2\cdot\frac{a}{2}\cdot\frac{a}{2}\cdot\frac{a}{2} - \frac{1}{a}3\cdot\frac{a}{2}2\frac{a}{2}\cdot\frac{a}{2}\cdot\frac{a}{2}\right\}$$

$$= \frac{4}{a^3}\left(\frac{a^3}{4} - \frac{3a^3}{8}\right)$$

$$= \frac{4}{a^3}\left(-\frac{a^3}{8}\right) = -\frac{1}{2}$$

or the probability is

$$P = |\langle n=2, l=2 | n=1, l=1 \rangle|^2$$

$$= 1/4$$

27. A 2-D Oscillator has the Hamiltonian

$$H = \frac{1}{2}(P_x^2 + P_y^2) + \frac{1}{2}(1 + \delta xy)(x^2+y^2)$$

Where $\hbar = 1$, $m = 1$ & $\delta \ll 1$.

Give the W.F.s for the three lowest energy levels for $\delta = 0$; Evaluate the first order perturbation of these levels for $\delta \neq 0$.

Solution

Here we have a perturbing potential

$$H^1 = \frac{1}{2}(\delta xy)(x^2+y^2)$$

Where as

$$H_0 = \frac{1}{2}(P_x^2+P_y^2) + \frac{1}{2}(x^2+y^2)$$

i.e., letting $\delta = 0$ we get H_0. We can then solve the eigenvalue prob of this Hamiltonian — treating the prob. as the case of two independent Harmonic Oscillators.

We will find

$$\psi_{00} = \frac{1}{\sqrt{\pi}} e^{-\frac{1}{2}(x^2+y^2)}$$

$$\psi_{10} = \frac{\sqrt{2}}{\sqrt{\pi}} y\, e^{-\frac{1}{2}(x^2+y^2)}$$

$$\psi_{01} = \frac{\sqrt{2}}{\sqrt{\pi}} x\, e^{-\frac{1}{2}(x^2+y^2)}$$

Which are the three lowest energy eigenfunctions.

In general,

$$E_{n_1, n_2} = (n_1 + n_2 + 1)$$

$$\Rightarrow E_{00} = 1$$

$$E_{10} = E_{01} = 2$$

For ψ_{00} the perturbation to first order will be

$$E^1_{00} = \langle 0,0 | H' | 00 \rangle$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{\delta xy}{2}\right)(x^2+y^2)\, e^{-(x^2+y^2)}\, dx\, dy$$

$$= \frac{\delta}{2\pi}\left(\int_{-\infty}^{\infty} y e^{-y^2} \int_{-\infty}^{\infty} x^3 e^{-x^2} dx + \int_{-\infty}^{\infty} x e^{-x^2} dx \int_{-\infty}^{\infty} y^3 e^{-y^2} dy\right)$$

$$= 0$$

Since in the integrand we have a product of even & odd functions integrated between symmetric limits.

But $(n_1, n_2) = (1,0), (0,1)$ are degenerate so that we should diagonalize H^1 i.e.,

$$H^1 = \begin{pmatrix} \langle 10|H'|10\rangle & \langle 10|H'|01\rangle \\ \langle 01|H'|10\rangle & \langle 01|H'|01\rangle \end{pmatrix}$$

The diagonal elements vanish by the same argument as for E^1_{00}.

$$\langle 10|H'|01\rangle = \langle 01|H'|10\rangle$$

$$= \frac{\delta}{2}\langle 01|xy(x^2+y^2)|10\rangle$$

$$= \frac{\delta}{2}\cdot\frac{2}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^2y^2(x^2+y^2)e^{-(x^2+y^2)}\,dx\,dy$$

$$= \frac{\delta}{\pi}\left(\int_{-\infty}^{\infty} y^2 e^{-y^2}dy\int_{-\infty}^{\infty} x^4 e^{-x^2}dx + \int_{-\infty}^{\infty} x^2 e^{-x^2}dx\int_{-\infty}^{\infty} y^4 e^{-y^2}dy\right)$$

$$= \frac{4\delta}{\pi}\left(\int_0^{\infty} y^2 e^{-y^2}\int_0^{\infty} x^4 e^{-x^2}dx + \int_0^{\infty} x^2 e^{-x^2}dx\int_0^{\infty} y^4 e^{-y^2}dy\right)$$

But $\int_0^{\infty} x^n e^{-\alpha x^2}dx = \frac{1}{2}\alpha^{-\left(\frac{n+1}{2}\right)}\Gamma\left(\frac{n+1}{2}\right)$

$\Rightarrow \langle 10|H'|01\rangle = \langle 01|H'|10\rangle$

$$= \frac{4\delta}{\pi}\left(\Gamma\left(\tfrac{2+1}{2}\right)\Gamma\left(\tfrac{4+1}{2}\right) + \Gamma\left(\tfrac{2+1}{2}\right)\Gamma\left(\tfrac{4+1}{2}\right)\right)$$

$$= \frac{8\delta}{\pi}\Gamma\left(\tfrac{2+1}{2}\right)\Gamma\left(\tfrac{4+1}{2}\right)$$

$$= \frac{8\delta}{\pi}\left(1\cdot\frac{\sqrt{\pi}}{2^2}\cdot 1\cdot 3\cdot\frac{\sqrt{\pi}}{2^4}\right)$$

$$= \frac{8\delta}{\pi}\cdot\frac{3\pi}{2^6}$$

$$= \frac{3\delta}{8}$$

i.e., $H' = \begin{pmatrix} 0 & 3\delta/8 \\ 3\delta/8 & 0 \end{pmatrix}$

Solving its eigenval prob.

$$(-\lambda)(-\lambda) - \frac{9\delta^2}{64} = 0$$

$$\lambda^2 = \frac{9\delta^2}{64}$$

$$\text{or } \lambda = \pm\frac{3\delta}{8}$$

That means the perturbation splits the degenerate levels, whose energies are now

$$E_+ = 2 + \frac{3\delta}{8}$$

for $|\psi\rangle_+ = \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle)$

$$E_- = 2 - \frac{3\delta}{8}$$

$$|\psi\rangle_- = \frac{1}{\sqrt{2}}(|10\rangle - |01\rangle)$$

28. Consider two identical linear oscillators with spring constant k. The interaction potential is given by $H' = C x_1 x_2$ where x_1 & x_2 are the oscillator variables.

a, find the exact energy levels

b, Assume $C \ll k$ and compute the lowest order pair of excited states in first order perturbation theory. (Given energy levels to first order and eigenfunctions to zeroth order).

Solution

a, The Hamiltonian of the system is

$$H = \frac{-\hbar^2}{2m}\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right) + \frac{1}{2}k(x_1^2 + x_2^2) + C x_1 x_2$$

We define, $x_1 = \dfrac{\gamma + \beta}{\sqrt{2}}$

$$x_2 = \frac{\gamma - \beta}{\sqrt{2}}$$

$$\Rightarrow \gamma = \frac{1}{\sqrt{2}}(x_1 + x_2)$$

$$\beta = \frac{1}{\sqrt{2}}(x_1 - x_2)$$

$$\frac{\partial}{\partial x_1} = \frac{\partial}{\partial \gamma}\frac{\partial \gamma}{\partial x_1} + \frac{\partial}{\partial \beta}\frac{\partial \beta}{\partial x_1}$$

$$= \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial \gamma} + \frac{\partial}{\partial \beta}\right)$$

$$\frac{\partial^2}{\partial x_1^2} = \frac{1}{\sqrt{2}}\frac{\partial}{\partial \gamma}\left(\frac{\partial}{\partial \gamma} + \frac{\partial}{\partial \beta}\right)\frac{\partial \gamma}{\partial x_1} + \frac{1}{\sqrt{2}}\frac{\partial}{\partial \beta}\left(\frac{\partial}{\partial \gamma} + \frac{\partial}{\partial \beta}\right)\frac{\partial \beta}{\partial x_1}$$

$$= \frac{1}{2}\left(\frac{\partial^2}{\partial \gamma^2} + \frac{\partial^2}{\partial \beta^2}\right) + \frac{\partial^2}{\partial \gamma \partial \beta}$$

Also,

$$\frac{\partial}{\partial x_2} = \frac{\partial}{\partial \gamma}\frac{\partial \gamma}{\partial x_2} + \frac{\partial}{\partial \beta}\frac{\partial \beta}{\partial x_2}$$

$$= \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial \gamma} - \frac{\partial}{\partial \beta}\right)$$

$$\frac{\partial^2}{\partial x_2^2} = \frac{1}{\sqrt{2}}\frac{\partial}{\partial \gamma}\left(\frac{\partial}{\partial \gamma} - \frac{\partial}{\partial \beta}\right)\frac{\partial \gamma}{\partial x_2} + \frac{1}{\sqrt{2}}\frac{\partial}{\partial \beta}\left(\frac{\partial}{\partial \gamma} - \frac{\partial}{\partial \beta}\right)\frac{\partial \beta}{\partial x_2}$$

$$= \frac{1}{2}\left(\frac{\partial^2}{\partial \gamma^2} + \frac{\partial^2}{\partial \beta^2}\right) - \frac{\partial^2}{\partial \gamma \partial \beta}$$

$$\frac{k}{2}(x_1^2 + x_2^2) + c\, x_1 x_2$$

$$= \frac{k}{2}\left\{\left(\frac{\gamma+\beta}{\sqrt{2}}\right)^2 + \left(\frac{\gamma-\beta}{\sqrt{2}}\right)^2\right\} + c\left(\frac{\gamma+\beta}{\sqrt{2}}\right)\left(\frac{\gamma-\beta}{\sqrt{2}}\right)$$

$$= \frac{1}{2}(k+c)\gamma^2 + \frac{1}{2}(k-c)\beta^2$$

i.e.,

$$H = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial \gamma^2} + \frac{\partial^2}{\partial \beta^2}\right) + \frac{1}{2}(k+c)\gamma^2 + \frac{1}{2}(k-c)\beta^2$$

$$= -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial \gamma^2} + \frac{1}{2}(k+c)\gamma^2 - \frac{\hbar^2}{2m}\frac{\partial^2}{\partial \beta^2} + \frac{1}{2}(k-c)\beta^2$$

Therefore the coupled Hamiltonian is decoupled in to that of two independent oscillator Hamiltonians – one oscillating about $\gamma = 0$ and the other $\beta = 0$ i.e.,

$$\gamma = 0 \Rightarrow x_{10} = \beta/\sqrt{2}$$
$$\beta = 0 \Rightarrow x_{20} = \gamma/\sqrt{2}$$

So now the energy levels of the system are given by the sum of the energy levels of the individual oscillators

$$E_{n_1,n_2} = (n_1 + \tfrac{1}{2})\hbar\omega_1 + (n_2 + \tfrac{1}{2})\hbar\omega_2$$

$n_1, n_2 = 0, 1, 2, 3, \ldots$

$$\omega_1 = \sqrt{\frac{k+c}{m}}, \quad \omega_2 = \sqrt{\frac{k-c}{m}}$$

b) If we treat $c x_1 x_2$ as a perturbation, we will have two independent harmonic oscillators (with same frequency) whose eigenfunctions we already known.

Here the wave functions are $\psi_{n_1}(x_1)\,\psi_{n_2}(x_2)$ with eigenvalues

$$E_{n_1,n_2} = \hbar\omega(n_1 + n_2 + 1), \quad \omega = \sqrt{\frac{k}{m}}$$

The first excited state is a two fold degenerate with energy eigenvalue $2\hbar\omega$. These degenerate states are

$$|n_1, n_2\rangle = \begin{matrix} |10\rangle \\ |01\rangle \end{matrix}$$

And the matrix of the perturbation $H' = C x_1 \cdot x_2$ in this degenerate subspace is

$$H' = \begin{pmatrix} 0 & \epsilon \\ \epsilon & 0 \end{pmatrix}$$

(rows n_2: 1, 0; columns n_1: 1, 0)

where $\epsilon = \langle 10|H'|01\rangle = \langle 01|H'|10\rangle$

$$= \langle 10|C x_1 x_2|01\rangle$$

$$= C\langle 0|x_1|1\rangle\langle 1|x_2|0\rangle$$

The Harmonic oscillator wave functions for $n=0$ & $n=1$ are

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega x^2}{2\hbar}}$$

$$\psi_1(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left(\frac{2m\omega}{\hbar}\right)^{1/2} \cdot x\, e^{-\frac{m\omega}{2\hbar}x^2}$$

$$\Rightarrow \langle 0|x|1\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left(\frac{2m\omega}{\hbar}\right)^{1/2} \int_{-\infty}^{\infty} x^2 e^{-\frac{m\omega x^2}{\hbar}}\,dx$$

$$= \left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\left(\frac{2m\omega}{\hbar}\right)^{1/2} \cdot \frac{1}{2}\left(\frac{\hbar}{m\omega}\right)\left(\frac{\pi\hbar}{m\omega}\right)^{1/2}$$

$$= \left(\frac{\hbar}{2m\omega}\right)^{1/2}$$. Similarly $\langle 1|x|0\rangle = \left(\frac{\hbar}{2m\omega}\right)^{1/2}$

i.e., $\epsilon = \frac{C\hbar}{2m\omega}$

Next we diagonalize H'

$$\begin{vmatrix} -\lambda & \epsilon \\ \epsilon & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 - \epsilon^2 = 0$$

or $\lambda = \pm\epsilon = \pm \frac{C\hbar}{2m\omega}$

The energy levels are thus

$$E_{\pm} = 2\hbar\omega \pm \frac{C\hbar}{2m\omega}$$

Corresponding to eigen functions

$$|\Psi_{\pm}\rangle = \frac{1}{\sqrt{2}}\left(|10\rangle \pm |01\rangle\right) \quad (i)$$

or w. functions

$$\frac{1}{\sqrt{2}}\left(\psi_1(x_1)\psi_0(x_2) \pm \psi_0(x_1)\psi_1(x_2)\right) \quad (ii)$$

29. a, A linear harmonic oscillator is acted upon by a uniform electric field which is considered to be a perturbation and which depends on time as

$$\mathcal{E}(t) = \frac{A}{\tau\sqrt{\pi}}\, e^{-(t/\tau)^2} \qquad (A \equiv \text{constant})$$

At $t = -\infty$, when the field is switched on, the oscillator is in the ground state. Evaluate to a first approximation the probability that it is excited at the end of the action of the field (i.e, $t = +\infty$). Can this prob. be solved exactly?

b, Solve the same prob. for a field and which $\mathcal{E}(t) \sim \frac{1}{t^2+\tau^2}$

corresponds to a given total classical imparted impulse P.

Solution

a, The perturbing Hamiltonian is

$$H' = -e\int \bar{E}\cdot d\bar{l}$$
$$= -eEX$$

assuming the oscillator has a charge e.

The perturbed W.F. may be written as

$$|\psi\rangle = \sum_n a_n(t)\, e^{-iE_n^0 t/\hbar}\, |n^0\rangle$$

Where $|n^0\rangle$ is the unperturbed state vector. The dependence of a_n on t is due to H'. Inserting the above in Schrödinger's eq. and dotting with $\langle f^0|$ and solving, we will get

$$a_f(t) = \delta_{fi} - \frac{i}{\hbar}\int_0^t \left(\langle f^0|H'(t')|i^0\rangle \cdot e^{i\omega_{fi}t'}\, dt'\right)$$

$$\omega_{fi} = \frac{E_f^0 - E_i^0}{\hbar}$$

for the ground state

$|i^0\rangle = |0^0\rangle$, at $t=-\infty$

and for $|f^0\rangle \neq |0^0\rangle$

$$a_f(\infty) = -i/\hbar \int_{-\infty}^{\infty} (-eE)\langle f^0|X|0^0\rangle\, e^{i\omega_{f0}t'}\, dt'$$

$$= \frac{ie}{\hbar}\frac{A}{\tau\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-t'^2/\tau^2}\langle f|X|0\rangle \cdot e^{i\omega_{f0}t'}\, dt'$$

$$X = \sqrt{\frac{\hbar}{2m\omega}}\,(a + a^{\dagger})$$

$$\Rightarrow a_f(\infty) = \frac{ieA}{\hbar\tau\sqrt{\pi}}\sqrt{\frac{\hbar}{2m\omega}}\int_{-\infty}^{\infty} e^{-t'^2/\tau^2} \langle f^0|(a+a^{\dagger})|0\rangle\, e^{if\omega t'}\, dt'$$

as $\frac{E_f - E_0}{\hbar} = f\omega$ $\quad (f+\tfrac{1}{2})\hbar\omega - \tfrac{1}{2}\hbar\omega$

$$\langle f^0|(a+a^{\dagger})|0\rangle = \langle f|1\rangle = \delta_{f1}$$
$$\Rightarrow f = 1.$$

$$a_1(\infty) = \frac{ieA}{\hbar\tau\sqrt{\pi}}\left(\frac{\hbar}{2m\omega}\right)^{1/2}\int_{-\infty}^{\infty} e^{-\frac{(t'^2 - i\omega\tau^2 t')}{\tau^2}}\, dt'$$

$$= \frac{ieA}{\hbar\tau\sqrt{\pi}}\left(\frac{\hbar}{2m\omega}\right)^{1/2}\cdot \tau\sqrt{\pi}\, e^{-\frac{\tau^2\omega^2}{4}}$$

$$= \frac{ieA}{\sqrt{2m\omega\hbar}}\, e^{-\frac{\tau^2\omega^2}{4}}$$

i.e., the probability of the transition 0 to 1 is

$$P_{0\to1} = |a_1(\infty)|^2 = \frac{e^2A^2}{2m\omega\hbar}\, e^{-\frac{\omega^2\tau^2}{2}}$$

(et tal pulse given by the field is $P = \int eE(t)\,dt = eA$ i.e., $P_{0\to1} = \frac{P^2}{2m\omega\hbar}e^{\dots}$)

b, Take $E(t) = \frac{1}{t^2+\tau^2}$

The total impulse transferred to the oscillator during the perturbation is

$$P = \int_{-\infty}^{\infty} eE(t)\, dt = eA\int_{-\infty}^{\infty}\frac{dt}{t^2+\tau^2}$$

Let $t = \tau \tan\gamma$

$$dt = \tau \sec^2\gamma \, d\gamma$$

$$t = \pm\infty \Rightarrow \gamma = \pm \pi/2$$

$$\Rightarrow p = \int_{-\pi/2}^{\pi/2} \frac{eA\tau \sec^2\gamma \, d\gamma}{\tau^2 \sec^2\gamma}$$

$$= \frac{eA}{\tau}\int_{-\pi/2}^{\pi/2} d\gamma$$

or $p = \frac{eA}{\tau}\pi \Rightarrow A = \frac{p\tau}{e\pi}$

$$\therefore E(t) = \frac{p\tau}{e\pi}\,\frac{1}{\tau^2+t^2}$$

Now,

$$d_f(\infty) = \frac{ie}{\hbar}\left(\frac{p\tau}{e\pi}\right)\int_{-\infty}^{\infty}\frac{1}{\tau^2+t'^2}\langle f^0|x|i^0\rangle \cdot e^{i\omega_{f0}t'}\,dt'$$

$\langle f^0|x|i^0\rangle = 0$ unless $f^0 = 1$

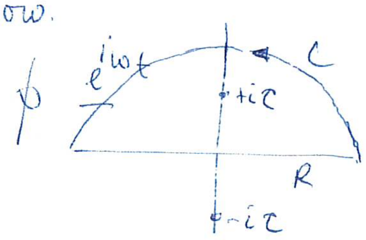
$$\omega_{f0} = f\omega$$

$$\Rightarrow d_1(\infty) = \frac{ie}{\hbar}\left(\frac{p\tau}{e\pi}\right)\sqrt{\frac{\hbar}{2m\omega}} \cdot \int_{-\infty}^{\infty}\frac{e^{i\omega t'}}{\tau^2+t'^2}\,dt'$$

To evaluate $d_1(\infty)$ consider the contour integral of

$$f(t) = \frac{e^{i\omega t}}{\tau^2+t^2}$$

over the semi circular path below.



$$\oint_C \frac{e^{i\omega t}}{\tau^2+t^2}\,dt = 2\pi i \,\mathrm{Res}\left(\frac{e^{i\omega t}}{\tau^2+t^2}\right)\Bigg|_{t=i\tau}$$

By the residue theorem

If $R \to \infty$,

$$\oint_C \frac{e^{i\omega t}}{\tau^2+t^2}\,dt \longrightarrow \int_{-\infty}^{\infty}\frac{e^{i\omega t}}{\tau^2+t^2}\,dt$$

(... Since the integral over the semicircular path vanishes as $R \to \infty$ due to the $e^{i\omega t}$ term.)

$$\text{i.e., } \int_{-\infty}^{\infty}\frac{e^{i\omega t}}{\tau^2+t^2}\,dt = 2\pi i \cdot \mathrm{Res}\,\frac{e^{i\omega t}}{\tau^2+t^2}\Bigg|_{t=i\tau}$$

$$= 2\pi i\,\frac{e^{-\omega\tau}}{2i\tau} = \frac{\pi e^{-\omega\tau}}{\tau}$$

$$\therefore d_1(\infty) = \frac{ie}{\hbar}\,\frac{p\tau}{e\pi}\sqrt{\frac{\hbar}{2m\omega}}\,\frac{\pi}{\tau}e^{-\omega\tau}$$

$$= \frac{ip}{\sqrt{2m\omega\hbar}}\,e^{-\omega\tau}$$

So that

$$P_{0\to1} = |d_1(\infty)|^2 = \frac{p^2}{2m\omega\hbar}\,e^{-2\omega\tau}$$

30. Two identical particles of spin ½ obey Fermi statistics. They are confined in a cubical box whose sides are 10^{-8} cm in length. There exists an attractive potential between the two particles of strength 10^{-3} eV acting whenever the dist. between the two particles is less than 10^{-10} cm. Using non relativistic perturbation theory calculate the ground state energy and W.F. (Take the individual mass of the particles to be the mass of the electron.)

(Take the mass of the individual particles to be the mass of the electron.)

Solution

Since the total spin commutes with the Hamiltonian of the system, the W.F. of this system is

$$\Psi = \Phi(\bar{r}_1, \bar{r}_2)\, \mathcal{X}(1,2)$$

where $\Phi(\bar{r}_1, \bar{r}_2)$ is function of space coordinates and $\mathcal{X}(1,2)$ is the spin W.F. of the two particles.

$$S = 1, 0$$

If $S = 1$, $\mathcal{X}(1,2)$ is a symmetric (triplet) W.F. Since Ψ is anti symmetric, Φ must be anti symmetric and for $S=0$, $\mathcal{X}(1,2)$ should be antisymmetric and $\Phi(\bar{r}_1,\bar{r}_2)$ symm.

If we neglect (or in the absence of) interaction $\Phi(\bar{r}_1,\bar{r}_2)$ should be ~~as~~ a symmetrized product

$$\Phi(\bar{r}_1, \bar{r}_2) = \Phi_{m_1,m_2,m_3}(\bar{r}_1)\, \Phi_{n_1,n_2,n_3}(\bar{r}_2) \pm \Phi_{n_1,n_2,n_3}(\bar{r}_1)\, \Phi_{m_1,m_2,m_3}(\bar{r}_2)$$

where

$$\Phi_{m_1,m_2,m_3}(\bar{r}) = \left(\frac{2}{d}\right)^{3/2} \sin\left(\frac{m_1 \pi x}{d}\right) \sin\left(\frac{m_2 \pi y}{d}\right) \cdot \sin\left(\frac{m_3 \pi z}{d}\right)$$

etc.

and
$$E_{m_1,m_2,m_3,\,n_1,n_2,n_3} = \frac{\pi^2 \hbar^2}{2md^2}(m_1^2 + m_2^2 + m_3^2 + n_1^2 + n_2^2 + n_3^2)$$

or
$$E_{mn} = \frac{\pi^2 \hbar^2}{2Md^2}(m^2 + n^2)$$

where $m = (m_1, m_2, m_3)$ & $n = (n_1, n_2, n_3)$

$$\left.\begin{matrix} m_1, m_2, m_3 \\ n_1, n_2, n_3 \end{matrix}\right\} = 1, 1, 2, \ldots$$

$$d = 10^{-8}\,\text{cm}$$

The interaction of the particles is

$$V(|\bar{r}_1 - \bar{r}_2|) = \begin{cases} 0 & \text{for } |\bar{r}_1 - \bar{r}_2| > a \\ -V_0 & \text{for } |\bar{r}_1 - \bar{r}_2| < a \end{cases}$$

where $a = 10^{-10}\,\text{cm}$

$V_0 = 10^{-3}\,\text{eV}$

$$\int V(|\bar{r}_1 - \bar{r}_2|)\, d^3\bar{r}_2 = -\frac{4\pi a^3}{3} V_0$$

We may ~~thus~~ take

$$V(|\bar{r}_1 - \bar{r}_2|) = -\frac{4\pi a^3}{3} V_0\, \delta^3(\bar{r}_1 - \bar{r}_2)$$

$$\int \delta^3(\bar{r}_1 - \bar{r}_2)\, d^3\bar{r}_2 = 1$$

So now the first order correction to the energy will be

$$E' = \int d^3\bar{r}_1\, d^3\bar{r}_2\, \Phi^*(\bar{r}_1, \bar{r}_2)\, V(|\bar{r}_1 - \bar{r}_2|) \cdot \Phi(\bar{r}_1, \bar{r}_2)$$

Since $V \sim \delta^3(\bar{r}_1 - \bar{r}_2)$ is non zero only when $\bar{r}_1 = \bar{r}_2$ and an antisymmetrical $\Phi(\bar{r}_1, \bar{r}_2)$ vanishes when $\bar{r}_1 = \bar{r}_2$ the energies of the $S=1$ states are unaffected (unshifted). But the symmetrical states have their energy shifted by

$$E_s' = -\left(\frac{4\pi a^3 V_0}{3}\right) \cdot \int d^3\bar{r}_1\, d^3\bar{r}_2\, \Phi^*(\bar{r}_1, \bar{r}_2)\, \delta^3(\bar{r}_1 - \bar{r}_2) \cdot \Phi(\bar{r}_1, \bar{r}_2)$$

$$= -\frac{4\pi a^3 V_0}{3} \int d^3\bar{r}_1\, |\Phi(\bar{r}_1, \bar{r}_1)|^2$$

$$\Rightarrow i\dot{c}_1 = H_{11}c_1 + H_{12}c_2$$
$$= E_1 c_1 + V_{12} c_2$$
$$i\dot{c}_2 = H_{21}c_1 + H_{22}c_2$$
$$= V_{12}^* c_1 + E_2 c_2$$

As a trial solution let

$$c_1(t) = A_1 e^{-i\omega t}$$
$$c_2(t) = A_2 e^{-i\omega t}$$

$A_1, A_2 \equiv$ constant

$$\Rightarrow \omega A_1 e^{-i\omega t} = (A_1 E_1 + A_2 V_{12}) e^{-i\omega t}$$
$$\omega A_2 e^{-i\omega t} = (A_1 V_{12}^* + E_2 A_2) e^{-i\omega t}$$

$$\Rightarrow \begin{cases} (\omega - E_1) A_1 - A_2 V_{12} = 0 \\ V_{12}^* A_1 + (E_2 - \omega) A_2 = 0 \end{cases}$$

$$\frac{A_1}{A_2} = \frac{V_{12}}{\omega - E_1} = \frac{\omega - E_2}{V_{12}^*}$$

$$\Rightarrow (\omega - E_1)(\omega - E_2) = |V_{12}|^2$$
$$\omega^2 - (E_1 + E_2)\omega + E_1 E_2 - |V_{12}|^2 = 0$$

$$\text{i.e., } \omega = \frac{E_1 + E_2 \pm \sqrt{(E_1+E_2)^2 - [4E_1E_2 - 4|V_{12}|^2]}}{2}$$

$$\omega_\pm = \frac{E_1 + E_2}{2} \pm \frac{\sqrt{(E_1 - E_2)^2 + 4|V_{12}|^2}}{2}$$

$$\therefore c_1 = A_1 e^{-i\omega_+ t} + B_1 e^{-i\omega_- t}$$
$$c_2 = A_2 e^{-i\omega_+ t} + B_2 e^{-i\omega_- t}$$

Using the ratio $\frac{A_1}{A_2} =$

$$\frac{V_{12}}{\omega_+ - E_1} = \frac{2V_{12}}{(E_2 - E_1) + \sqrt{(E_1 - E_2)^2 + 4|V_{12}|^2}} \quad \text{---(1)}$$

$$\frac{B_1}{B_2} = \frac{V_{12}}{\omega_- - E_1} = \frac{2V_{12}}{(E_2 - E_1) - \sqrt{(E_1 - E_2)^2 + 4|V_{12}|^2}} \quad \text{---(2)}$$

Initial conditions:

$$c_1(0) = A_1 + B_1 = 1 \quad \ldots\ldots \square$$

(As it is a prob. amplitude and the system was in state 1 at t=0.)

$$c_2(0) = A_2 + B_2 = 0 \quad \ldots\ldots \square\square$$

Also, since the probabilities should add to ~~zero~~ one

$$A_1^2 + B_1^2 + A_2^2 + B_2^2 = 1 \quad \ldots\ldots \square\square\square$$

and $(A_1 + B_1)^2 + (A_2 + B_2)^2 = 1$

$$A_1^2 + B_1^2 + A_2^2 + B_2^2 + 2(A_1B_1 + A_2B_2) = 1$$

$$\Rightarrow A_1B_1 + A_2B_2 = 0 \quad \ldots\ldots \square\square\square\square$$

Define $E_2 - E_1 = \alpha$

$$\sqrt{(E_1 - E_2)^2 + 4|V_{12}|^2} = \beta$$

From (1) & (2)

$$\frac{A_1}{A_2} = \frac{2V_{12}}{\alpha + \beta} \Rightarrow A_2 = \frac{\alpha + \beta}{2V_{12}} A_1 \quad \text{--(3)}$$

$$\frac{B_1}{B_2} = \frac{2V_{12}}{\alpha - \beta} \Rightarrow B_2 = \frac{\alpha - \beta}{2V_{12}} B_1 \quad \text{--(4)}$$

(3) + (4)

$$\Rightarrow A_1 \frac{(\alpha + \beta)}{2V_{12}} + B_1 \frac{(\alpha - \beta)}{2V_{12}} = A_2 + B_2 = 0, \text{ by } \square\square$$

$$B_1 = 1 - A_1 \quad \text{by } \square$$

$$\Rightarrow A_1 \frac{(\alpha + \beta)}{2V_{12}} + (1 - A_1)\frac{(\alpha - \beta)}{2V_{12}} = 0$$

$\Rightarrow A_1(2\beta) = \beta - \alpha$

$\Rightarrow A_1 = \dfrac{\beta-\alpha}{2\beta}$

$B_1 = 1 - A_1 = \dfrac{\beta+\alpha}{2\beta}$

$A_2 = \dfrac{\alpha+\beta}{2V_{12}} A_1 = \dfrac{\beta^2-\alpha^2}{2V_{12}\beta} = -B_2$

or $A_1 = \dfrac{\beta^2-\alpha^2}{2\beta(\beta+\alpha)}$

substituting back for α & β

$$A_1 = \frac{2|V_{12}|^2}{(E_1-E_2)^2+4|V_{12}|^2+(E_2-E_1)\sqrt{(E_1-E_2)^2+4|V_{12}|^2}}$$

$$B_1 = \frac{\beta^2-\alpha^2}{2\beta(\beta-\alpha)}$$

$$= \frac{2|V_{12}|^2}{(E_1-E_2)^2+4|V_{12}|^2-(E_2-E_1)\sqrt{(E_1-E_2)^2+4|V_{12}|^2}}$$

$$A_2 = -B_2 = \frac{V_{12}^*}{\sqrt{(E_1-E_2)^2+4|V_{12}|^2}}$$

32. Use the variation principle to estimate the ground state energy of a particle in the potential

$V = \infty$ for $x < 0$

$= cx$ for $x > 0$

Take xe^{-ax} as a trial function.

solution

we have $\psi(x) = \begin{cases} 0 & x<0 \\ xe^{-ax}, & x>0 \end{cases}$

The energy eigenvalue for this function is

$$E = \frac{\int_0^\infty xe^{-ax}\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+cx\right)xe^{-ax}dx}{\int_0^\infty x^2e^{-2ax}dx}$$

... the denominator is because ψ is not normalized.

$$\frac{d^2}{dx^2}(xe^{-ax}) = \frac{d}{dx}\{(1-ax)e^{-ax}\}$$

$$= -ae^{-ax} - a(1-ax)e^{-ax}$$

$$= a(ax-2)e^{-ax}$$

$$\Rightarrow E = \frac{\int_0^\infty xe^{-2ax}\left\{-\frac{\hbar^2 a}{2m}(ax-2)+cx^2\right\}dx}{\int_0^\infty x^2e^{-2ax}dx}$$

$$= \frac{\left[-\frac{\hbar^2 a}{2m}\int_0^\infty x(ax-2)e^{-2ax}dx + c\int_0^\infty x^3e^{-2ax}dx\right]}{\int_0^\infty x^2e^{-2ax}}$$

$$= \frac{-\frac{\hbar^2 a}{2m}\left(-\frac{1}{4a^2}\right)+\frac{3c}{8a^4}}{1/4a^3}$$

$$\bar{E} = \left(\frac{\hbar^2}{8ma} + \frac{3c}{8a^4}\right)4a^3$$

$$= \frac{\hbar^2 a^2}{2m} + \frac{3c}{2a}$$

To find the min. value of $\bar{E}$ we treat a as a parameter

$$\frac{d\bar{E}}{da} = 0$$

$$\Rightarrow + \frac{\hbar^2 a}{m} - \frac{3c}{2a^2} = 0$$

$$\frac{\hbar^2 a}{m} - \frac{3c}{2a^2} = 0$$

or $a = \left(\frac{3mc}{2\hbar^2}\right)^{1/3}$

$$\Rightarrow \bar{E} = \frac{\hbar^2}{2m}\left(\frac{3mc}{2\hbar^2}\right)^{2/3} + \frac{3c}{2}\left(\frac{2\hbar^2}{3mc}\right)^{1/3}$$

$$= \left(\frac{3mc\,\hbar^3}{2^{3/2}\, m^{3/2}\hbar^2}\right)^{2/3}$$

$$+ \left(\frac{9c^3 \cdot 2\hbar^2}{8 \cdot 3mc}\right)^{1/3}$$

$$= \frac{\hbar^2}{2m}\left(\frac{9m^2c^2}{4\hbar^4}\right)^{1/3} + \frac{3}{2}\left(\frac{2\hbar^2c^3}{3mc}\right)^{1/3}$$

$$= \frac{1}{2}\left(\frac{9}{4}\frac{\hbar^6 m^2 c^2}{m^3 \hbar^4}\right)^{1/3} + \frac{3}{2}\left(\frac{2\hbar^2 c^3}{3mc}\right)^{1/3}$$

$$= \frac{1}{2}\left(\frac{9}{4}\frac{2\cdot 3}{3\cdot 2}\frac{\hbar^2 c^2}{m}\right)^{1/3} + \frac{3}{2}\left(\frac{2\hbar^2 c^2}{3m}\right)^{1/3}$$

$$= \frac{1}{2}\left(\frac{27}{8}\,\frac{2\hbar^2c^2}{3m}\right)^{1/3} + \frac{3}{2}\left(\frac{2\hbar^2c^2}{3m}\right)^{1/3}$$

$$= \left(\frac{3}{4} + \frac{3}{2}\right)\left(\frac{2\hbar^2c^2}{3m}\right)^{1/3}$$

$$= \frac{9}{4}\left(\frac{2\hbar^2c^2}{3m}\right)^{1/3}$$

So since xe^{-ax} more or less approximates the ground state energy, $\bar{E}$ gives the upper limit of the ground state energy E_0 (i.e.)

$$E_0 \leq \frac{9}{4}\left(\frac{2\hbar^2c^2}{3m}\right)^{1/3}$$

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Find the frequency and the period of oscillation of a charge q placed between two similar charges of magnitude Q in a linear approximation.

Solution

Since the two charges are identical and of the same sign, the charge q would oscillate about the center of the system, i.e., about the midpoint of the segment joining the other two charges.

Suppose the distance between one of the charges and the center is a and q be deviated by x from this point.

At point the site of q the potential of the system is

$$U = \frac{Qq}{4\pi\epsilon_0(a+x)} + \frac{Qq}{4\pi\epsilon_0(a-x)}$$

$$= \frac{Qq}{4\pi\epsilon_0}\left\{\frac{1}{a+x} + \frac{1}{a-x}\right\}$$

Since x is small we can use a binomial expansion and consider only upto the quadratic term which means the linear term of the force, i.e., linear approximation. Then,

$$\frac{1}{a+x} = \frac{1}{a(1+\frac{x}{a})} = \frac{1}{a}\left(1+\frac{x}{a}\right)^{-1}$$

$$= \frac{1}{a}\left(1 - \frac{x}{a} + \left(\frac{x}{a}\right)^2 + \dots\right)$$

or $\frac{1}{a+x} \approx \frac{1}{a}\left[1 - \frac{x}{a} + \left(\frac{x}{a}\right)^2\right]$

$$\frac{1}{a-x} = \frac{1}{a(1-\frac{x}{a})} = \frac{1}{a}\left(1 - \frac{x}{a}\right)^{-1}$$

$$= \frac{1}{a}\left(1 + \frac{x}{a} + \left(\frac{x}{a}\right)^2 + \dots\right)$$

i.e., $\frac{1}{a-x} \approx \frac{1}{a}\left[1 + \frac{x}{a} + \left(\frac{x}{a}\right)^2\right]$

So now we obtain for U

$$U = \frac{Qq}{4\pi\epsilon_0}\left\{\frac{1}{a}\left(1 - \frac{x}{a} + \left(\frac{x}{a}\right)^2\right) + \frac{1}{a}\left(1 + \frac{x}{a} + \left(\frac{x}{a}\right)^2\right)\right\}$$

$$= \frac{Qq}{4\pi\epsilon_0 a}\left(1 - \frac{x}{a} + \left(\frac{x}{a}\right)^2 + 1 + \frac{x}{a} + \left(\frac{x}{a}\right)^2\right)$$

$$= \frac{Qq}{4\pi\epsilon_0 a}\left(2 + 2\left(\frac{x}{a}\right)^2\right)$$

$$= \frac{Qq}{2\pi\epsilon_0 a^3}(a^2 + x^2)$$

The force acting on q is thus

$$F = -\frac{\partial U}{\partial x} = -\frac{Qq}{\pi\epsilon_0 a^3}x$$

Newton's 2nd law for the motion of the particle tells us that

$$m\ddot{x} = -\frac{Qq}{\pi\epsilon_0 a^3}x$$

$$m\ddot{x} + \frac{Qq}{\pi\epsilon_0 a^3}x = 0$$

or $$\ddot{x} + \frac{Qq}{m\pi\epsilon_0 a^3}x = 0$$

An equation which indeed describes oscillation.

$$\therefore \omega^2 = \frac{Qq}{m\pi\epsilon_0 a^3}$$ ✓

$$\omega = \sqrt{\frac{Qq}{m\pi\epsilon_0 a^3}}$$

which in turn gives the period of oscillation to be

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m\pi\epsilon_0 a^3}{Qq}}$$

Or the period of oscillation could also be found as follows.

The system is conservative i.e., we have electrostatic interaction. This would mean

$$\tfrac{1}{2}m\dot{x}^2 + U(x) = E, \text{ constant}$$

where $U(x) = \frac{Qq}{2\pi\epsilon_0 a^3}(a^2 + x^2)$, as found.

$$\Rightarrow \dot{x} = \sqrt{\tfrac{2}{m}(E - u(x))} \quad \cdots\cdots (*)$$

$$\frac{dx}{dt} = \sqrt{\frac{2}{m}(E - u(x))}$$

$$dt = \frac{dx}{\sqrt{\frac{2}{m}(E-u(x))}}$$

Let x_0 be the maximum displacement (amplitude of oscillation of the charge) from the equilibrium position. x_0 is determined by the root of eq. (*) i.e., when $\dot{x} = 0$. So the period is given by

$$\tau = \int_0^{\tau} dt = 2\int_{-x_0}^{x_0} \frac{dx}{\sqrt{\frac{2}{m}(E-u(x))}}$$

$$= 4\int_0^{x_0} \frac{dx}{\sqrt{\frac{2}{m}(E-u(x))}}$$

Substituting $u(x) = \frac{Q\varepsilon}{2\pi\epsilon_0 a^3}(a^2+x^2)$

$$\tau = 4\int_0^{x_0} \frac{dx}{\sqrt{\frac{2}{m}\left(E - \frac{Q\varepsilon}{2\pi\epsilon_0 a^3}(a^2+x^2)\right)}}$$

$$= 4\int_0^{x_0} \frac{dx}{\sqrt{\frac{2}{m}\left(E - \frac{Q\varepsilon a^2}{2\pi\epsilon_0 a^3} - \frac{Q\varepsilon}{2\pi\epsilon_0 a^3}x^2\right)}}$$

$$= 4\int_0^{x_0} \frac{dx}{\sqrt{\frac{Q\varepsilon}{m\pi\epsilon_0 a^3}\left(\frac{2\pi\epsilon_0 a^3 E - a^2 Q\varepsilon}{Q\varepsilon}\right) - x^2}}$$

$$= 4\sqrt{\frac{m\pi\epsilon_0 a^3}{Q\varepsilon}} \int_0^{x_0} \frac{dx}{\sqrt{\left(\frac{2\pi\epsilon_0 a^3 E - a^2 Q\varepsilon}{Q\varepsilon}\right) - x^2}}$$

Let $\lambda^2 = \frac{2\pi\epsilon_0 a^3 E - a^2 Q\varepsilon}{Q\varepsilon}$ = constant

$$\Rightarrow \tau = 4\sqrt{\frac{m\pi\epsilon_0 a^3}{Q\varepsilon}} \int_0^{x_0} \frac{dx}{\sqrt{\lambda^2 - x^2}}$$

$$= 4\sqrt{\frac{m\pi\epsilon_0 a^3}{Q\varepsilon}} \arcsin \frac{x}{\lambda}\Bigg|_0^{x_0}$$

or

$$\tau = 4\sqrt{\frac{m\pi\epsilon_0 a^3}{Q\varepsilon}} \cdot \arcsin \frac{x}{\sqrt{\frac{2\pi\epsilon_0 a^3 - a^2 Q\varepsilon}{Q\varepsilon}}}\Bigg|_0^{x_0}$$

Let us find x_0, max. displacement

$$E = u(x_0) = \frac{Q\varepsilon}{2\pi\epsilon_0 a^3}(a^2 + x_0^2)$$

$$\Rightarrow x_0 = \sqrt{\frac{2\pi\epsilon_0 a^3 E - a^2 Q\varepsilon}{Q\varepsilon}}$$

i.e.,

$$\tau = 4\sqrt{\frac{m\pi\epsilon_0 a^3}{Q\varepsilon}} \cdot \arcsin \frac{\sqrt{\frac{2\pi\epsilon_0 a^3 E - a^2 Q\varepsilon}{Q\varepsilon}}}{\sqrt{\frac{2\pi\epsilon_0 a^3 E - a^2 Q\varepsilon}{Q\varepsilon}}} - 0$$

$$= 4\sqrt{\frac{m\pi\epsilon_0 a^3}{Q\varepsilon}} \arcsin 1$$

$$= 4\sqrt{\frac{m\pi\epsilon_0 a^3}{Q\varepsilon}} \cdot \frac{\pi}{2}$$

or

$$\tau = 2\pi\sqrt{\frac{m\pi\epsilon_0 a^3}{Q\varepsilon}}$$

which is the previous result.

So for the particle

$$\tau = 2\pi\sqrt{\frac{m\pi\epsilon_0 a^3}{Q\varepsilon}}$$

$$\omega = \sqrt{\frac{Q\varepsilon}{m\pi\epsilon_0 a^3}}$$

where a is the distance between the center and one of the charges (i.e.) half the distance between the two charges.

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The equation of the ellipse in the xy frame is given to be

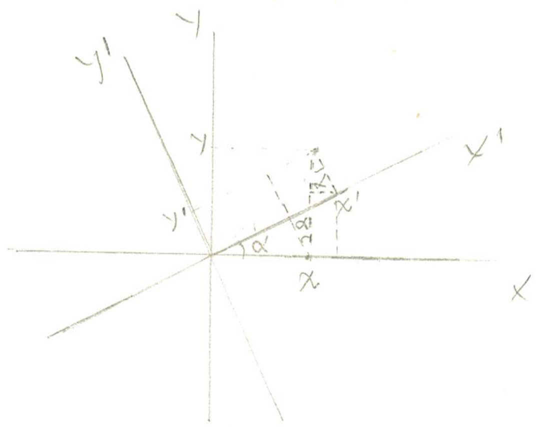
$$x^2 - 2\frac{v_{ox}}{v_{oy}} xy + \left[\frac{v_{ox}^2}{v_{oy}^2} + r_0^2\left(\frac{w_0}{v_{oy}}\right)^2\right] y^2 = r_0^2$$

Let $a = \frac{v_{ox}}{v_{oy}}$

$b = \frac{v_{ox}^2}{v_{oy}^2} + r_0^2 \frac{w_0^2}{v_{oy}^2}$

$\Rightarrow x^2 - 2axy + by^2 = r_0^2$

This eq. can be written for a frame that is rotated by an angle α so that the xy term can be elliminated.



As can be seen from the figure

$x = x'\cos\alpha - y'\sin\alpha$

and $y = x'\sin\alpha + y'\cos\alpha$

$\Rightarrow x^2 = x'^2\cos^2\alpha + y'^2\sin^2\alpha - 2x'y'\sin\alpha\cos\alpha$

$y^2 = x'^2\sin^2\alpha + y'^2\cos^2\alpha + 2x'y'\sin\alpha\cos\alpha$

$xy = x'^2\sin\alpha\cos\alpha - y'^2\sin\alpha\cos\alpha + x'y'(\cos^2\alpha - \sin^2\alpha)$

Let us use these expressions in the original equation of the ellipse

$$x'^2\cos^2\alpha + y'^2\sin^2\alpha - 2x'y'\sin\alpha\cos\alpha - 2a(x'^2\sin\alpha\cos\alpha - y'^2\sin\alpha\cos\alpha + x'y'(\cos^2\alpha - \sin^2\alpha)) + b(x'^2\sin^2\alpha + y'^2\cos^2\alpha + 2x'y'\sin\alpha\cos\alpha) = r_0^2$$

Collecting terms separately multiplied by x'^2, y'^2 and $x'y'$ separately

$$(\cos^2\alpha + b\sin^2\alpha - 2a\sin\alpha\cos\alpha)x'^2 + (\sin^2\alpha + b\cos^2\alpha + 2a\sin\alpha\cos\alpha)y'^2 + (-2\sin\alpha\cos\alpha - 2a(\cos^2\alpha - \sin^2\alpha) + 2b\sin\alpha\cos\alpha)x'y' = r_0^2$$

$$(\cos^2\alpha + b\sin^2\alpha - a\sin 2\alpha)x'^2 + (\sin^2\alpha + b\cos^2\alpha + a\sin 2\alpha)y'^2 + ((b-1)\sin 2\alpha - 2a\cos 2\alpha)x'y' = r_0^2$$

This equation is of the form

$$A'x'^2 + B'y'^2 + C'x'y' = r_0^2$$

where

$A' = \cos^2\alpha + b\sin^2\alpha - a\sin 2\alpha$

$B' = \sin^2\alpha + b\cos^2\alpha + a\sin 2\alpha$

$C' = (b-1)\sin 2\alpha - 2a\cos 2\alpha$

$\Rightarrow A' + B' = 1 + b$ is invariant

Also under rotation of the x-y frame the other invariant is

$4a^2 - 4b = C'^2 - 4A'B'$

Now, in the rotated frame for the equation to represent an ellipse the $x'y'$ term must vanish, i.e.,

$C' = 0$

$\Rightarrow (b-1)\sin 2\alpha - 2a\cos 2\alpha = 0$

or $\tan 2\alpha = \frac{2a}{b-1} = \frac{-2a}{1-b}$

To find A' & B' in terms of a & b.

$A' + B' = 1 + b$

$-4A'B' = 4(a^2 - b)$, $C' = 0$

$B' = 1 + b - A'$

$\Rightarrow -4A'(1 + b - A') = 4(a^2 - b)$

$-4(1+b)A' + 4A'^2 = 4(a^2 - b)$

$$A'^2 - (1+b)A' - (a^2-b) = 0$$

$$A' = \frac{(1+b) \pm \sqrt{(1+b)^2 + 4(a^2-b)}}{2}$$

* $(1+b)^2 + 4(a^2-b)$

$= 1 + 2b + b^2 + 4a^2 - 4b$

$= 1 - 2b + b^2 + 4a^2$

$= (1-b)^2 + 4a^2$

Now, $A' = \dfrac{(1+b) \pm \sqrt{(1-b)^2 + 4a^2}}{2}$

$A_1' = \dfrac{(1+b) + \sqrt{(1-b)^2 + 4a^2}}{2}$

$A_2' = \dfrac{(1+b) - \sqrt{(1-b)^2 + 4a^2}}{2}$

$B' = 1 + b - A'$

$B_1' = (1+b) - \left\{ \dfrac{(1+b) + \sqrt{(1-b)^2 + 4a^2}}{2} \right\}$

$= \dfrac{(1+b) - \sqrt{(1-b)^2 + 4a^2}}{2}$

$= A_1'$

$B_2' = (1+b) - \left\{ \dfrac{(1+b) - \sqrt{(1-b)^2 + 4a^2}}{2} \right\}$

$= \dfrac{(1+b) + \sqrt{(1-b)^2 + 4a^2}}{2}$

$= A_1'$

i.e., $A' = A_1' \Rightarrow B' = A_2'$

$A' = A_2' \Rightarrow B' = A_1'$

The equation in the rotated frame becomes

$A'x'^2 + B'y'^2 = r_0^2, \; C' = 0$

$A_1'x'^2 + A_2'y'^2 = r_0^2$

$$\frac{x'^2}{(r_0^2/A_1')} + \frac{y'^2}{(r_0^2/A_2')} = 1$$

if $a'^2 = r_0^2/A_1'$

$b'^2 = r_0^2/A_2'$

we obtain

$$\frac{x'^2}{a'^2} + \frac{y'^2}{b'^2} = 1$$

And $A_2'x'^2 + A_1'y'^2 = r_0^2$

$$\frac{x'^2}{r_0^2/A_2'} + \frac{y'^2}{r_0^2/A_1'} = 1$$

$$\frac{x'^2}{b'^2} + \frac{y'^2}{a'^2} = 1$$

That means in general we have two solutions

$$\frac{x'^2}{a'^2} + \frac{y'^2}{b'^2} = 1$$

or $\dfrac{x'^2}{b'^2} + \dfrac{y'^2}{a'^2} = 1$

But knowing α one can reject one of these equations.

$a'^2 = r_0^2/A_1'$

$= \dfrac{2r_0^2}{(1+b) + \sqrt{(1-b)^2 + 4a^2}}$

$b'^2 = r_0^2/A_2'$

$= \dfrac{2r_0^2}{(1+b) - \sqrt{(1-b)^2 + 4a^2}}$

$b'^2 > a'^2$

So if the semimajor axis lies along the x'-axis the equation would be

$$\frac{x'^2}{b'^2} + \frac{y'^2}{a'^2} = 1$$

And if it lies along the y'-axis

$$\frac{x'^2}{a'^2} + \frac{y'^2}{b'^2} = 1$$

Note: $\tan 2\alpha = \dfrac{-2a}{1-b}$

where a & b are as given.

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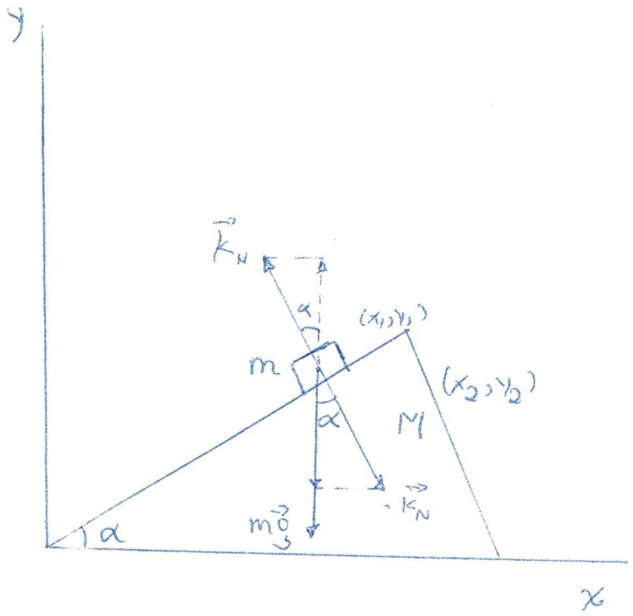

$\vec{K}_N$ is the normal force acting on the surface of contact of the block and the inclined plane.

The coordinates of the block are (x_1, y_1) and of the inclined plane are (x_2, y_2).

The forces which result in the motion of the bodies are:

(i) the force of gravity $m\vec{g}$

(ii) the normal force $\vec{K}_N$

So now Newton's 2nd law applied to the system i.e.) to block and the inclined plane (or wedge) gives

$$m\ddot{x}_1 = -K_N \sin\alpha \quad \text{----- (1)}$$

$$m\ddot{y}_1 = -mg + K_N \cos\alpha \quad \text{----- (2)}$$

$$M\ddot{x}_2 = K_N \sin\alpha \quad \text{----- (3)}$$

$$M\ddot{y}_2 = 0, \quad \text{----- (4)}$$

There is no motion of the inclined plane in the y-direction since y_2 is fixed.

The fact that the mass m stays on the inclined plane is given by the equation of the constraining surface, i.e.)

$$\frac{y_1}{x_1 - x_2} = \tan\alpha$$

or $y_1 = (x_1 - x_2)\tan\alpha$ ----- (5)

from equations (1) and (2)

$$m\ddot{x}_1 = -M\ddot{x}_2 \quad \text{----- (6)}$$

Differentiating eq. (5)

$$\ddot{y}_1 = (\ddot{x}_1 - \ddot{x}_2)\tan\alpha$$

$$\Rightarrow \ddot{x}_2 = \ddot{x}_1 - \ddot{y}_1 \cot\alpha$$

But from eq. (2)

$$\ddot{y}_1 = -g + (K_N/m)\cos\alpha$$

and from eq. (1)

$$K_N = -\frac{m}{\sin\alpha}\ddot{x}_1$$

$$\Rightarrow \ddot{y}_1 = -g - \ddot{x}_1 \cot\alpha \quad \text{----- (7)}$$

Now;

$$\ddot{x}_2 = \ddot{x}_1 - \ddot{y}_1 \cot\alpha$$

$$= \ddot{x}_1 + (g + \ddot{x}_1 \cot\alpha)\cot\alpha$$

Using this expression of $\ddot{x}_2$ in eq.(6) we obtain

$$m\ddot{x}_1 = -M\{\ddot{x}_1(1 + \cot^2\alpha) + g\cot\alpha\}$$

$$\ddot{x}_1(m + M(1 + \cot^2\alpha)) = -Mg\cot\alpha$$

$$\text{or } \ddot{x}_1 = \frac{-Mg\cot\alpha}{m + M + M\cot^2\alpha} \quad \text{--- (8)}$$

from eq. (7)

$$\ddot{y}_1 = -g + \frac{Mg\cot^2\alpha}{m + M + M\cot^2\alpha} \quad \text{--- (9)}$$

from eq. (6)

$$\ddot{x}_2 = -\frac{m}{M}\ddot{x}_1$$

$$\text{or } \ddot{x}_2 = \frac{mg\cot\alpha}{m + M + M\cot^2\alpha} \quad \text{--- (10)}$$

It follows then the system is governed by three equations of motion:

from eq. 8

$$\boxed{\ddot{x}_1 + \frac{Mg\cot\alpha}{m + M + M\cot^2\alpha} = 0}$$

from eq. (9)

$$\boxed{\ddot{y}_1 + g - \frac{Mg\cot^2\alpha}{m + M + M\cot^2\alpha} = 0}$$

from eq (10)

$$\boxed{\ddot{x}_2 - \frac{mg\cot\alpha}{m + M + M\cot^2\alpha} = 0}$$

y_2 is fixed

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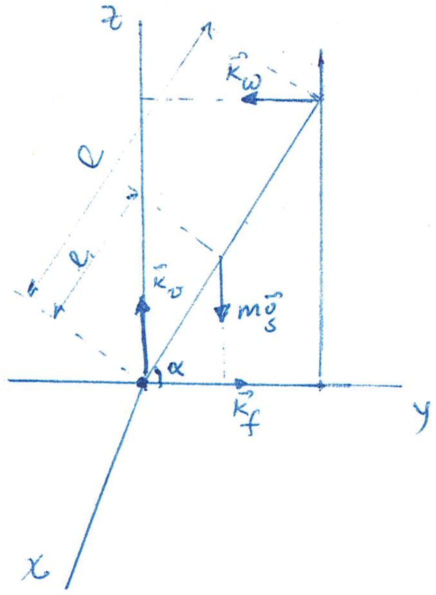

As the wall and the floor are frictionless, their reaction forces are perpendicular to their own faces. $\vec{K}_\omega$ is the reaction force of the wall and $\vec{K}_v$ is that of the floor.

Since the system is in equilibrium

$$K_f - K_\omega = 0 \quad \text{.....(1)}$$

$$K_v - mg_s = 0 \quad \text{.....(2)}$$

In addition, the sum of the torques about the x-axis is zero, i.e.,

$$\vec{M}_x = \vec{r}_\omega \times \vec{K}_\omega + \vec{r}_{mg} \times m\vec{g}_s = 0$$

or $M_x = K_\omega l \sin\alpha - mg_s l_1 \cos\alpha = 0$

$$\Rightarrow K_\omega = \frac{mg\, l_1 \cos\alpha}{l \sin\alpha}$$

$$= \frac{mg\, l_1}{l} \cot\alpha$$

But from eq. (1), $K_f = K_\omega$, whence

$$K_f = \frac{mg_s\, l_1}{l} \cot\alpha$$

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$\vec{E} = -\partial \vec{A}/\partial t - \text{grad } V(\vec{r},t)$

$\vec{B} = \text{Curl } \vec{A}(\vec{r},t)$

Where $\vec{A}$ is the magnetic vector potential and V is the electric potential

The Lagrangian of a particle of mass m and charge q moving in an electromagnetic field is

$$L = \tfrac{1}{2} m\vec{v}^2 + q\vec{v}\vec{A}(\vec{r},t) - qV(\vec{r},t)$$

to find the force acting on the charge.

$$L = \tfrac{1}{2} m\dot{\vec{r}}^2 + q\dot{\vec{r}}\vec{A}(\vec{r},t) - qV(\vec{r},t)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\vec{r}}} - \frac{\partial L}{\partial \vec{r}} = 0$$

$$\frac{\partial L}{\partial \dot{\vec{r}}} = m\dot{\vec{r}} + q\vec{A}(\vec{r},t)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\vec{r}}} = m\ddot{\vec{r}} + q\frac{d\vec{A}}{dt}$$

Since $\vec{A} = \vec{A}(\vec{r},t)$,

$$\frac{d\vec{A}}{dt} = \frac{\partial \vec{A}}{\partial t} + \frac{\partial \vec{A}}{\partial x}\frac{dx}{dt} + \frac{\partial \vec{A}}{\partial y}\frac{dy}{dt} + \frac{\partial \vec{A}}{\partial z}\frac{dz}{dt}$$

$$= \frac{\partial \vec{A}}{\partial t} + v_x\frac{\partial \vec{A}}{\partial x} + v_y\frac{\partial \vec{A}}{\partial y} + v_z\frac{\partial \vec{A}}{\partial z}$$

$$= \frac{\partial \vec{A}}{\partial t} + (\vec{v}\cdot\nabla)\vec{A}$$

$$= \partial \vec{A}/\partial t + (\dot{\vec{r}}\cdot\nabla)\vec{A}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\vec{r}}} = m\ddot{\vec{r}} + q\frac{\partial \vec{A}}{\partial t} + q(\dot{\vec{r}}\cdot\nabla)\vec{A}$$

$$\frac{\partial L}{\partial \vec{r}} = q\dot{\vec{r}}\frac{\partial \vec{A}}{\partial \vec{r}} - q\frac{\partial V}{\partial \vec{r}}$$

So now the equation of motion becomes

$$m\ddot{\vec{r}} + q\frac{\partial \vec{A}}{\partial t} + q(\dot{\vec{r}}\cdot\nabla)\vec{A} - q\dot{\vec{r}}\frac{\partial \vec{A}}{\partial \vec{r}} + q\frac{\partial V}{\partial \vec{r}} = 0$$

But $\frac{\partial \vec{A}}{\partial t} = -(\vec{E} + \text{grad } V)$

$$m\ddot{\vec{r}} - q(\vec{E} + \text{grad } V) + q(\dot{\vec{r}}\cdot\nabla)\vec{A} - q\dot{\vec{r}}\frac{\partial \vec{A}}{\partial \vec{r}} + q\frac{\partial V}{\partial \vec{r}} = 0$$

$$(*)\quad m\ddot{\vec{r}} - q\vec{E} + q\left\{(\dot{\vec{r}}\cdot\nabla)\vec{A} - \dot{\vec{r}}\frac{\partial \vec{A}}{\partial \vec{r}}\right\} - q\left(\text{grad } V - \frac{\partial V}{\partial \vec{r}}\right) = 0$$

Consider $\dot{\vec{r}} \times (\nabla \times \vec{A})$

from vector analysis

$$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$$

$$\Rightarrow \dot{\vec{r}} \times (\nabla \times \vec{A}) = \nabla(\dot{\vec{r}}\cdot\vec{A}) - (\dot{\vec{r}}\cdot\nabla)\vec{A}$$

that is taking care of the vector and operator nature of ∇.

But ∇ does not act on $\dot{\vec{r}}$ as $\dot{\vec{r}} \neq \dot{\vec{r}}(x,y,z)$ and considering ∇ in spherical coordinates and noting that $\vec{A} = \vec{A}(\vec{r},t)$

$$\dot{\vec{r}} \times (\nabla \times \vec{A}) = \nabla(\dot{\vec{r}}\cdot\vec{A}) - (\dot{\vec{r}}\cdot\nabla)\vec{A}$$

$$= \dot{\vec{r}}\frac{\partial \vec{A}}{\partial \vec{r}} - (\dot{\vec{r}}\cdot\nabla)\vec{A}$$

or $(\dot{\vec{r}}\cdot\nabla)\vec{A} - \dot{\vec{r}}\frac{\partial \vec{A}}{\partial \vec{r}} = -\dot{\vec{r}}\times(\nabla\times\vec{A})$

Using this result in $(*)$

$$m\ddot{\vec{r}} - q\vec{E} - q\dot{\vec{r}}\times(\nabla\times\vec{A}) - q\left(\text{grad } V - \frac{\partial V}{\partial \vec{r}}\right) = 0$$

But $\text{grad } V = \vec{i}\frac{\partial V}{\partial x} + \vec{j}\frac{\partial V}{\partial y} + \vec{k}\frac{\partial V}{\partial z}$

or $\text{grad } V = \vec{i}\frac{\partial V}{\partial \vec{r}}\frac{\partial r}{\partial x} + \vec{j}\frac{\partial V}{\partial r}\frac{\partial r}{\partial y} + \vec{k}\frac{\partial V}{\partial r}\frac{\partial r}{\partial x}$

$$\text{grad } V = \frac{\partial V}{\partial r}\left(\vec{i}\frac{x}{r} + \vec{j}\frac{y}{r} + \vec{k}\frac{z}{r}\right)$$

$$= \frac{\partial V}{\partial r}\frac{\vec{r}}{r}$$

$$= \frac{\partial V}{\partial r}\vec{e}_r$$

$$\equiv \frac{\partial V}{\partial \vec{r}}$$

$$m\ddot{\vec{r}} - q\vec{E} - q\dot{\vec{r}}\times(\nabla\times\vec{A})$$

$$- q\left(\frac{\partial V}{\partial \vec{r}} - \frac{\partial V}{\partial \vec{r}}\right)$$

$$= 0$$

$$m\ddot{\vec{r}} - q\vec{E} - q\dot{\vec{r}}\times(\nabla\times\vec{A}) = 0$$

But $\nabla\times\vec{A} = \text{curl}\,\vec{A} = \vec{B}$

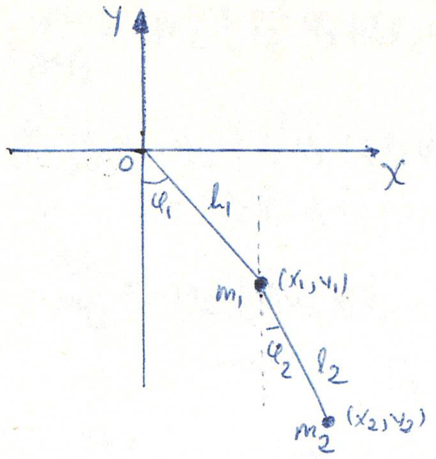
(i.e.)

$$m\ddot{\vec{r}} - q\vec{E} - q\dot{\vec{r}}\times\vec{B} = 0$$

or $$m\ddot{\vec{r}} = q\vec{E} + q\dot{\vec{r}}\times\vec{B}$$

$$\dot{\vec{r}} = \vec{v}$$

$$\therefore m\ddot{\vec{r}} = q\vec{E} + q\vec{v}\times\vec{B}$$



$x_1 = l_1 \sin\varphi_1 \ , \ y_1 = -l_1 \cos\varphi_1$

$x_2 = x_1 + l_2 \sin\varphi_2$

$\quad = l_1 \sin\varphi_1 + l_2 \sin\varphi_2$

$y_2 = y_1 - l_2 \cos\varphi_2$

$\quad = -l_1 \cos\varphi_1 - l_2 \cos\varphi_2$

$\dot{x}_1 = l_1 \dot{\varphi}_1 \cos\varphi_1$

$\dot{y}_1 = l \dot{\varphi}_1 \sin\varphi_1$

$\dot{x}_2 = l_1 \dot{\varphi}_1 \cos\varphi_1 + l_2 \dot{\varphi}_2 \cos\varphi_2$

$\dot{y}_2 = l_1 \dot{\varphi}_1 \sin\varphi_1 + l_2 \dot{\varphi}_2 \sin\varphi_2$

$T = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2)$

$T = \frac{1}{2} m_1 (l_1^2 \dot{\varphi}_1^2 \cos^2\varphi_1 + l_1^2 \dot{\varphi}_1^2 \sin^2\varphi_1)$

$+ \frac{1}{2} m_2 (l_1^2 \dot{\varphi}_1^2 \cos^2\varphi_1 + l_2^2 \dot{\varphi}_2^2 \cos^2\varphi_2$

$+ 2 l_1 l_2 \dot{\varphi}_1 \dot{\varphi}_2 \cos\varphi_1 \cos\varphi_2$

$+ l_1^2 \dot{\varphi}_1^2 \sin^2\varphi_1 + l_2^2 \dot{\varphi}_2^2 \sin^2\varphi_2$

$+ 2 l_1 l_2 \dot{\varphi}_1 \dot{\varphi}_2 \sin\varphi_1 \sin\varphi_2)$

$= \frac{1}{2} m_1 l_1^2 \dot{\varphi}_1^2$

$+ \frac{1}{2} m_2 (l_1^2 \dot{\varphi}_1^2 + l_2^2 \dot{\varphi}_2^2$

$+ 2 l_1 l_2 \dot{\varphi}_1 \dot{\varphi}_2 (\cos\varphi_1 \cos\varphi_2 + \sin\varphi_1 \sin\varphi_2))$

$= \frac{1}{2} m_1 l_1^2 \dot{\varphi}_1^2$

$+ \frac{1}{2} m_2 \{ l_1^2 \dot{\varphi}_1^2 + l_2^2 \dot{\varphi}_2^2$

$+ 2 l_1 l_2 \dot{\varphi}_1 \dot{\varphi}_2 \cos(\varphi_1 - \varphi_2) \}$

$U = m_1 g y_1 + m_2 g y_2$

$= -m_1 g l_1 \cos\varphi_1 - m_2 g (l_1 \cos\varphi_1 + l_2 \cos\varphi_2)$

$L = T - U$

$= \frac{1}{2} m_1 l_1^2 \dot{\varphi}_1^2$

$+ \frac{1}{2} m_2 \{ l_1^2 \dot{\varphi}_1^2 + l_2^2 \dot{\varphi}_2^2$

$+ 2 l_1 l_2 \dot{\varphi}_1 \dot{\varphi}_2 \cos(\varphi_1 - \varphi_2) \}$

$+ m_1 g l_1 \cos\varphi_1 + m_2 g (l_1 \cos\varphi_1 + l_2 \cos\varphi_2)$

$\frac{d}{dt} \frac{\partial L}{\partial \dot{\varphi}_1} - \frac{\partial L}{\partial \varphi_1} = 0$

$\frac{\partial L}{\partial \dot{\varphi}_1} = m_1 l_1^2 \dot{\varphi}_1 + m_2 \{ l_1^2 \dot{\varphi}_1$

$+ l_1 l_2 \dot{\varphi}_2 \cos(\varphi_1 - \varphi_2) \}$

$\frac{d}{dt} \frac{\partial L}{\partial \dot{\varphi}_1} = m_1 l_1^2 \ddot{\varphi}_1$

$+ m_2 \{ l_1^2 \ddot{\varphi}_1 + l_1 l_2 \ddot{\varphi}_2 \cos(\varphi_1 - \varphi_2)$

$- l_1 l_2 \dot{\varphi}_2 (\dot{\varphi}_1 - \dot{\varphi}_2) \sin(\varphi_1 - \varphi_2) \}$

$\frac{\partial L}{\partial \varphi_1} = -m_2 \{ l_1 l_2 \dot{\varphi}_1 \dot{\varphi}_2 \sin(\varphi_1 - \varphi_2)$

$+ g l_1 \sin\varphi_1 \}$

$- m_1 g l_1 \sin\varphi_1$

So the equation of motion corresponding to φ_1 becomes

$m_1 l_1^2 \ddot{\varphi}_1 + m_2 \{ l_1^2 \ddot{\varphi}_1 + l_1 l_2 \ddot{\varphi}_2 \cos(\varphi_1 - \varphi_2)$

$- l_1 l_2 \dot{\varphi}_2 (\dot{\varphi}_1 - \dot{\varphi}_2) \sin(\varphi_1 - \varphi_2) \}$

$+ m_2 \{ l_1 l_2 \dot{\varphi}_1 \dot{\varphi}_2 \sin(\varphi_1 - \varphi_2) + g l_1 \sin\varphi_1 \}$

$+ m_1 g l_1 \sin\varphi_1 = 0$

$m_1 l_1 \ddot{\varphi}_1 + m_2 \{ l_1 \ddot{\varphi}_1 + l_2 \ddot{\varphi}_2 \cos(\varphi_1 - \varphi_2)$

$+ l_2 \dot{\varphi}_2^2 \sin(\varphi_1 - \varphi_2) + g \sin\varphi_1 \}$

$+ m_1 g \sin\varphi_1 = 0$

$(m_1 + m_2) l_1 \ddot{\varphi}_1 + (m_1 + m_2) g \sin\varphi_1$

$+ m_2 l_2 \{ \ddot{\varphi}_2 \cos(\varphi_1 - \varphi_2) + \dot{\varphi}_2^2 \sin(\varphi_1 - \varphi_2) \}$

$= 0$

$\quad$ -- (1)

$\frac{d}{dt} \frac{\partial L}{\partial \dot{\varphi}_2} - \frac{\partial L}{\partial \varphi_2} = 0$

$$\frac{\partial L}{\partial \dot\varphi_2} = m_2\{l_2^2\dot\varphi_2 + l_1 l_2\dot\varphi_1\cos(\varphi_1-\varphi_2)\}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot\varphi_2} = m_2\{l_2^2\ddot\varphi_2 + l_1 l_2\ddot\varphi_1\cos(\varphi_1-\varphi_2) - l_1 l_2\dot\varphi_1(\dot\varphi_1-\dot\varphi_2)\sin(\varphi_1-\varphi_2)\}$$

$$\frac{\partial L}{\partial \varphi_2} = m_2\{l_1 l_2\dot\varphi_1\dot\varphi_2\sin(\varphi_1-\varphi_2) - g l_2\sin\varphi_2\}$$

The equation of motion corresponding to φ_2 becomes

$$m_2\{l_2^2\ddot\varphi_2 + l_1 l_2\ddot\varphi_1\cos(\varphi_1-\varphi_2) - l_1 l_2\dot\varphi_1(\dot\varphi_1-\dot\varphi_2)\sin(\varphi_1-\varphi_2) - l_1 l_2\dot\varphi_1\dot\varphi_2\sin(\varphi_1-\varphi_2) + g l_2\sin\varphi_2\} = 0$$

$$l_2\ddot\varphi_2 + l_1\ddot\varphi_1\cos(\varphi_1-\varphi_2) - l\dot\varphi_1^2\sin(\varphi_1-\varphi_2) + g\sin\varphi_2 = 0$$

--- (2)

Thus the general equations of motion of the system are

$$(m_1+m_2)l_1\ddot\varphi_1 + (m_1+m_2)g\sin\varphi_1 + m_2 l_2\{\ddot\varphi_2\cos(\varphi_1-\varphi_2) + \dot\varphi_2^2\sin(\varphi_1-\varphi_2)\} = 0$$

$$l_1\ddot\varphi_1\cos(\varphi_1-\varphi_2) - l_1\dot\varphi_1^2\sin(\varphi_1-\varphi_2) + l_2\ddot\varphi_2 + g\sin\varphi_2 = 0$$

Consider then the case of small oscillation where

$\sin\varphi_1 \approx \varphi_1$, $\sin\varphi_2 \approx \varphi_2$

$\sin(\varphi_1-\varphi_2) \approx \varphi_1-\varphi_2$, $\cos(\varphi_1-\varphi_2) \approx 1$

So now the equations reduce to

$$(m_1+m_2)l_1\ddot\varphi_1 + (m_1+m_2)g\varphi_1 + m_2 l_2\{\ddot\varphi_2 + \dot\varphi_2^2(\varphi_1-\varphi_2)\} = 0$$

$$l_1\ddot\varphi_1 - l_1\dot\varphi_1^2(\varphi_1-\varphi_2) + l_2\ddot\varphi_2 + g\varphi_2 = 0$$

Since φ_1 & φ_2 are small, $\dot\varphi_1$ & $\dot\varphi_2$ are also small so that we neglect terms of order two or greater than two whereby

$$(m_1+m_2)l_1\ddot\varphi_1 + (m_1+m_2)g\varphi_1 + m_2 l_2\ddot\varphi_2 = 0$$

$$l_1\ddot\varphi_1 + l_2\ddot\varphi_2 + g\varphi_2 = 0$$

Suppose

$$\varphi_1 = Ae^{i\lambda t}$$

$$\varphi_2 = Be^{i\lambda t}$$

$$\Rightarrow -(m_1+m_2)l_1\lambda^2 A + (m_1+m_2)gA - (m_2 l_2\lambda^2 B) = 0$$

$$-l_1\lambda^2 A - l_2\lambda^2 B + gB = 0$$

$$\frac{A}{B} = \frac{m_2 l_2\lambda^2}{(m_1+m_2)\{g-l_1\lambda^2\}} \quad \text{---- (3)}$$

$$\text{or } \frac{A}{B} = \frac{g-l_2\lambda^2}{l_1\lambda^2} \quad \text{------ (4)}$$

Then,

$$\frac{m_2 l_2\lambda^2}{(m_1+m_2)(g-l_1\lambda^2)} = \frac{g-l_2\lambda^2}{l_1\lambda^2}$$

$$m_2 l_1 l_2\lambda^4 = (m_1+m_2)(g-l_1\lambda^2)(g-l_2\lambda^2) = (m_1+m_2)(g^2 - gl_1\lambda^2 - gl_2\lambda^2 + l_1 l_2\lambda^4)$$

$$m_1 l_1 l_2\lambda^4 - (m_1+m_2)g(l_1+l_2)\lambda^2 + (m_1+m_2)g^2 = 0$$

i.e.,

$$\lambda^2 = \frac{(m_1+m_2)g(l_1+l_2)}{2m_1 l_1 l_2} \pm \frac{\sqrt{(m_1+m_2)^2 g^2(l_1+l_2)^2 - 4m_1(m_1+m_2)g^2 l_1 l_2}}{2m_1 l_1 l_2}$$

--- (*)

Cases:

I. $m_1 >> m_2$

$$\lambda^2 = \tfrac{1}{2}(1+\tfrac{m_2}{m_1})(g/l_1+g/l_2) \pm \tfrac{1}{2}\left[(1+m_2/m_1)^2(g/l_1+g/l_2)^2 - 4(1+\tfrac{m_2}{m_1})\tfrac{g}{l_1}\cdot\tfrac{g}{l_2}\right]^{1/2}$$

Let $\mu = m_2/m_1$, $\omega_{01}^2 = g/l$, $\omega_{02}^2 = g/l_2$

$$\lambda^2 = \frac{1+\mu}{2}(\omega_{01}^2+\omega_{02}^2) \pm \tfrac{1}{2}\left[(1+\mu)^2(\omega_{01}^2+\omega_{02}^2)^2 - 4(1+\mu)\omega_{01}^2\omega_{02}^2\right]^{1/2}$$

Since $m_1 >> m_2$, $\mu^2 <<< 1$ and neglecting μ^2 we obtain

$$\lambda^2 = \frac{1+\mu}{2}(\omega_{01}^2+\omega_{02}^2) \pm \tfrac{1}{2}\left[(1+2\mu)(\omega_{01}^2+\omega_{02}^2)^2 - 4(1+\mu)\omega_{01}^2\omega_{02}^2\right]^{1/2}$$

$$= \frac{1+\mu}{2}(\omega_{01}^2+\omega_{02}^2) \pm \tfrac{1}{2}\left[(1+2\mu)(\omega_{01}^4+\omega_{02}^4) - 2\omega_{01}^2\omega_{02}^2\right]^{1/2}$$

$$= \frac{1+\mu}{2}(\omega_{01}^2+\omega_{02}^2) \pm \tfrac{1}{2}\left[(\omega_{02}^2-\omega_{01}^2)^2 + 2\mu(\omega_{01}^4+\omega_{02}^4)\right]^{1/2}$$

i.e., $$\lambda^2 = \frac{1+\mu}{2}(\omega_{01}^2+\omega_{02}^2) - \tfrac{1}{2}\left[(\omega_{02}^2-\omega_{01}^2)^2 + 2\mu(\omega_{01}^4+\omega_{02}^4)\right]^{1/2} = \omega_1^2$$

or $$\lambda^2 = \frac{1+\mu}{2}(\omega_{01}^2+\omega_{02}^2) + \tfrac{1}{2}\left[(\omega_{02}^2-\omega_{01}^2) + 2\mu(\omega_{01}^4+\omega_{02}^4)\right]^{1/2} = \omega_2^2$$

$\Rightarrow$ $\lambda = \pm\omega_1$ or $\lambda = \pm\omega_2$

$\therefore$ $$\varphi_1 = A_1 e^{i\omega_1 t} + A_2 e^{-i\omega_1 t} + A_3 e^{i\omega_2 t} + A_4 e^{-i\omega_2 t}$$

$$\varphi_2 = B_1 e^{i\omega_1 t} + B_2 e^{-i\omega_1 t} + B_3 e^{i\omega_2 t} + B_4 e^{-i\omega_2 t}$$

or writing in terms of sine and cosine functions

$$\varphi_1 = a_1 \cos\omega_1 t + b_1 \sin\omega_1 t + a_2 \cos\omega_2 t + b_2 \sin\omega_2 t$$

$$\varphi_2 = \gamma_1 a_1 \cos\omega_1 t + \gamma_1 b_1 \sin\omega_1 t + \gamma_2 a_2 \cos\omega_2 t + \gamma_2 b_2 \sin\omega_2 t$$

Where γ_1 and γ_2 are the values of A/B which result for $\lambda = \omega_1$ and $\lambda = \omega_2$ respectively.

Suppose $\varphi_1(0) = 0$, $\dot\varphi_1(0) = \dot\varphi_{01}$, $\varphi_2(0) = 0$, $\dot\varphi_2(0) = 0$

It follows,

$$\left.\begin{aligned} a_1 + a_2 &= 0 \\ \gamma_1 a_1 + \gamma_2 a_2 &= 0 \end{aligned}\right\} \Rightarrow a_1 = a_2 = 0$$

$$\left.\begin{aligned} \omega_1 b_1 + \omega_2 b_2 &= 0 \\ \gamma_1\omega_1 b_1 + \gamma_2\omega_2 b_2 &= \dot\varphi_{01} \end{aligned}\right\}$$

$$\Rightarrow b_1 = \frac{\dot\varphi_{01}}{\omega_1(\gamma_1-\gamma_2)}$$

$$b_2 = \frac{\dot\varphi_{01}}{\omega_2(\gamma_2-\gamma_1)}$$

$$\therefore \varphi_1 = \frac{\dot\varphi_{01}}{\gamma_1-\gamma_2}\left(\frac{\sin\omega_1 t}{\omega_1} - \frac{\sin\omega_2 t}{\omega_2}\right)$$

$$\varphi_2 = \frac{\dot\varphi_{01}}{\gamma_2-\gamma_1}\left(\frac{\gamma_1}{\omega_1}\sin\omega_1 t - \frac{\gamma_2}{\omega_2}\sin\omega_2 t\right)$$

II $m_2 >> m_1$

$$\lambda^2 = \frac{1+\mu}{2}(\omega_{01}^2+\omega_{02}^2) \pm \tfrac{1}{2}\left[(1+\mu)^2(\omega_{01}^2+\omega_{02}^2)^2 - 4(1+\mu)\omega_{01}^2\omega_{02}^2\right]^{1/2}$$

In this case $\mu = \frac{m_2}{m_1} >> 1$ that we neglect 1.

$$\lambda^2 = \frac{\mu}{2}(\omega_{01}^2+\omega_{02}^2) \pm \tfrac{1}{2}\left[\mu^2(\omega_{01}^2+\omega_{02}^2)^2 - 4\mu\omega_{01}^2\omega_{02}^2\right]^{1/2}$$

(i.e.) $$\lambda^2 = \frac{\mu}{2}(\omega_{01}^2+\omega_{02}^2) - \tfrac{1}{2}\left[\mu^2(\omega_{01}^2+\omega_{02}^2)^2 - 4\mu\omega_{01}^2\omega_{02}^2\right]^{1/2} = \omega_1'^2$$

or $$\lambda^2 = \frac{\mu}{2}(\omega_{01}^2+\omega_{02}^2) + \tfrac{1}{2}\left[\mu^2(\omega_{01}^2+\omega_{02}^2)^2 - 4\mu\omega_{01}^2\omega_{02}^2\right]^{1/2} = \omega_2'^2$$

$\Rightarrow$ $\lambda = \pm\omega_1'$ or $\lambda = \pm\omega_2'$

$$\therefore \varphi_1 = A_1 e^{i\omega_1' t} + A_2' e^{-i\omega_1' t} + A_3' e^{i\omega_2' t} + A_4' e^{-i\omega_2' t}$$

$$\varphi_2 = B_1' e^{i\omega_1' t} + B_2' e^{-i\omega_1' t} + B_3 e^{i\omega_2' t} + B_4 e^{-i\omega_2' t}$$

or
$$\varphi_1 = a_1' \cos\omega_1' t + b_1' \sin\omega_1' t + a_2' \cos\omega_2' t + b_2' \sin\omega_2' t$$

$$\varphi_2 = \gamma_1' a_1' \cos\omega_1' t + \gamma_1' b_1' \sin\omega_1' t + \gamma_2' a_2' \cos\omega_2' t + \gamma_2' b_2' \sin\omega_2 t$$

where γ_1' and γ_2' are the values of A/B for $\lambda = \omega_1'$ and $\lambda = \omega_2'$ respectively.

Once again it is possible to apply initial conditions to determine the constants.

Name: Rēru Isogaye

$$Q = \ln(1+\sqrt{q}\cos p)$$
$$P = 2(1+\sqrt{q}\cos p)\sqrt{q}\sin p \quad (*)$$

To show

(i) the given equations represent a canonical transformation

(ii). the generating function is

$$F = -(e^{Q}-1)^2 \tan p$$

where q and p are canonical variables.

Solution

(i) If the given equations represent a canonical transformation we must have

$$\dot{Q} = \frac{\partial H}{\partial P} \quad \text{---(1)}$$
$$\dot{P} = -\frac{\partial H}{\partial Q} \quad \text{---(2)}$$

From $(*)$ we can have an invert relation for q and p, whence

$$q = q(Q,P)$$
$$p = p(Q,P)$$

$$\Rightarrow \dot{q} = \frac{\partial q}{\partial Q}\dot{Q} + \frac{\partial q}{\partial P}\dot{P} = \frac{\partial H}{\partial p}$$
$$\dot{p} = \frac{\partial p}{\partial Q}\dot{Q} + \frac{\partial p}{\partial P}\dot{P} = -\frac{\partial H}{\partial q}$$

... as q and p are canonical variables.

$$\frac{\partial H}{\partial p} = \frac{\partial H}{\partial Q}\frac{\partial Q}{\partial p} + \frac{\partial H}{\partial P}\frac{\partial P}{\partial p}$$
$$\frac{\partial H}{\partial q} = \frac{\partial H}{\partial Q}\frac{\partial Q}{\partial q} + \frac{\partial H}{\partial P}\frac{\partial P}{\partial q}$$

$$\Rightarrow \dot{q} = \frac{\partial q}{\partial Q}\dot{Q} + \frac{\partial q}{\partial P}\dot{P}$$
$$= \frac{\partial H}{\partial Q}\frac{\partial Q}{\partial p} + \frac{\partial H}{\partial P}\frac{\partial P}{\partial p}$$
$$\dot{p} = \frac{\partial p}{\partial Q}\dot{Q} + \frac{\partial p}{\partial P}\dot{P}$$
$$= -\frac{\partial H}{\partial Q}\frac{\partial Q}{\partial q} - \frac{\partial H}{\partial P}\frac{\partial P}{\partial q}$$

... a system of two linear equations in $\dot{Q}$ and $\dot{P}$.

A unique solution can be obtained if only the determinant

$$\Delta = \begin{vmatrix} \frac{\partial q}{\partial Q} & \frac{\partial q}{\partial P} \\ \frac{\partial p}{\partial Q} & \frac{\partial p}{\partial P} \end{vmatrix} \neq 0$$

Then,

$$\dot{Q} = \frac{1}{\Delta}\begin{vmatrix} \frac{\partial H}{\partial Q}\frac{\partial Q}{\partial p} + \frac{\partial H}{\partial P}\frac{\partial P}{\partial p} & \frac{\partial q}{\partial P} \\ -\frac{\partial H}{\partial Q}\frac{\partial Q}{\partial q} - \frac{\partial H}{\partial P}\frac{\partial P}{\partial q} & \frac{\partial p}{\partial P} \end{vmatrix}$$

$$= \frac{1}{\Delta}\left\{\frac{\partial H}{\partial Q}\frac{\partial Q}{\partial p}\frac{\partial p}{\partial P} + \frac{\partial H}{\partial P}\frac{\partial P}{\partial p}\frac{\partial p}{\partial P} + \frac{\partial H}{\partial Q}\frac{\partial Q}{\partial q}\frac{\partial q}{\partial P} + \frac{\partial H}{\partial P}\frac{\partial P}{\partial q}\frac{\partial q}{\partial P}\right\}$$

$$= \frac{1}{\Delta}\left\{\frac{\partial H}{\partial Q}\left[\frac{\partial Q}{\partial p}\frac{\partial p}{\partial P} + \frac{\partial Q}{\partial q}\frac{\partial q}{\partial P}\right] + \frac{\partial H}{\partial P}\left[\frac{\partial P}{\partial p}\frac{\partial p}{\partial P} + \frac{\partial P}{\partial q}\frac{\partial q}{\partial P}\right]\right\}$$

$$= \frac{1}{\Delta}\left\{\frac{\partial H}{\partial Q}\frac{dQ}{dP} + \frac{\partial H}{\partial P}\frac{dP}{dP}\right\}$$

$$= \frac{1}{\Delta}\frac{\partial H}{\partial P}$$

$$\dot{P} = \frac{1}{\Delta}\begin{vmatrix} \frac{\partial q}{\partial Q} & \frac{\partial H}{\partial Q}\frac{\partial Q}{\partial p} + \frac{\partial H}{\partial P}\frac{\partial P}{\partial p} \\ \frac{\partial p}{\partial Q} & -\frac{\partial H}{\partial Q}\frac{\partial Q}{\partial q} - \frac{\partial H}{\partial P}\frac{\partial P}{\partial q} \end{vmatrix}$$

$$= -\frac{1}{\Delta}\left\{ \frac{\partial H}{\partial Q}\left[\frac{\partial Q}{\partial q}\frac{\partial q}{\partial Q} + \frac{\partial Q}{\partial p}\frac{\partial p}{\partial Q}\right] + \frac{\partial H}{\partial P}\left[\frac{\partial P}{\partial q}\frac{\partial q}{\partial Q} + \frac{\partial P}{\partial p}\frac{\partial p}{\partial Q}\right]\right\}$$

$$= -\frac{1}{\Delta}\left\{\frac{\partial H}{\partial Q}\frac{dQ}{dQ} + \frac{\partial H}{\partial P}\frac{dP}{dQ}\right\}$$

$$= -\frac{1}{\Delta}\frac{\partial H}{\partial Q}$$

so it is obtained that

$$\dot{Q} = \frac{1}{\Delta}\frac{\partial H}{\partial P} \quad \text{------} \quad (3)$$

$$\dot{P} = -\frac{1}{\Delta}\frac{\partial H}{\partial Q} \quad \text{----} \quad (4)$$

for (3) and (4) to be identical with (1) and (2), respectively,

$$\Delta = 1$$

Δ is the Jacobian of the transformation from the p, q system to that of the P, Q, i.e.,

$$\Delta = \frac{\partial(q,p)}{\partial(Q,P)}$$

since the inverse transformation is possible,

$$\frac{\partial(q,p)}{\partial(Q,P)} \cdot \frac{\partial(Q,P)}{\partial(q,p)} = 1$$

$$\Rightarrow \frac{\partial(Q,P)}{\partial(q,p)} = 1$$

$$\frac{\partial(Q,P)}{\partial(q,p)} = \begin{vmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial q} & \frac{\partial P}{\partial p} \end{vmatrix} = 1$$

So now

$$Q = \ln(1+\sqrt{q}\cos p)$$

$$P = 2(1+\sqrt{q}\cos p)\sqrt{q}\sin p$$

$$\Rightarrow \frac{\partial Q}{\partial q} = \frac{\cos p}{2\sqrt{q}(1+\sqrt{q}\cos p)}$$

$$\frac{\partial Q}{\partial p} = \frac{-\sqrt{q}\sin p}{1+\sqrt{q}\cos p}$$

$$\frac{\partial P}{\partial q} = 2\left(\frac{\sin p}{2\sqrt{q}} + \sin p\cos p\right)$$

$$\frac{\partial P}{\partial p} = 2[\sqrt{q}\cos p + q(\cos^2 p - \sin^2 p)]$$

$$\frac{\partial(Q,P)}{\partial(q,p)} = \begin{vmatrix} \frac{\cos p}{2\sqrt{q}(1+\sqrt{q}\cos p)} & \frac{-\sqrt{q}\sin p}{1+\sqrt{q}\cos p} \\ 2\sin p\left(\frac{1}{2\sqrt{q}}+\cos p\right) & 2[\sqrt{q}\cos p + q(\cos^2 p - \sin^2 p)] \end{vmatrix}$$

$$= \frac{\cos p}{\sqrt{q}(1+\sqrt{q}\cos p)} \cdot [\sqrt{q}\cos p + q(\cos^2 p - \sin^2 p)] + \frac{2\sqrt{q}\sin^2 p\left(\frac{1}{2\sqrt{q}} + \cos p\right)}{1+\sqrt{q}\cos p}$$

$$= \frac{\cos p[\cos p + \sqrt{q}(\cos^2 p - \sin^2 p)] + \sin^2 p(1+2\sqrt{q}\cos p)}{1+\sqrt{q}\cos p}$$

$$= \frac{[\cos^2 p + \sqrt{q}(\cos^3 p - \cos p\sin^2 p) + \sin^2 p + 2\sqrt{q}\cos p\sin^2 p]}{1+\sqrt{q}\cos p}$$

$$= \frac{1+\sqrt{q}(\cos^3 p + \cos p\sin^2 p)}{1+\sqrt{q}\cos p}$$

$$= \frac{1+\sqrt{q}\cos p(\cos^2 p + \sin^2 p)}{1+\sqrt{q}\cos p}$$

$$= \frac{1+\sqrt{q}\cos p}{1+\sqrt{q}\cos p}$$

$$= 1$$

i.e., $\dot{Q} = \frac{\partial H}{\partial P}$

$\dot{P} = -\frac{\partial H}{\partial Q}$

as $\Delta = 1$.

Therefore, the given transformation equations are canonical transformations

(ii) All the four forms of the generating functions are possible, since one can replace anyone of the four variables in terms of any two of the remaining three. But ~~F₃ which~~ F_2 will give the simplest possible expression.

$$\frac{\partial F_2}{\partial Q} = -p$$

$$\Rightarrow \int \frac{\partial F_2}{\partial Q} dQ = -\int p\, dQ$$

$$F_2 = -\int p\, dQ$$

$$Q = \ln(1+\sqrt{q}\cos p)$$

$$P = 2(1+\sqrt{q}\cos p)\sqrt{q}\sin p$$

$$\text{i.e., } p = 2e^{Q}\frac{(e^{Q}-1)}{\cos p}\sin p$$

$$= 2(e^{2Q}-e^{Q})\tan p$$

$$F_2 = -\int 2(e^{2Q}-e^{Q})\tan p\, dQ$$

$$= -\tan p \int d(e^{Q}-1)^2$$

$$= -\tan p\,(e^{Q}-1)^2$$

So the generating function is given by

$$\boxed{F_2 = -(e^{Q}-1)^2 \tan p}$$

Name : Biru Tsegaye

Consider a system of n particles.
The angular momentum of the system as a whole is given by

$$\bar{D} = \sum_{k=1}^{n} (\vec{r}_k \times \vec{P}_k)$$

$$\vec{r}_k \times \vec{P}_k = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ x_k & y_k & z_k \\ P_{kx} & P_{ky} & P_{kz} \end{vmatrix}$$

$$= \bar{i}(y_k P_{kz} - z_k P_{ky}) - \bar{j}(x_k P_{kz} - z_k P_{kx}) + \bar{k}(x_k P_{ky} - y_k P_{kx})$$

$$\Rightarrow \bar{D} = \bar{i}\left\{\sum_{k=1}^{n}(y_k P_{kz} - z_k P_{ky})\right\} - \bar{j}\left\{\sum_{k=1}^{n}(x_k P_{kz} - z_k P_{kx})\right\} + \bar{k}\left\{\sum_{k=1}^{n}(x_k P_{ky} - y_k P_{kx})\right\}$$

i.e., $D_x = \sum_{k=1}^{n}(y_k P_{kz} - z_k P_{ky})$

$D_y = \sum_{k=1}^{n}(x_k P_{kz} - z_k P_{kx})$

$D_z = \sum_{k=1}^{n}(x_k P_{ky} - y_k P_{kx})$

$$\bar{D}^2 = D_x^2 + D_y^2 + D_z^2 = \left\{\sum_{k=1}^{n}(y_k P_{kz} - z_k P_{ky})\right\}^2 + \left\{\sum_{k=1}^{n}(x_k P_{kz} - z_k P_{kx})\right\}^2 + \left\{\sum_{k=1}^{n}(x_k P_{ky} - y_k P_{kx})\right\}^2$$

For two functions F and G depending on the generalized coordinates q_j and the generalized momenta P_j, the poisson bracket is given by

$$\{F, G\} = \sum_{j=1}^{f}\left(\frac{\partial F}{\partial P_j}\frac{\partial G}{\partial q_j} - \frac{\partial F}{\partial q_j}\frac{\partial G}{\partial P_j}\right)$$

Treating x_k, y_k and z_k as generalized coordinates and P_{kx}, P_{ky} and P_{kz} as generalized momenta:

1. $$\{\bar{D}^2, D_x\} = \sum_{k=1}^{n}\left\{\left(\frac{\partial \bar{D}^2}{\partial P_{kx}}\frac{\partial D_x}{\partial x_k} + \frac{\partial \bar{D}^2}{\partial P_{ky}}\frac{\partial D_x}{\partial y_k} + \frac{\partial \bar{D}^2}{\partial P_{kz}}\frac{\partial D_x}{\partial z_k}\right) - \left(\frac{\partial \bar{D}^2}{\partial x_k}\frac{\partial D_x}{\partial P_{kx}} + \frac{\partial \bar{D}^2}{\partial y_k}\frac{\partial D_x}{\partial P_{ky}} + \frac{\partial \bar{D}^2}{\partial z_k}\frac{\partial D_x}{\partial P_{kz}}\right)\right\}$$

$$= \sum_{k=1}^{n}\left\{\left(\frac{\partial \bar{D}^2}{\partial P_{ky}}\frac{\partial D_x}{\partial y_k} + \frac{\partial \bar{D}^2}{\partial P_{kz}}\frac{\partial D_x}{\partial z_k}\right) - \left(\frac{\partial \bar{D}^2}{\partial y_k}\frac{\partial D_x}{\partial P_{ky}} + \frac{\partial \bar{D}^2}{\partial z_k}\frac{\partial D_x}{\partial P_{kz}}\right)\right\}$$

$$= 2\sum_{k=1}^{n}\Big\{\big[(y_k P_{kz} - z_k P_{ky})(-z_k) + (x_k P_{ky} - y_k P_{kx})x_k\big]P_{kz} + \big[(y_k P_{kz} - z_k P_{ky})y_k + (x_k P_{kz} - z_k P_{kx})x_k\big](-P_{ky}) - \big[(y_k P_{kz} - z_k P_{ky})P_{kz} + (x_k P_{ky} - y_k P_{kx})(-P_{kx})\big](-z_k) - \big[(y_k P_{kz} - z_k P_{ky})(-P_{ky}) + (x_k P_{kz} - z_k P_{kx})(-P_{kx})\big]y_k\Big\}$$

$$= 2\sum_{k=1}^{n}\big(-y_k z_k P_{kz}^2 + z_k^2 P_{ky}P_{kz} + x_k^2 P_{ky}P_{kz} - x_k y_k P_{kx}P_{kz} - y_k^2 P_{ky}P_{kz} + y_k z_k P_{ky}^2 - x_k^2 P_{ky}P_{kz} + x_k z_k P_{kx}P_{ky} + y_k z_k P_{kz}^2 - z_k^2 P_{ky}P_{kz} - x_k z_k P_{kx}P_{ky} + y_k z_k P_{kx}^2 + y_k^2 P_{ky}P_{kz} - y_k z_k P_{ky}^2 + x_k y_k P_{kx}P_{kz} - y_k z_k P_{kx}^2\big)$$

$= 0$

$\{\bar{D}^2, D_x\} = 0$

2. $$\{\bar{D}^2, D_y\} = \sum_{k=1}^{n}\left\{\left(\frac{\partial \bar{D}^2}{\partial P_{kx}}\frac{\partial D_y}{\partial x_k} + \frac{\partial \bar{D}^2}{\partial P_{ky}}\frac{\partial D_y}{\partial y_k} + \frac{\partial \bar{D}^2}{\partial P_{kz}}\frac{\partial D_y}{\partial z_k}\right) - \left(\frac{\partial \bar{D}^2}{\partial x_k}\frac{\partial D_y}{\partial P_{kx}} + \frac{\partial \bar{D}^2}{\partial y_k}\frac{\partial D_y}{\partial P_{ky}} + \frac{\partial \bar{D}^2}{\partial z_k}\frac{\partial D_y}{\partial P_{kz}}\right)\right\}$$

$$= \sum_{k=1}^{n}\left\{\left(\frac{\partial \bar{D}^2}{\partial P_{kx}}\frac{\partial D_y}{\partial x_k} + \frac{\partial \bar{D}^2}{\partial P_{kz}}\frac{\partial D_y}{\partial z_k}\right) - \left(\frac{\partial \bar{D}^2}{\partial x_k}\frac{\partial D_y}{\partial P_{kx}} + \frac{\partial \bar{D}^2}{\partial z_k}\frac{\partial D_y}{\partial P_{kz}}\right)\right\}$$

$\{\bar{D}^2, D_y\}$

$$= 2\sum_{k=1}^{n}\Big\{\big[(x_k p_{kz} - z_k p_{kx})(-z_k) + (x_k p_{ky} - y_k p_{kx})(-y_k)\big] p_{kz}$$
$$+ \big[(y_k p_{kz} - z_k p_{ky}) y_k + (x_k p_{kz} - z_k p_{kx}) x_k\big](-p_{kx})$$
$$- \big[(x_k p_{kz} - z_k p_{kx}) p_{kz} + (x_k p_{ky} - y_k p_{kx}) p_{ky}\big](-z_k)$$
$$- \big[(y_k p_{kz} - z_k p_{ky})(-p_{ky}) + (x_k p_{kz} - z_k p_{kx})(-p_{kx})\big] x_k\Big\}$$

$$= 2\sum_{k=1}^{n}\big(-x_k z_k p_{kz}^2 + z_k^2 p_{kx} p_{kz} - x_k y_k p_{ky} p_{kz} + y_k^2 p_{kx} p_{kz} - y_k^2 p_{kx} p_{kz} + y_k z_k p_{kx} p_{ky} - x_k^2 p_{kx} p_{kz} + x_k z_k p_{kx}^2$$
$$+ x_k z_k p_{kz}^2 - z_k^2 p_{kx} p_{kz} + x_k z_k p_{ky}^2 - y_k z_k p_{kx} p_{ky} + x_k y_k p_{ky} p_{kz} - x_k z_k p_{ky}^2 + x_k^2 p_{kx} p_{kz} - x_k z_k p_{kx}^2\big)$$

$$= 0$$

3. $$\{\bar{D}^2, D_z\} = \sum_{k=1}^{n}\Big\{\Big(\frac{\partial \bar{D}^2}{\partial p_{kx}}\frac{\partial D_z}{\partial x_k} + \frac{\partial \bar{D}^2}{\partial p_{ky}}\frac{\partial D_z}{\partial y_k} + \frac{\partial \bar{D}^2}{\partial p_{kz}}\frac{\partial D_z}{\partial z_k}\Big) - \Big(\frac{\partial \bar{D}^2}{\partial x_k}\frac{\partial D_z}{\partial p_{kx}} + \frac{\partial \bar{D}^2}{\partial y_k}\frac{\partial D_z}{\partial p_{ky}} + \frac{\partial \bar{D}^2}{\partial z_k}\frac{\partial D_z}{\partial p_{kz}}\Big)\Big\}$$

$$= \sum_{k=1}^{n}\Big\{\Big(\frac{\partial \bar{D}^2}{\partial p_{kx}}\frac{\partial D_z}{\partial x_k} + \frac{\partial \bar{D}^2}{\partial p_{ky}}\frac{\partial D_z}{\partial y_k}\Big) - \Big(\frac{\partial \bar{D}^2}{\partial x_k}\frac{\partial D_z}{\partial p_{kx}} + \frac{\partial \bar{D}^2}{\partial y_k}\frac{\partial D_z}{\partial p_{ky}}\Big)\Big\}$$

$$= \sum_{k=1}^{n}\Big\{\big[(x_k p_{kz} - z_k p_{kx})(-z_k) + (x_k p_{ky} - y_k p_{kx})(-y_k)\big] p_{ky}$$
$$+ \big[(y_k p_{kz} - z_k p_{ky})(-z_k) + (x_k p_{ky} - y_k p_{kx}) x_k\big](-p_{kx})$$
$$- \big[(x_k p_{kz} - z_k p_{kx})(p_{kz}) + (x_k p_{ky} - y_k p_{kx}) p_{ky}\big](-y_k)$$
$$- \big[(y_k p_{kz} - z_k p_{ky}) p_{kz} + (x_k p_{ky} - y_k p_{kx})(-p_{kx})\big] x_k$$

$\{\bar{D}^2, D_z\}$

$$= 2\sum_{k=1}^{n}\big(-x_k z_k p_{ky} p_{kz} + z_k^2 p_{kx} p_{ky} - x_k y_k p_{ky}^2 + y_k^2 p_{kx} p_{ky} + y_k z_k p_{kx} p_{kz} - z_k^2 p_{kx} p_{ky} - x_k^2 p_{kx} p_{ky} + x_k y_k p_{kx}^2$$
$$+ x_k y_k p_{kz}^2 - y_k z_k p_{kx} p_{kz} + x_k y_k p_{ky}^2 - y_k^2 p_{kx} p_{ky} - x_k y_k p_{kz}^2 + x_k z_k p_{ky} p_{kz} + x_k^2 p_{kx} p_{ky} - x_k y_k p_{kx}^2\big)$$

$$= 0$$

4. $$\{D_x, D_y\} = \sum_{k=1}^{n}\Big\{\Big(\frac{\partial D_x}{\partial p_{kx}}\frac{\partial D_y}{\partial x_k} + \frac{\partial D_x}{\partial p_{ky}}\frac{\partial D_y}{\partial y_k} + \frac{\partial D_x}{\partial p_{kz}}\frac{\partial D_y}{\partial z_k}\Big) - \Big(\frac{\partial D_x}{\partial x_k}\frac{\partial D_y}{\partial p_{kx}} + \frac{\partial D_x}{\partial y_k}\frac{\partial D_y}{\partial p_{ky}} + \frac{\partial D_x}{\partial z_k}\frac{\partial D_y}{\partial p_{kz}}\Big)\Big\}$$

$$= \sum_{k=1}^{n}\Big(\frac{\partial D_x}{\partial p_{kz}}\frac{\partial D_y}{\partial z_k} - \frac{\partial D_x}{\partial z_k}\frac{\partial D_y}{\partial p_{kz}}\Big)$$

$$= \sum_{k=1}^{n}(-y_k p_{kx} + p_{ky} x_k)$$

$$= \sum_{k=1}^{n}(x_k p_{ky} - y_k p_{kx})$$

$$= D_z$$

5. $$\{D_y, D_z\} = \sum_{k=1}^{n}\Big(\frac{\partial D_y}{\partial p_{kx}}\frac{\partial D_z}{\partial x_k} - \frac{\partial D_y}{\partial x_k}\frac{\partial D_z}{\partial p_{kx}}\Big)$$

$$= \sum_{k=1}^{n}(-z_k p_{ky} + p_{kz} y_k)$$

$$= \sum_{k=1}^{n}(y_k p_{kz} - z_k p_{ky})$$

$$= D_x$$

6. $$\{D_z, D_x\} = \sum_{k=1}^{n}\Big(\frac{\partial D_z}{\partial p_{ky}}\frac{\partial D_x}{\partial y_k} - \frac{\partial D_z}{\partial y_k}\frac{\partial D_x}{\partial p_{ky}}\Big)$$

$$= \sum_{k=1}^{n}(x_k p_{kz} - p_{kx} z_k)$$

$$= \sum_{k=1}^{n}(x_k p_{kz} - z_k p_{kx})$$

$$= D_y$$

The same result holds true for one particle.

In Quantum mechanics the angular momentum corresponding to its x- y- and z-components has respective operators

$$\hat{L}_x = -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)$$

$$\hat{L}_y = -i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)$$

$$\hat{L}_z = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$$

and $\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$

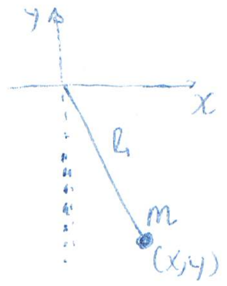
The result for the commutator operators is as follows:

1. $[\hat{L}^2, \hat{L}_x] = 0$
2. $[\hat{L}^2, \hat{L}_y] = 0$
3. $[\hat{L}^2, \hat{L}_z] = 0$
4. $[\hat{L}_x, \hat{L}_y] = i\hbar\hat{L}_z$
5. $[\hat{L}_y, \hat{L}_z] = i\hbar\hat{L}_x$
6. $[\hat{L}_z, \hat{L}_x] = i\hbar\hat{L}_y$

As can be seen, the six poissons brackets calculated above have got analogus results to the six commutator operators here. That means there is structural similarity between the poissons brackets and the ~~comm~~ quantum mechanical commutator operators corresponding to the angular momentum.

Name: Biru Tsegaye

The Mathematical pendulum



$x = l \sin\varphi$

$y = l - l\cos\varphi \sim -l\cos\varphi$

$\dot{x} = l\dot{\varphi}\cos\varphi$

$\dot{y} = l\dot{\varphi}\sin\varphi$

$T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2)$, $\quad U \sim -mgl\cos\varphi$

$= \frac{1}{2}ml^2\dot{\varphi}^2$

$L = T - U$

$= \frac{1}{2}ml^2\dot{\varphi}^2 + mgl\cos\varphi$

(1) $H = \dot{\varphi}P_\varphi - L$

$= \dot{\varphi}P_\varphi - \frac{1}{2}ml^2\dot{\varphi}^2 - mgl\cos\varphi$

$P_\varphi = \frac{\partial L}{\partial\dot{\varphi}} = ml^2\dot{\varphi}$

$\dot{\varphi} = \frac{P_\varphi}{ml^2}$

i.e. $H = \frac{P_\varphi^2}{ml^2} - \frac{1}{2}ml^2\frac{P_\varphi^2}{m^2l^4} - mgl\cos\varphi$

$= \frac{P_\varphi^2}{ml^2} - \frac{P_\varphi^2}{2ml^2} - mgl\cos\varphi$

or $H = \frac{P_\varphi^2}{2ml^2} - mgl\cos\varphi$

(2) The Hamilton-Jacobi eq.

$$\frac{1}{2ml^2}\left(\frac{\partial W}{\partial\varphi}\right)^2 - mgl\cos\varphi + \frac{\partial W}{\partial t} = 0$$

(3) The reduced equation

$W(\varphi,\alpha,t) = S(\varphi,\alpha) - \alpha t$

$$\frac{1}{2ml^2}\left(\frac{\partial S}{\partial\varphi}\right)^2 - mgl\cos\varphi - \alpha = 0$$

$\Rightarrow S = l\sqrt{2m}\int\sqrt{\alpha + mgl\cos\varphi}\,d\varphi$

Then, $W = l\sqrt{2m}\int\sqrt{\alpha+mgl\cos\varphi}\,d\varphi - \alpha t$

(4) $\beta = \frac{\partial W}{\partial\alpha} = l\sqrt{\frac{m}{2}}\int\frac{d\varphi}{\sqrt{\alpha+mgl\cos\varphi}} - t$

$\beta + t = l\sqrt{\frac{m}{2}}\int\frac{d\varphi}{\sqrt{\alpha+mgl\cos\varphi}}$

We know that the equation of motion of the mathematical pendulum

$$\ddot{\varphi} + (g/l)\sin\varphi = 0$$

does not have analytic solution unless we resort to certain approximations. So also the above integral cannot have analytic solution.

Consider then the case of small oscillations.

$\cos\varphi = 1 - \frac{\varphi^2}{2!} + \frac{\varphi^4}{4!} - \frac{\varphi^6}{6!} + \dots$

$\approx 1 - \frac{\varphi^2}{2}$ (for small φ)

i.e. $\beta + t = l\sqrt{\frac{m}{2}}\int\frac{d\varphi}{\sqrt{\alpha+mgl(1-\frac{\varphi^2}{2})}}$

$= l\sqrt{\frac{m}{2}}\sqrt{\frac{2}{mgl}}\int\frac{d\varphi}{\sqrt{\frac{2\alpha+2mgl}{mgl}-\varphi^2}}$

$= \sqrt{\frac{l}{g}}\int\frac{d\varphi}{\sqrt{\frac{2\alpha+2mgl}{mgl}-\varphi^2}}$

Let $a^2 = \frac{2\alpha+2mgl}{mgl}$

$$\beta + t = \sqrt{l/g} \int \frac{d\varphi}{\sqrt{a^2 - \varphi^2}}$$

$$= -\sqrt{l/g} \arccos \frac{\varphi}{a}$$

$$\varphi = a \cos\left[-\sqrt{g/l}\,(\beta + t)\right]$$

$$= a \cos \sqrt{g/l}\,(\beta + t)$$

cosine is even function

$$\sqrt{g/l} = \omega_0$$

and using the value of a

$$(*) \quad \boxed{\varphi = \sqrt{\frac{2\alpha + 2mgl}{mgl}} \cos \omega_0 (\beta + t)}$$

φ is given in terms of the new variables α and β and the time t. We know that this is exactly the solution for φ as a function of time when the pendulum undergoes small oscillations.

5. Determination of the constants

from the Hamilton-Jacobi equation in (2)

$$\left(\frac{\partial W}{\partial \varphi}\right)_{t=0} = P_\varphi \Big|_{t=0} = 0$$

$$-mgl\cos\varphi_0 - \alpha = 0$$

$$\alpha = -mgl\cos\varphi_0$$

where φ_0 is the initial displacement of the pendulum from its equilibrium position.

Also, from the same eq.

$$H + \frac{\partial W}{\partial t} = 0$$

$$H - \alpha = 0$$

or $\alpha = H$

That means the total energy of the system is a constant.

$$\alpha = H = E \text{ (total energy)}$$

from $(*)$ and using

$$\alpha = -mgl\cos\varphi_0$$

$$\varphi = \sqrt{\frac{-2mgl\cos\varphi_0 + 2mgl}{mgl}} \cos \omega_0 (\beta + t)$$

$$= \sqrt{-2\cos\varphi_0 + 2} \cos \omega_0 (\beta + t)$$

$$\boxed{\varphi(t) = \sqrt{2(1 - \cos\varphi_0)} \cos \omega_0 (\beta + t)}$$

Remark: Using a similar approximation ω can be found from the equation in (3).

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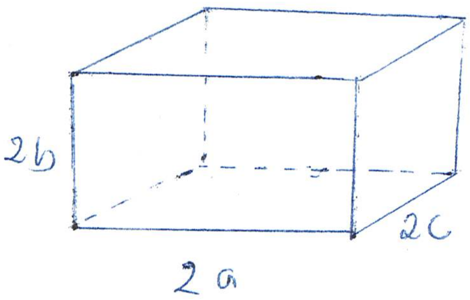
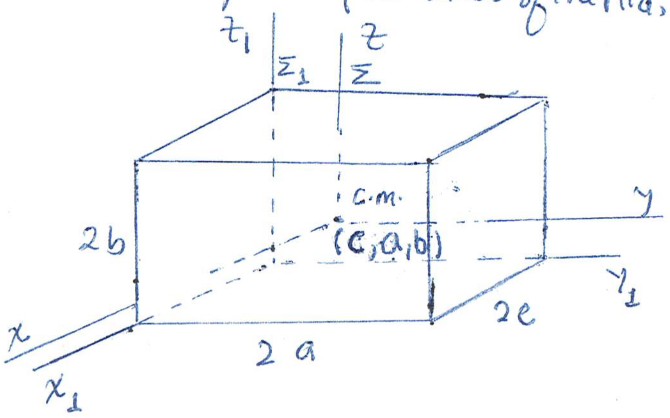


$a < b < c$

It is possible to take either the center of mass or an arbitrary point of a rigid body as the origin of the principal axes of inertia. But it must in both cases be true that the products of inertia like θ_{xy}, etc. are zero. This happens if the axes x, y, z of a frame of reference are taken along the principal diameters of the ellipsoid of inertia.

So then for the above parallepiped, three axes, whose origin lies at the c.m., and are perpendicular to the faces of the parallepiped will lie along the principal diameters of the ellipsoid of inertia and thus can serve as the principal axes of inertia.



Σ_1 is with origin at one corner of the parallepiped and Σ is with origin at the c.m. and axes lying along the principal diameters of the ellipsoid of inertia as can be verified by computing θ_{xy}, θ_{xz} & θ_{yz} and showing they have a zero value.

The volume of the parallepiped is

$$V = 2c \cdot 2a \cdot 2b = 8cab$$

1. To find the principal moments of inertia

Remark : It is possible to find the moments of inertia w.r.t. X_1, Y_1 and Z_1 and later apply Steiner's theorem to find the principal moments of inertia.

But let us directly find the principal moments of inertia.

Assume the parallelpiped is homogeneous, $\rho =$ constant.

The coordinates of the c.m. (c, a, b) are w.r.t Σ_1. But with Σ we may have $(0,0,0)$. Whence

$$-c \le x \le c$$
$$-a \le y \le a$$
$$-b \le z \le b$$

$$\theta_{xx} = \rho \int_{-c}^{c}\int_{-a}^{a}\int_{-b}^{b} (y^2+z^2)\, dx\, dy\, dz$$

$$= \rho \int_{-c}^{c} dx \int_{-a}^{a} y^2 dy \int_{-b}^{b} dz$$

$$+ \rho \int_{-c}^{c} dx \int_{-a}^{a} dy \int_{-b}^{b} z^2 dz$$

$$\Theta_{xx} = \rho [x]_{-c}^{c} \left[\frac{y^3}{3}\right]_{-a}^{a} [z]_{-b}^{b}$$

$$+ \rho [x]_{-c}^{c} [y]_{-a}^{a} \left[\frac{z^3}{3}\right]_{-b}^{b}$$

$$= \rho \cdot 2c \cdot \frac{2a^3}{3} \cdot 2b$$

$$+ \rho \cdot 2c \cdot 2a \cdot \frac{2b^3}{3}$$

$$= \frac{1}{3} \rho \, 2c \cdot 2a \cdot 2b \, (a^2 + b^2)$$

$$= \frac{1}{3} \rho V (a^2 + b^2)$$

$$= \frac{1}{3} M (a^2 + b^2)$$

$$\Theta_{yy} = \rho \int_{-c}^{c} \int_{-a}^{a} \int_{-b}^{b} (x^2 + z^2) \, dx\, dy\, dz$$

$$= \rho \int_{-c}^{c} x^2 dx \int_{-a}^{a} dy \int_{-b}^{b} dz$$

$$+ \rho \int_{-c}^{c} dx \int_{-a}^{a} dy \int_{-b}^{b} z^2 dz$$

$$= \rho \left[\frac{x^3}{3}\right]_{-c}^{c} [y]_{-a}^{a} [z]_{-b}^{b}$$

$$+ \rho [x]_{-c}^{c} [y]_{-a}^{a} \left[\frac{z^3}{3}\right]_{-b}^{b}$$

$$= \rho \cdot \frac{2c^3}{3} \cdot 2a \cdot 2b$$

$$+ \rho \cdot 2c \cdot 2a \cdot \frac{2b^3}{3}$$

$$= \frac{1}{3} \rho \cdot 2c \cdot 2a \cdot 2b \, (c^2 + b^2)$$

$$= \frac{1}{3} M (b^2 + c^2)$$

$$\Theta_{zz} = \rho \int_{-c}^{c} \int_{-a}^{a} \int_{-b}^{b} (x^2 + y^2) \, dx\, dy\, dz$$

$$= \rho \int_{-c}^{c} x^2 dx \int_{-a}^{a} dy \int_{-b}^{b} dz$$

$$+ \rho \int_{-c}^{c} dx \int_{-a}^{a} y^2 dy \int_{-b}^{b} dz$$

$$\Theta_{zz} = \rho \left[\frac{x^3}{3}\right]_{-c}^{c} [y]_{-a}^{a} [z]_{-b}^{b}$$

$$+ \rho [x]_{-c}^{c} \left[\frac{y^3}{3}\right]_{-a}^{a} [z]_{-b}^{b}$$

$$= \rho \cdot \frac{2c^3}{3} \cdot 2a \cdot 2b$$

$$+ \rho \cdot 2c \cdot \frac{2a^3}{3} \cdot 2b$$

$$= \frac{1}{3} \rho \cdot 2c \cdot 2a \cdot 2b \, (c^2 + a^2)$$

$$= \frac{1}{3} M (a^2 + c^2)$$

Products of Inertia

$$\Theta_{xy} = \rho \int_{-c}^{c} \int_{-a}^{a} \int_{-b}^{b} xy \, dx\, dy\, dz$$

$$= \rho \int_{-c}^{c} x\,dx \int_{-a}^{a} y\,dy \int_{-b}^{b} dz$$

$$= \rho \left[\frac{x^2}{2}\right]_{-c}^{c} \left[\frac{y^2}{2}\right]_{-a}^{a} [z]_{-b}^{b}$$

$$= \rho \cdot 0 \cdot 0 \cdot 2b$$

$$= 0$$

Similarly $\Theta_{xz} = 0$, $\Theta_{yz} = 0$

Therefore the principal moments of inertia are

$$\Theta_{xx} = \Theta_1 = \frac{1}{3} M (a^2 + b^2)$$

$$\Theta_{yy} = \Theta_2 = \frac{1}{3} M (b^2 + c^2)$$

$$\Theta_{zz} = \Theta_3 = \frac{1}{3} M (a^2 + c^2)$$

(2) To determine axes of equilibrium

From Euler's equations of motion for a rigid body and considering free rotation we have:

$$\theta_1\dot{\omega}_1-\omega_2\omega_3(\theta_2-\theta_3)=0$$
$$\theta_2\dot{\omega}_2-\omega_3\omega_1(\theta_3-\theta_1)=0$$
$$\theta_3\dot{\omega}_3-\omega_1\omega_2(\theta_1-\theta_2)=0$$

a) consider rotation about the principal axis x, $\omega_2=\omega_3=0$ and $\omega_1=\omega_1^0$. Now for small perturbation we may have

$$\omega_1=\omega_1^0+\delta\omega_1$$
$$\omega_2=\delta\omega_2$$
$$\omega_3=\delta\omega_3$$

$$\Rightarrow \theta_1\delta\dot{\omega}_1-\delta\omega_2\delta\omega_3(\theta_2-\theta_3)=0$$
$$\theta_2\delta\dot{\omega}_2-\delta\omega_3(\omega_1^0+\delta\omega_1)(\theta_3-\theta_1)=0$$
$$\theta_3\delta\dot{\omega}_3-(\omega_1^0+\delta\omega_1)\delta\omega_2(\theta_1-\theta_2)=0$$

Since we assume small perturbations we neglect terms of order two so as to obtain

$$\theta_1\delta\dot{\omega}_1=0$$
$$\theta_2\delta\dot{\omega}_2-\omega_1^0\delta\omega_3(\theta_3-\theta_1)=0$$
$$\theta_3\delta\dot{\omega}_3-\omega_1^0\delta\omega_2(\theta_1-\theta_2)=0$$

The first equation implies that ω_1 remains constant regardless of the perturbation and hence it cannot tell about the condition of stability and instability that we reject it.

$$\theta_2\delta\dot{\omega}_2-\omega_1^0\delta\omega_3(\theta_3-\theta_1)=0$$
$$\theta_3\delta\dot{\omega}_3-\omega_1^0\delta\omega_2(\theta_1-\theta_2)=0$$

$$\delta\omega_2=Be^{\lambda t},\quad \delta\omega_3=ce^{\lambda t}$$

$$\theta_2\lambda\,\delta\omega_2-\omega_1^0\delta\omega_3(\theta_3-\theta_1)=0$$
$$\theta_3\lambda\,\delta\omega_3-\omega_1^0\delta\omega_2(\theta_1-\theta_2)=0$$

A system of linear equations in $\delta\omega_2$ and $\delta\omega_3$

$$\theta_2\lambda\,\delta\omega_2-\omega_1^0\delta\omega_3(\theta_3-\theta_1)=0$$
$$-\omega_1^0(\theta_1-\theta_2)\delta\omega_2+\theta_3\lambda\,\delta\omega_3=0$$

For nontrivial solution

$$\begin{vmatrix}\theta_2\lambda & -\omega_1^0(\theta_3-\theta_1)\\ -\omega_1^0(\theta_1-\theta_2) & \theta_3\lambda\end{vmatrix}=0$$

$$\theta_2\theta_3\lambda^2=\omega_1^{0\,2}(\theta_1-\theta_2)(\theta_3-\theta_1)$$

$$\lambda^2=\frac{\omega_1^{0\,2}}{\theta_2\theta_3}(\theta_1-\theta_2)(\theta_3-\theta_1)$$

$$\lambda=\pm\frac{\omega_1^0}{\sqrt{\theta_2\theta_3}}\left[(\theta_1-\theta_2)(\theta_3-\theta_1)\right]^{1/2}$$

$$\theta_1-\theta_2=\tfrac{1}{3}M(a^2+b^2-b^2-c^2)$$
$$=\tfrac{1}{3}M(a^2-c^2)<0$$
since $a<c$

$$\theta_3-\theta_1=\tfrac{1}{3}M(a^2+c^2-a^2-b^2)$$
$$=\tfrac{1}{3}M(c^2-b^2)>0$$
since $b<c$.

(i.e.) $(\theta_1-\theta_2)(\theta_3-\theta_1)<0$

$$\therefore\ \lambda=\pm\frac{i\omega_0}{\sqrt{\theta_2\theta_3}}\left[|(\theta_1-\theta_2)(\theta_3-\theta_1)|\right]^{1/2}$$

That means upon perturbation there is oscillation about the x-axis. Thus the principal axis x corresponds to stable equilibrium.

b) consider rotation about the principal axis y, $\omega_1=0$, $\omega_2=\omega_2^0$, $\omega_3=0$.

For small perturbation

$\omega_1 = \delta\omega_1$

$\omega_2 = \omega_2^0 + \delta\omega_2$

$\omega_3 = \delta\omega_3$

The Euler equations for free rotations becomes

$$\theta_1 \delta\dot{\omega}_1 - (\omega_2^0 + \delta\omega_2)\delta\omega_3(\theta_2 - \theta_3) = 0$$
$$\theta_2 \delta\dot{\omega}_2 - \delta\omega_3 \delta\omega_1 (\theta_3 - \theta_1) = 0$$
$$\theta_3 \delta\dot{\omega}_3 - \delta\omega_1(\omega_2^0 + \delta\omega_2)(\theta_1 - \theta_2) = 0$$

Once again neglecting the second order terms we obtain

$$\theta_1 \delta\dot{\omega}_1 - \omega_2^0 \delta\omega_3(\theta_2 - \theta_3) = 0$$
$$\theta_2 \delta\dot{\omega}_2 = 0$$
$$\theta_3 \delta\dot{\omega}_3 - \omega_2^0 \delta\omega_1(\theta_1 - \theta_2) = 0$$

Leaving the 2nd eq.

$$\theta_1 \delta\dot{\omega}_1 - \omega_2^0 \delta\omega_3(\theta_2 - \theta_3) = 0$$
$$\theta_3 \delta\dot{\omega}_3 - \omega_2^0 \delta\omega_1(\theta_1 - \theta_2) = 0$$

Let $\delta\omega_1 = A e^{\lambda t}$, $\delta\omega_3 = c e^{\lambda t}$

$$\theta_1 \lambda \delta\omega_1 - \omega_2^0 \delta\omega_3(\theta_2 - \theta_3) = 0$$
$$\theta_3 \lambda \delta\omega_3 - \omega_2^0 \delta\omega_1(\theta_1 - \theta_2) = 0$$

$$\theta_1 \lambda \delta\omega_1 - \omega_2^0 \delta\omega_3(\theta_2 - \theta_3) = 0$$
$$-\omega_2^0 \delta\omega_1(\theta_1 - \theta_2) + \theta_3 \lambda \delta\omega_3 = 0$$

For nontrivial solution

$$\begin{vmatrix} \theta_1 \lambda & -\omega_2^0(\theta_2 - \theta_3) \\ -\omega_2^0(\theta_1 - \theta_2) & \theta_3 \lambda \end{vmatrix} = 0$$

$$\theta_1 \theta_3 \lambda^2 = \omega_2^{0\,2}(\theta_1 - \theta_2)(\theta_2 - \theta_3)$$

$$\lambda^2 = \frac{\omega_2^{0\,2}}{\theta_1 \theta_3}(\theta_1 - \theta_2)(\theta_2 - \theta_3)$$

$$\lambda = \frac{\pm \omega_2^0}{\sqrt{\theta_1 \theta_3}}\left[(\theta_1 - \theta_2)(\theta_2 - \theta_3)\right]^{1/2}$$

$\theta_1 - \theta_2 < 0$, from (a)

$$\theta_2 - \theta_3 = \tfrac{1}{3}M(b^2 + c^2 - a^2 - c^2)$$
$$= \tfrac{1}{3}M(b^2 - a^2) > 0, \quad a < b$$

i.e., $(\theta_1 - \theta_2)(\theta_2 - \theta_3) < 0$

$$\therefore \lambda = \frac{\pm i \omega_2^0}{\sqrt{\theta_1 \theta_3}}\left[|(\theta_1 - \theta_2)(\theta_2 - \theta_3)|\right]^{1/2}$$

There is oscillation about the principal axis y and hence it corresponds to stable equilibrium.

c) Rotation about the principal axis z. $\omega_1 = 0, \omega_2 = 0$, $\omega_3 = \omega_3^0$

For small perturbation

$\omega_1 = \delta\omega_1$

$\omega_2 = \delta\omega_2$

$\omega_3 = \omega_3^0 + \delta\omega_3$

The Euler equations for free rotation takes on the form

$$\theta_1 \delta\dot{\omega}_1 - \delta\omega_2(\omega_3^0 + \delta\omega_3)(\theta_2 - \theta_3) = 0$$
$$\theta_2 \delta\dot{\omega}_2 - (\omega_3^0 + \delta\omega_3)\delta\omega_1(\theta_3 - \theta_1) = 0$$
$$\theta_3 \delta\dot{\omega}_3 - \delta\omega_1 \delta\omega_2 = 0$$

Neglecting 2nd order small terms

$$\theta_1 \delta\dot{\omega}_1 - \omega_3^0 \delta\omega_2(\theta_2 - \theta_3) = 0$$
$$\theta_2 \delta\dot{\omega}_2 - \omega_3^0 \delta\omega_1(\theta_3 - \theta_1) = 0$$
$$\theta_3 \delta\dot{\omega}_3 = 0$$

Leaving the last eq.

$$\theta_1 \delta\dot{\omega}_1 - \omega_3^0(\theta_2-\theta_3)\delta\omega_2 = 0$$

$$-\omega_3^0(\theta_3-\theta_1)\delta\omega_1 + \theta_2\delta\dot{\omega}_2 = 0$$

$\delta\omega_1 = A e^{\lambda t}$, $\delta\omega_2 = B e^{\lambda t}$

$$\Rightarrow \theta_1 \lambda \delta\omega_1 - \omega_3^0(\theta_2-\theta_3)\delta\omega_2 = 0$$

$$-\omega_3^0(\theta_3-\theta_1)\delta\omega_1 + \theta_2\lambda\delta\omega_2 = 0$$

For non trivial solution

$$\begin{vmatrix} \theta_1\lambda & -\omega_3^0(\theta_2-\theta_3) \\ -\omega_3^0(\theta_3-\theta_1) & \theta_2\lambda \end{vmatrix} = 0$$

$$\theta_1\theta_2\lambda^2 = \omega_3^{0\,2}(\theta_3-\theta_1)(\theta_2-\theta_3)$$

$$\lambda^2 = \frac{\omega_3^{0\,2}}{\theta_1\theta_2}(\theta_3-\theta_1)(\theta_2-\theta_3)$$

$$\lambda = \frac{\pm\omega_3^0}{\sqrt{\theta_1\theta_2}}\left[(\theta_3-\theta_1)(\theta_2-\theta_3)\right]^{\frac{1}{2}}$$

$\theta_2 - \theta_3 > 0$ from b

$(\theta_3-\theta_1) > 0$ from a

i.e., $(\theta_3-\theta_1)(\theta_2-\theta_3) > 0$

and $\lambda = \frac{\pm\omega_3^0}{\sqrt{\theta_1\theta_2}}\left[(\theta_3-\theta_1)(\theta_2-\theta_3)\right]^{\frac{1}{2}}$

is negative or positive real number.

This implies that the perturbation will grow with time and thus the principal axis z does not correspond to stable rotation.

Assignment I

1. Show that $\vec{A} \perp \vec{B}$ if $|\vec{A}+\vec{B}| = |\vec{A}-\vec{B}|$

2. Show that an angle α inserted in a semicircle is the right angle

3. Direction cosines of $\vec{A}$ are $\alpha_1, \alpha_2, \alpha_3$ and those of $\vec{B}$ are $\beta_1, \beta_2, \beta_3$. Show that

$$\cos\varphi = \alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3$$

φ being the angle between $\vec{A}, \vec{B}$.

4. Let

$a_{11} = 1 \quad a_{12} = -1 \quad a_{13} = 0$

$a_{21} = -2 \quad a_{22} = 3 \quad a_{23} = 1$

$a_{31} = 2 \quad a_{32} = 0 \quad a_{33} = 4$

Show that

$$\Sigma a_{ii} = 8 \qquad \Sigma a_{i1} a_{i2} = -7 \qquad \Sigma a_{i2} a_{i3} = 3$$

5. Let $b_1 = 1 \quad b_2 = -1 \quad b_3 = 4$

with a_{ij} from problem 4, show that

$$\sum_i a_{ii} b_i = 2, \quad \sum_j a_{j1} b_j = 11 \quad \sum_i \sum_j a_{ij} b_j = 49$$

6. Show that

$$\sum_j \delta_{ij} b_j = \sum_j \delta_{ji} b_j = b_i$$

7. With a_{ij} from prob. 4, evaluate

$$\sum_j a_{ij}\delta_{1j} \qquad \sum_i a_{12}\delta_{ii} \qquad \sum_i \sum_k a_{1i} a_{2k} \delta_{ik}$$

8. Show that $\vec{C} = \vec{A} \times \vec{B}$ obeys the vector transformation law

$$c_i' = \sum_k a_{ik} c_k$$

9. Determine the following expressions

$(\vec{A}-\vec{B})\cdot(\vec{A}+\vec{B})$, $(\vec{A}-\vec{B})\times(\vec{A}+\vec{B})$

10. using the vectors

$\vec{P} = \vec{i}\cos\theta + \vec{j}\sin\theta$

$\vec{Q} = \vec{i}\cos\varphi - \vec{j}\sin\varphi$

$\vec{R} = \vec{i}\cos\varphi + \vec{j}\sin\varphi$

Prove the trigonometric identities

$\cos(\theta+\varphi) = \cos\theta\cos\varphi - \sin\theta\sin\varphi$

$\sin(\theta+\varphi) = \sin\theta\cos\varphi + \cos\theta\sin\varphi$

?* 11. Find the ~~unit~~ form for the vector $\vec{r}$ satisfying the eq.

$\vec{r} \times (1,1,1) = (2,-4,2)$

12. Show that for all scalars l

$(\vec{a}+l\vec{b})\cdot(\vec{b}\times\vec{c}) = \vec{a}\cdot(\vec{b}\times\vec{c})$

13. Show that $(\vec{a}+\vec{b}+\vec{c})\cdot(\vec{b}\times\vec{c}) = \vec{a}\cdot(\vec{b}\times\vec{c})$

14. " " $\vec{a}\times[\vec{b}\times(\vec{c}\times\vec{a})] = (\vec{a}\cdot\vec{b})\,\vec{a}\times\vec{c}$

15. " " $\vec{A}\times(\vec{B}\times\vec{C}) + \vec{B}\times(\vec{C}\times\vec{A}) + \vec{C}\times(\vec{A}\times\vec{B}) = 0$

16. A vector $\vec{A}$ is decomposed into a radial vector $\vec{A}_r$ and a tangential $\vec{A}_t$. Let $\vec{r}^{\,0}$ is a unit vector in the radial direction. Show that

$\vec{A}_r = \vec{r}^{\,0}(\vec{A}\cdot\vec{r}^{\,0})$

$\vec{A}_t = -\vec{r}^{\,0}\times(\vec{r}^{\,0}\times\vec{A})$

17. Show that any vector $\vec{V}$ may be expressed in terms of the reciprocal vectors $\vec{a}^{\,1}, \vec{b}^{\,1}, \vec{c}^{\,1}$ by

$\vec{V} = (\vec{V}\cdot\vec{a})\,\vec{a}^{\,1} + (\vec{V}\cdot\vec{b})\,\vec{b}^{\,1} + (\vec{V}\cdot\vec{c})\,\vec{c}^{\,1}$

Rough calculations to problems on Assignment I

1. $\vec{A} \perp \vec{B} \Rightarrow \vec{A}\cdot\vec{B} = 0,\ \vec{A}, \vec{B} \neq 0$

$|\vec{A}+\vec{B}| = \{(\vec{A}+\vec{B})\cdot(\vec{A}+\vec{B})\}^{1/2}$

$= (\vec{A}\cdot\vec{A} + \vec{A}\cdot\vec{B} + \vec{B}\cdot\vec{A} + \vec{B}\cdot\vec{B})^{1/2}$

$= (A^2 + B^2 + 2\vec{A}\cdot\vec{B})^{1/2}$

$|\vec{A}-\vec{B}| = \{(\vec{A}-\vec{B})\cdot(\vec{A}-\vec{B})\}^{1/2}$

$= (\vec{A}\cdot\vec{A} - \vec{A}\cdot\vec{B} - \vec{B}\cdot\vec{A} + \vec{B}\cdot\vec{B})^{1/2}$

$= (A^2 + B^2 - 2\vec{A}\cdot\vec{B})^{1/2}$

$|\vec{A}+\vec{B}| = |\vec{A}-\vec{B}|$

$\Rightarrow (A^2+B^2+2\vec{A}\cdot\vec{B})^{1/2} = (A^2+B^2-2\vec{A}\cdot\vec{B})^{1/2}$

$A^2+B^2+2\vec{A}\cdot\vec{B} = A^2+B^2-2\vec{A}\cdot\vec{B}$

$\vec{A}\cdot\vec{B} = -\vec{A}\cdot\vec{B}$

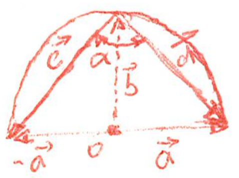
$2\vec{A}\cdot\vec{B} = 0$

$\vec{A}\cdot\vec{B} = 0$

(i.e.) $\vec{A} \perp \vec{B}$

$\therefore \vec{A} \perp \vec{B}$ if $|\vec{A}+\vec{B}| = |\vec{A}-\vec{B}|$

2.



if $\alpha = \pi/2$ then $\vec{c}\cdot\vec{d} = 0$

$|\vec{a}| = |-\vec{a}| = |\vec{b}| = r$, radius of the circle

$-\vec{a} = \vec{b} + \vec{c}$, $\quad \vec{a} = \vec{b} + \vec{d}$

$\Rightarrow \vec{c} = -(\vec{a}+\vec{b})$ $\quad \Rightarrow \vec{d} = \vec{a} - \vec{b}$

Now, $\vec{c}\cdot\vec{d} = -(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b})$

$= -a^2 + \vec{a}\cdot\vec{b} - \vec{b}\cdot\vec{a} + b^2$

$= b^2 - a^2$

$= r^2 - r^2$

$= 0$

But $\vec{c}\cdot\vec{d} = |\vec{c}||\vec{d}|\cos\alpha$

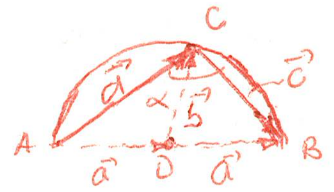
$\Rightarrow |\vec{c}||\vec{d}|\cos\alpha = 0$

$\cos\alpha = 0$

(i.e.) $\alpha = \pi/2$

$\therefore$ the angle inscribed in a semicircle is right angle.

or



if $\alpha = 90^0$, $\vec{c}\cdot\vec{d} = 0$

$|\vec{a}| = |\vec{b}| = r$, radius of the circle

$\vec{d} = \vec{a} + \vec{b}$ ------ (1)

$\vec{a} = \vec{b} + \vec{c}$

$\Rightarrow \vec{c} = \vec{a} - \vec{b}$ ------ (2)

Using (1) and (2)

$$\vec{c}\cdot\vec{d} = (\vec{a}-\vec{b})\cdot(\vec{a}+\vec{b})$$
$$= a^2 + \vec{a}\cdot\vec{b} - \vec{b}\cdot\vec{a} - b^2$$
$$= a^2 - b^2$$
$$= r^2 - r^2 = 0$$

$$\Rightarrow |\vec{c}||\vec{d}|\cos\alpha = 0$$
$$\cos\alpha = 0$$
i.e.) $\alpha = 90^0$

∴ The angle that is inscribed in a semicircle is a right angle.

3) from the def. for direction cosine

$\alpha_1 = \frac{A_x}{A}$, $\alpha_2 = A_y/A$, $\alpha_3 = A_z/A$

$\beta_1 = B_x/B$, $\beta_2 = B_y/B$, $\beta_3 = B_z/B$

$\vec{A}\cdot\vec{B} = A_x B_x + A_y B_y + A_z B_z$

But $A_x = \alpha_1 A$, $A_y = \alpha_2 A$, $A_z = \alpha_3 A$
$B_x = \beta_1 B$, $B_y = \beta_2 B$, $B_z = \beta_3 B$

$\Rightarrow \vec{A}\cdot\vec{B} = \alpha_1 A \beta_1 B + \alpha_2 A \beta_2 B + \alpha_3 A \beta_3 B$
$= AB(\alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3)$

i.e.) $AB\cos\varphi = AB(\alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3)$

∴ $\cos\varphi = \alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3$

4.

$\sum a_{ii} = a_{11} + a_{22} + a_{33} = 1 + 3 + 4 = 8$

$\sum a_{i1} a_{i2} = a_{11} a_{12} + a_{21} a_{22} + a_{31} a_{32} = (1)(-1) + (-2)(3) + (2)(0)$
$= -1 - 6 + 0 = -7$

$\sum a_{i2} a_{i3} = a_{12} a_{13} + a_{22} a_{23} + a_{32} a_{33} = (-1)(0) + (3)(1) + (0)(4)$
$= 0 + 3 + 0 = 3$

5.

$\sum a_{1i} b_i = a_{11} b_1 + a_{12} b_2 + a_{13} b_3 = 1 \cdot 1 + 1 \cdot 1 + 0 \cdot 4 = 2$

$\sum a_{j1} b_j = a_{11} b_1 + a_{21} b_2 + a_{31} b_3 = 1 \cdot 1 + 2 \cdot 1 + 2 \cdot 4 = 11$

$\sum_i \sum_j a_{ji} a_{ij} b_j = \sum_i (a_{1i} a_{i1} b_1 + a_{2i} a_{i2} b_2 + a_{3i} a_{i3} b_3)$

$= a_{11} a_{11} b_1 + a_{21} a_{12} b_2 + a_{31} a_{13} b_3$
$+ a_{12} a_{21} b_1 + a_{22} a_{22} b_2 + a_{32} a_{23} b_3$
$+ a_{13} a_{31} b_1 + a_{23} a_{32} b_2 + a_{33} a_{33} b_3$

$= 1 \cdot 1 \cdot 1 - 2 \cdot 1 \cdot 1 + 2 \cdot 0 \cdot 4$
$+ 1 \cdot 2 \cdot 1 - 3 \cdot 3 \cdot 1 + 0 \cdot 1 \cdot 4$
$+ 0 \cdot 2 \cdot 1 + 1 \cdot 0 \cdot 1 + 4 \cdot 4 \cdot 4$
$= 1 - 2 + 2 - 6 + 0 + 0 - 0 + 0 + 64$
$64 - 5 = 59$

6.

$\sum_j \delta_{ij} b_j \neq 0$ if $j = i$

$\sum_j \delta_{ij} b_j = \sum_i \delta_{ii} b_i = \sum_i b_i$

$\sum_j \delta_{ji} b_j = \sum_i \delta_{ii} b_i = \sum b_i$

$\therefore \sum_j \delta_{ij} b_j = \sum_j \delta_{ji} b_j = \sum_i b_i$

7.

$\sum_j a_{1j} \delta_{1j} \neq 0$ if $j = 1$

$\sum_j a_{1j} \delta_{1j} = a_{11} \delta_{11} = 1 \cdot 1 = 1$

$\sum_i a_{12} \delta_{ii} = a_{12} \sum_i \delta_{ii} = a_{12} (\delta_{11} + \delta_{22} + \delta_{33}) = -1(1+1+1) = -3$

$\sum_i \sum_k a_{1i} a_{2k} \delta_{ik} = \sum_i (a_{1i} a_{21} \delta_{i1} + a_{1i} a_{22} \delta_{i2} + a_{1i} a_{23} \delta_{i3}$
~~$+ a_{1i} a_{23} \delta_{i3}$~~ $)$

$= a_{12}a_{21}\delta_{11} + a_{12}a_{22}\delta_{22} + a_{12}a_{23}\delta_{33}$

$= +1\cdot 2\cdot 1 - 1\cdot 3\cdot 1 - 1\cdot 1\cdot 1$

$= 2 - 3 - 1$

$= -2$

8\. $\vec{C} = \vec{A}\times\vec{B} \Rightarrow \vec{C} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$, Then, $\vec{C} = \vec{i}(A_yB_z - A_zB_y) + \vec{j}(A_zB_x - A_xB_z) + \vec{k}(A_xB_y - A_yB_x)$

Now, $C_x = A_yB_z - A_zB_y$, $C_y = A_zB_x - A_xB_z$, $C_z = A_xB_y - A_yB_x$

i.e., $C_i = A_jB_k - A_kB_j$

$\Rightarrow C_i' = \sum_l a_{jl}A_l \sum_m a_{km}B_m - \sum_l a_{kl}A_l \sum_m a_{jm}B_m$

$= \sum_{l,m}(a_{jl}A_l a_{km}B_m - a_{kl}a_{jm}A_lB_m)$

$= \sum_{l,m}(a_{jl}a_{km} - a_{kl}a_{jm})A_lB_m$ (*)

~~The combination of direction cosines in parentheses will vanish for m=l, i.e.,~~

The combination $(a_{jl}a_{km} - a_{kl}a_{jm})$ will vanish if $m = l$, i.e.,

$a_{jl}a_{kl} - a_{kl}a_{jl} = 0$

Thus j and k assume fixed values depending on the value assigned for i, and six combinations of l & m. Take $i = 2$, then $j = 3$, $k = 1$ (cyclic order), and we obtain the following combination.

$a_{33}a_{11} - a_{13}a_{31} = a_{22}$

$a_{31}a_{12} - a_{11}a_{32} = a_{23}$

$a_{32}a_{13} - a_{12}a_{33} = a_{21}$

and their negative

Also l & m take cyclic values: $l=1 \Rightarrow m=2$, $l=2 \Rightarrow m=3$, $l=3 \Rightarrow m=1$.

Using this in (*)

$C_2' = a_{22}A_3B_1 + a_{23}A_1B_2 + a_{21}A_2B_3$
$\quad - a_{22}A_1B_3 - a_{23}A_2B_1 - a_{21}A_3B_2$

$= a_{22}C_2 + a_{23}C_3 + a_{21}C_1$

$= a_{21}C_1 + a_{22}C_2 + a_{23}C_3$

$= \sum_n a_{2n}C_n$

In general, $C_r' = \sum_n a_{rn}C_n$

9.

9. i: $(\vec{A}-\vec{B})\cdot(\vec{A}+\vec{B})$

dot product is distributive

$\Rightarrow (\vec{A}-\vec{B})\cdot(\vec{A}+\vec{B}) = \vec{A}\cdot(\vec{A}+\vec{B}) - \vec{B}\cdot(\vec{A}+\vec{B})$

$= \vec{A}\cdot\vec{A} + \vec{A}\cdot\vec{B} - \vec{B}\cdot\vec{A} - \vec{B}\cdot\vec{B} = A^2 - B^2$

ii: $(\vec{A}-\vec{B})\times(\vec{A}+\vec{B})$

cross product is distributive

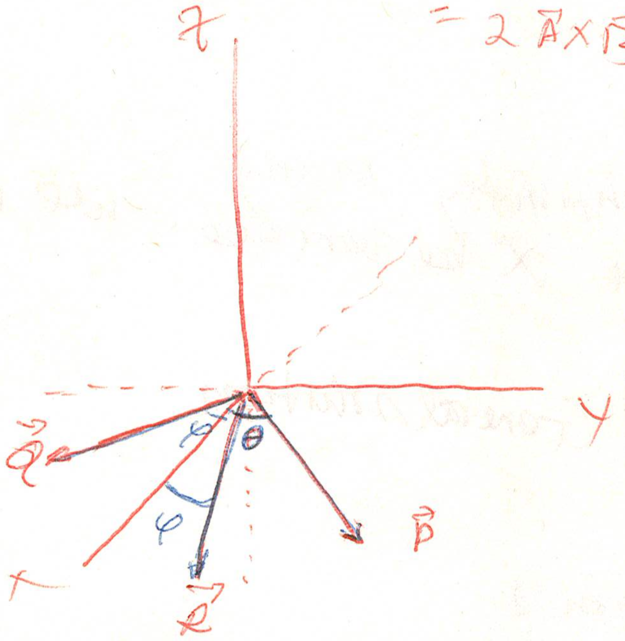
$\Rightarrow (\vec{A}-\vec{B})\times(\vec{A}+\vec{B}) = \vec{A}\times(\vec{A}+\vec{B}) - \vec{B}\times(\vec{A}+\vec{B})$

$= \underbrace{\vec{A}\times\vec{A}}_{0} + \vec{A}\times\vec{B} - \vec{B}\times\vec{A} - \underbrace{\vec{B}\times\vec{B}}_{0}$

$= \vec{A}\times\vec{B} + \vec{A}\wedge\vec{B}$

$= 2\vec{A}\times\vec{B}$

10.



The angle between $\vec{P}$ & $\vec{Q}$ is $(\theta+\varphi)$

$\vec{P}\cdot\vec{Q} = 1\cdot1\cos(\theta+\varphi) = \cos\theta\cos\varphi - \sin\theta\sin\varphi$

i.e.) $\cos(\theta+\varphi) = \cos\theta\cos\varphi - \sin\theta\sin\varphi$

$(\vec{Q}\times\vec{P}) = 1\cdot1\sin(\theta+\varphi)\vec{k} = (\cos\theta\sin\varphi + \sin\theta\cos\varphi)\vec{k}$

i.e.) $\sin(\theta+\varphi) = \sin\theta\cos\varphi + \sin\varphi\cos\theta$

11. $\vec{r}\times(1,1,1) = (2,-4,2)$

Let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$\Rightarrow \vec{r}\times(1,1,1) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix} = \vec{i}(y-z) + \vec{j}(z-x) + \vec{k}(x-y)$

i.e.) $\vec{i}(y-z) + \vec{j}(z-x) + \vec{k}(x-y) = (2,-4,2)$

$\Rightarrow$ $y - z = 2$

$z - x = -4$

$x - y = 2$

Rearranging

$x - y + 0z = 2$ --- 1

$-x + z = -4$ --- 2

$y - z = 2$ --- 3

adding (1) + (2)

$-y + z = -2$

or $y - z = 2$

So we can take

$x - y = 2$

$y - z = 2$

This system has infinitely many solution so, let x be variable. We have

x - variable
$y = x - 2$ | General solution
$z = y - 2$

Specific solutions

Example:

1. Let $x = 1$

$\Rightarrow$ $y = 1 - 2 = -1$

$z = y - 2 = -1 - 2 = -3$

i.e., $\vec{r} = (1, -1, -3)$

Check $\vec{r} \times (1,1,1) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & -3 \\ 1 & 1 & 1 \end{vmatrix}$

$= \vec{i}(-1+3) + \vec{j}(-3-1) + \vec{k}(1+1)$

$= 2\vec{i} - 4\vec{j} + 2\vec{k}$

$= (2, -4, 2)$ verified

2. Take $x = 2$

$y = 2 - 2 = 0$

$z = 0 - 2 = -2$

$\vec{r} \times (1,1,1) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 0 & -2 \\ 1 & 1 & 1 \end{vmatrix}$

$= \vec{i}(0+2) + \vec{j}(-2-2) + \vec{k}(2-0)$

$= 2\vec{i} - 4\vec{j} + 2\vec{k}$

$= (2, -4, 2)$

Q.E.D.

12. $(\vec{a} + l\vec{b}) \cdot (\vec{b} \times \vec{c})$

$= \vec{a} \cdot (\vec{b} \times \vec{c}) + l\vec{b} \cdot (\vec{b} \times \vec{c})$, dot product is distributive

$= \vec{a} \cdot \vec{b} \times \vec{c} + l[\vec{b} \cdot \vec{b} \times \vec{c}]$, l is scalar factor

$= \vec{a} \cdot \vec{b} \times \vec{c}$, as $\vec{b} \times \vec{c} \perp \vec{b}$, $\vec{b} \cdot \vec{b} \times \vec{c} = 0$

$\therefore (\vec{a} + l\vec{b}) \cdot (\vec{b} \times \vec{c}) = \vec{a} \cdot \vec{b} \times \vec{c}$

13. $(\vec{a} + \vec{b} + \vec{c}) \cdot (\vec{b} \times \vec{c})$

$= \vec{a} \cdot (\vec{b} \times \vec{c}) + \vec{b} \cdot (\vec{b} \times \vec{c}) + \vec{c} \cdot \vec{b} \times \vec{c}$

$= \vec{a} \cdot \vec{b} \times \vec{c}$, since $\vec{b} \cdot (\vec{b} \times \vec{c}) = 0$ & $\vec{c} \cdot (\vec{b} \times \vec{c}) = 0$

i.e.) $(\vec{b} \times \vec{c}) \perp \vec{b}$ & $(\vec{b} \times \vec{c}) \perp \vec{c}$.

14. $\vec{a} \times [\vec{b} \times (\vec{c} \times \vec{a})]$

$= \vec{b}(\vec{a} \cdot \vec{c} \times \vec{a}) - (\vec{c} \times \vec{a})(\vec{a} \cdot \vec{b})$

$= -(\vec{c} \times \vec{a})(\vec{a} \cdot \vec{b})$, as $(\vec{c} \times \vec{a}) \perp \vec{a}$, $\vec{a} \cdot \vec{c} \times \vec{a} = 0$

$= +(\vec{a} \times \vec{c})(\vec{a} \cdot \vec{b})$

$= (\vec{a} \cdot \vec{b}) \cdot \vec{a} \times \vec{c}$

15. $\vec{A} \times (\vec{B} \times \vec{C}) + \vec{B} \times (\vec{C} \times \vec{A}) + \vec{C} \times (\vec{A} \times \vec{B})$

$= \{\vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})\} + \{\vec{C}(\vec{B} \cdot \vec{A}) - \vec{A}(\vec{B} \cdot \vec{C})\} + \{\vec{A}(\vec{C} \cdot \vec{B}) - \vec{B}(\vec{C} \cdot \vec{A})\}$

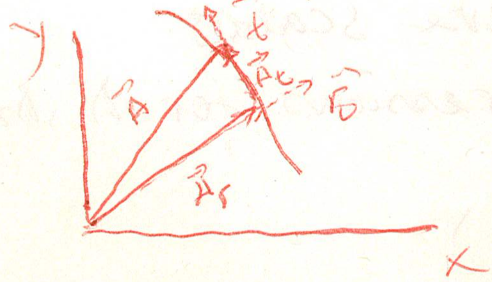
$= \{\vec{B}(\vec{A} \cdot \vec{C}) - \vec{B}(\vec{C} \cdot \vec{A})\} + \{\vec{C}(\vec{B} \cdot \vec{A}) - \vec{C}(\vec{A} \cdot \vec{B})\} + \{\vec{A}(\vec{C} \cdot \vec{B}) - \vec{A}(\vec{B} \cdot \vec{C})\}$

$= \{\vec{B}(\vec{A} \cdot \vec{C}) - \vec{B}(\vec{A} \cdot \vec{C})\} + \{\vec{C}(\vec{A} \cdot \vec{B}) - \vec{C}(\vec{A} \cdot \vec{B})\} + \{\vec{A}(\vec{B} \cdot \vec{C}) - \vec{A}(\vec{B} \cdot \vec{C})\}$

$= 0 + 0 + 0$

$= 0$

16.



As can be seen from the figure

$\vec{A} = A_r \vec{r}_0 + A_t \vec{t}$

Taking dot product of $\vec{A}$ with $\vec{r}_0$

$\vec{A} \cdot \vec{r}_0 = A_r \vec{r}_0 \cdot \vec{r}_0 + A_t \vec{t} \cdot \vec{r}_0$

$= A_r$, $\vec{t} \cdot \vec{r}_0 = 0$ as $\vec{t} \perp \vec{r}_0$

$A_r = \vec{A} \cdot \vec{r}_0$

$A_r \vec{r}_0 = (\vec{A} \cdot \vec{r}_0) \vec{r}_0$

i.e.) $\vec{A}_r = (\vec{A} \cdot \vec{r}_0) \vec{r}_0 = \vec{r}_0 (\vec{A} \cdot \vec{r}_0)$

Taking the cross product of $\vec{A}$ with $\vec{r_0}$

$$\vec{r_0} \times \vec{A} = \vec{r_0} \times (A_r \vec{r_0}) + \vec{r_0} \times (A_t \vec{t_0})$$

$$= A_r (\vec{r_0} \times \vec{r_0}) + A_t (\vec{r_0} \times \vec{t})$$

~~$= A_t \vec{n}$, $\vec{n}$ is the~~ normal to the plane of $\vec{r_0}$ & $\vec{t}$ ~~and it~~ is a unit vector

~~$A_t \vec{n} = \vec{r_0} \times \vec{A}$~~

~~Taking cross product with $\vec{r_0}$~~

~~$\vec{r_0}$~~

once again take cross product of the above eq. with $\vec{r_0}$

$$\vec{r_0} \times (\vec{r_0} \times \vec{A}) = \vec{r_0} \times A_t (\vec{r_0} \times \vec{t})$$

or/ $\vec{r_0} \times A_t (\vec{r_0} \times \vec{t}) = \vec{r_0} \times (\vec{r_0} \times \vec{A})$

$$= A_t [\vec{r_0} \times (\vec{r_0} \times \vec{t})$$

$$= A_t \{ (\vec{r_0} \cdot \vec{t}) \vec{r_0} - (\vec{r_0} \cdot \vec{r_0}) \vec{t} \}$$

$$= -A_t \vec{t}$$

$$= -\vec{A_t}$$

or $\quad -\vec{A_t} = \vec{r_0} \times (\vec{r_0} \times \vec{A})$

(i.e.) $\quad \vec{A_t} = -\vec{r_0} \times (\vec{r_0} \times \vec{A})$

17. Let $\vec{a}' = \vec{a}^{-1}$, $\vec{b}' = \vec{b}^{-1}$, $\vec{c}' = \vec{c}^{-1}$ and let $\vec{v}$ be given by

$$\vec{v} = \lambda_1 \vec{a}' + \lambda_2 \vec{b}' + \lambda_3 \vec{c}' \quad \text{--- (1)}$$

where λ_1, λ_2 & λ_3 are scalars.

To determine the expressions for λ_1, λ_2 & λ_3 eq. (1) suppose that

$$\vec{v} = (v_x, v_y, v_z)$$

$$\vec{a}' = (a_x', a_y', a_z') \quad \text{---- (2)}$$

$$\vec{b}' = (b_x', b_y', b_z')$$

$$\vec{c}' = (c_x', c_y', c_z')$$

Using the component expressions, eq. (2) ~~[illegible]~~ results in

$$v_x = \lambda_1 a'_x + \lambda_2 b'_x + \lambda_3 c'_x$$
$$v_y = \lambda_1 a'_y + \lambda_2 b'_y + \lambda_3 c'_y \quad \text{------} \quad 3$$
$$v_z = \lambda_1 a'_z + \lambda_2 b'_z + \lambda_3 c'_z$$

This is a system of linear eqs in λ_1, λ_2 & λ_3

$$\lambda_1 = \frac{\begin{vmatrix} v_x & b'_x & c'_x \\ v_y & b'_y & c'_y \\ v_z & b'_z & c'_z \end{vmatrix}}{\begin{vmatrix} a'_x & b'_x & c'_x \\ a'_y & b'_y & c'_y \\ a'_z & b'_z & c'_z \end{vmatrix}} = \frac{\begin{vmatrix} v_x & v_y & v_z \\ b'_x & b'_y & b'_z \\ c'_x & c'_y & c'_z \end{vmatrix}}{\begin{vmatrix} a'_x & a'_y & a'_z \\ b'_x & b'_y & b'_z \\ c'_x & c'_y & c'_z \end{vmatrix}} = \frac{\vec{v}\cdot\vec{b}'\times\vec{c}'}{\vec{a}'\cdot\vec{b}'\times\vec{c}'}$$

But $\dfrac{\vec{b}'\times\vec{c}'}{\vec{a}'\cdot\vec{b}'\times\vec{c}'} = \vec{a} \quad \Rightarrow \quad \lambda_1 = \vec{v}\cdot\vec{a}$

$$\lambda_2 = \frac{\begin{vmatrix} a'_x & v_x & c'_x \\ a'_y & v_y & c'_y \\ a'_z & v_z & c'_z \end{vmatrix}}{\vec{a}'\cdot\vec{b}'\times\vec{c}'} = \frac{\begin{vmatrix} a'_x & a'_y & a'_z \\ v_x & v_y & v_z \\ c'_x & c'_y & c'_z \end{vmatrix}}{\vec{a}'\cdot\vec{b}'\times\vec{c}'} = \frac{\vec{a}'\cdot\vec{v}\times\vec{c}'}{\vec{a}'\cdot\vec{b}'\times\vec{c}'} = \frac{\vec{v}\cdot\vec{c}'\times\vec{a}'}{\vec{a}'\cdot\vec{b}'\times\vec{c}'}$$

But $\dfrac{\vec{c}'\times\vec{a}'}{\vec{a}'\cdot\vec{b}'\times\vec{c}'} = \vec{b} \quad \Rightarrow \quad \lambda_2 = \vec{v}\cdot\vec{b}$

$$\lambda_3 = \frac{\begin{vmatrix} a'_x & b'_x & v_x \\ a'_y & b'_y & v_y \\ a'_z & b'_z & v_z \end{vmatrix}}{\vec{a}'\cdot\vec{b}'\times\vec{c}'} = \frac{\begin{vmatrix} a'_x & a'_y & a'_z \\ b'_x & b'_y & b'_z \\ v_x & v_y & v_z \end{vmatrix}}{\vec{a}'\cdot\vec{b}'\times\vec{c}'} = \frac{\vec{a}'\cdot\vec{b}'\times\vec{v}}{\vec{a}'\cdot\vec{b}'\times\vec{c}'} = \frac{\vec{v}\cdot\vec{a}'\times\vec{b}'}{\vec{a}'\cdot\vec{b}'\times\vec{c}'}$$

But $\dfrac{\vec{a}'\times\vec{b}'}{\vec{a}'\cdot\vec{b}'\times\vec{c}'} = \vec{c} \quad \Rightarrow \quad \lambda_3 = \vec{v}\cdot\vec{c}$

Then $\vec{v} = \lambda_1\vec{a}' + \lambda_2\vec{b}' + \lambda_3\vec{c}'$

$\therefore \vec{v} = (\vec{v}\cdot\vec{a})\vec{a}' + (\vec{v}\cdot\vec{b})\vec{b}' + (\vec{v}\cdot\vec{c})\vec{c}'$

or $\vec{v} = (\vec{v}\cdot\vec{a})\vec{a}^{-1} + (\vec{v}\cdot\vec{b})\vec{b}^{-1} + (\vec{v}\cdot\vec{c})\vec{c}^{-1}$

Assignment II

1. The continuous parameter t can take all real values. Sketch the curves whose parametric equations are resp.:
   a) $\vec{r} = (2\cos\pi t, \sin\pi t, 0)$
   b) $\vec{r} = (2\sin\pi t, 0, 0)$
   c) $\vec{r} = (t, |t|, 0)$
   d) $\vec{r} = (t^2, t^3 - t, 0)$
   e) $\vec{r} = \begin{cases} (t, -t, 0) & \text{for } -\infty < t \le 0 \\ (t, -t^2, 0) & \text{for } 0 < t < \infty \end{cases}$

2. Given $d\vec{r}/dt = \{-e^{-t}(\cos t + \sin t), e^{-t}(\cos t - \sin t), 0\}$. For $t=0$, the $\vec{r}$ vector is $\vec{r} = (1,0,0)$. Determine $\vec{r}$. Sketch the locus of the point with position vector for $t \ge 0$

3. Show in a diagram the direction of $d\vec{r}/dt$ at: $t=0$, $t=1$, $t=-1$, for the following curves
   a) $\vec{r} = (2\cos\frac{\pi t}{2}, \sin\frac{\pi t}{2}, 0)$ $t \in [-2, 2]$
   b) $\vec{r} = (t^2, t^3 - t, 0)$ $t \in (-\infty, \infty)$

4. Determine the unit tangent for the curve $\vec{r} = (3, t, t^2)$

5. Determine the unit tangent vector and the curvature for the curve $\vec{r} = (4\cos t, \cos 2t, \sin 2t)$

6. By finding $\vec{t}, \vec{n}$ and $\vec{b}$ verify that the plane parabolic curve $\vec{r} = (t, t^2/2, 0)$ have zero torsion.

7. Show that $d\vec{n}/ds = -\kappa\vec{t} + \tau\vec{b}$

8. A particle moves with velocity $\vec{v}$ and acc. $\vec{a}$, show that the radius of curvature of its path is

$$\rho = v^3 / |\vec{v} \times \vec{a}|$$

Determine the radius of curvature at 0 for $\vec{r} = (t, t^2, t^3)$.

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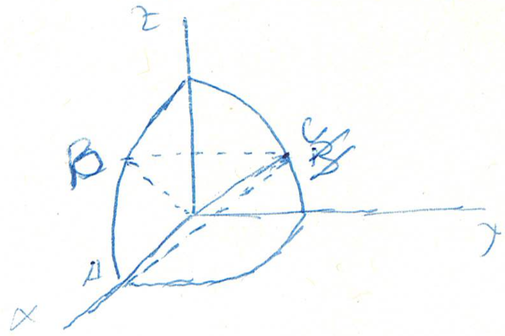
Assignment III

19. Find the angles of the spherical triangle defined by three vectors
$\vec{A} = \vec{l} = (1, 0, 0)$, $\vec{B} = \vec{m} = (1/\sqrt{2}, 0, 1/\sqrt{2})$, $\vec{C} = \vec{n} = (0, 1/\sqrt{2}, 1/\sqrt{2})$.
Each vector starts from the origin.
(Ans. $\angle A = 45^\circ$, $\angle B = 54.74^\circ$, $\angle C = 125.26^\circ$)



20. prove the following equation:
$$\text{grad } f(r) = \vec{r}_0 \, df/dr$$
where $\vec{r}_0$ is the radial unit vector, $r = \sqrt{x^2+y^2+z^2}$ distance from the origin.

21. Show that $\nabla(uv) = u\nabla v + v\nabla u$ where u & v are differentiable scalar functions of x, y, z.

22. Let $S(x,y,z) = (x^2+y^2+z^2)^{3/2}$. Determine ∇S, $|\nabla S|$ and direction cosines of ∇S at the point $(1,2,3)$.

23. Show that a necessary and sufficient condition that $u(x,y,z)$ and $v(x,y,z)$ are related by some function $f(u,v) = 0$ is that $\nabla u \times \nabla v = 0$.

24. If a vector function $\vec{A}(x,y,z,t)$ depends both on space coordinates x, y, z and time t, show that
$$d\vec{A} = (d\vec{r} \cdot \nabla)\vec{A} + \frac{\partial \vec{A}}{\partial t} dt$$

25. Let $u = 3x^2y$, $v = xz^2 - 2y$. Evaluate grad[(grad u)·(grad v)]

26. Calculate $\nabla \cdot \vec{r}$

27. calculate $\nabla \cdot (\vec{r} f(r))$

28. Show that $\nabla \cdot \vec{r}\, r^{n-1} = (n+2) r^{n-1}$

29. prove that $\nabla \cdot (f\vec{v}) = \nabla f \cdot \vec{v} + f \nabla \cdot \vec{v}$

31. A rigid body is rotating with constant angular velocity ω. Show that the linear velocity $\vec{v}$ is solenoidal.

32. Be sure that you understand the geometrical meaning of
a) grad φ b) div $\vec{A}$ c) curl $\vec{A}$ d) $\int_S \vec{A} \cdot d\vec{s}$ e) $\int_C \vec{A} \cdot d\vec{l}$
f) $\oint_C \vec{A} \cdot d\vec{l}$

33. Express $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ in spherical polar coordinates

34. From the result of (33) show that $x\,\partial/\partial y - y\,\partial/\partial x = \partial/\partial\varphi$. Express $-i\hbar\,\partial/\partial\varphi$ is an imp. operator in Q.M.

35. Verify the following vector operator equations
a) $\nabla = \vec{e}_r(\vec{e}_r \cdot \nabla) - \vec{e}_r \times (\vec{e}_r \times \nabla)$ b) $\nabla \times (\vec{r} \times \nabla) = \vec{r}\nabla^2 - \nabla(1 + r\,\partial/\partial r)$

* 36. A certain force field is given by $\vec{F} = \vec{e}_r \frac{2P\cos\theta}{r^3} + \vec{e}_\theta \frac{P}{r^3}\sin\theta$, $r \geq P/2$ in spherical polar coordinates.
a) Examine $\nabla \times \vec{F}$ to see if a potential exists.
b) Calculate $\oint \vec{F} \cdot d\vec{l}$ for a unit circle in the plane $\theta = \pi/2$. What does this indicate about the force field? Is it

conservative or nonconservative?

37. Be sure that the following relations of spherical polar unit vectors into their Cartesian components can be obtained quite easily.

$\vec{e}_r = \vec{i}\sin\theta\cos\varphi + \vec{j}\sin\theta\sin\varphi + \vec{k}\cos\theta$

$\vec{e}_\theta = \vec{i}\cos\theta\sin\varphi + \vec{j}\cos\theta\sin\varphi - \vec{k}\sin\theta$

$\vec{e}_\varphi = -\vec{i}\sin\varphi + \vec{j}\cos\varphi$

Hint: Close your eyes and visualize the necessary projection.

38. By using reciprocal matrix (in our case transpose matrix) of
* the transform in (37), resolve the cartesian unit vectors into their polar components.

39. A particle is moving through space. Find the spherical components of its velocity and acceleration.

Ans. $v_r = \dot{r}$, $v_\theta = r\dot{\theta}$, $v_\varphi = r\sin\theta\,\dot{\varphi}$

$a_r = \ddot{r} - r\dot{\theta}^2 - r\sin\theta\,\dot{\varphi}^2$

$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\sin\theta\cos\theta\,\dot{\varphi}^2$

$a_\varphi = r\sin\theta\,\ddot{\varphi} + 2\dot{r}\dot{\varphi}\sin\theta + 2r\dot{\theta}\dot{\varphi}\cos\theta$

? 40. prove that

$$A = \begin{pmatrix} y^2 & -xy \\ -xy & x^2 \end{pmatrix} \quad B = \begin{pmatrix} -xy & x^2 \\ -y^2 & xy \end{pmatrix}$$ are tensors.

41. prove that $C = \begin{pmatrix} y^2 & xy \\ xy & x^2 \end{pmatrix}$ $D = \begin{pmatrix} xy & y^2 \\ x^2 & -xy \end{pmatrix}$ are not tensors

42. Separate the tensor $\begin{pmatrix} -xy & x^2 \\ -y^2 & xy \end{pmatrix}$ into symmetric and antisymmetric parts.

43. Show that if the components of any tensor of any rank vanish in some particular coordinate system, they vanish in all coordinate systems.

44. The components of tensor A are equal to the corresponding components of tensor B in one particular system of coordinates, $A^0_{ij} = B^0_{ij}$. Show that tensor A is equal to tensor B in all coordinate systems i.e., $A_{ij} = B_{ij}$

45. T_{iklm} is antisymmetric wrt all pair of indices. How many components independent components has it? (in 3D)

46. Given the three pauli-spin matrices

$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -j \\ j & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

show that a, $\sigma_i^2 = 1$ b, $\sigma_i\sigma_j = j\sigma_k$ $(i,j,k) = \begin{matrix}(1,2,3)\\(2,3,1)\\(3,1,2)\end{matrix}$

c, $\sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij}E$, E is unit matrix.

47. Given the three matrices

$M_x = 1/\sqrt{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ $M_y = 1/\sqrt{2}\begin{pmatrix} 0 & -j & 0 \\ j & 0 & -j \\ 0 & j & 0 \end{pmatrix}$ $M_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

Show that

a, the commutator $[M_x, M_y] = jM_z$ and cyclic permutation of indices when E is a unit matrix.

b, $M^2 = M_x^2 + M_y^2 + M_z^2 = 2E$ c, $[M^2, M_i] = 0$ d, $[M_z, L^+] = L^+$

e, $[L^+, L^-] = 2M_z$ where $L^+ = M_x + jM_y$; $L^- = M_x - jM_y$

N.B. M_x, M_y, M_z can be used for description of spin 1.

48. Repeat (47) with

$$M_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix} \quad M_y = \frac{i}{\sqrt{2}}\begin{pmatrix} 0 & -\sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & -2 & 0 \\ 0 & 2 & 0 & -\sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix} \quad M_z = \frac{1}{2}\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}$$

N.B. M_x, M_y, M_z can be used for description of spin $3/2$.

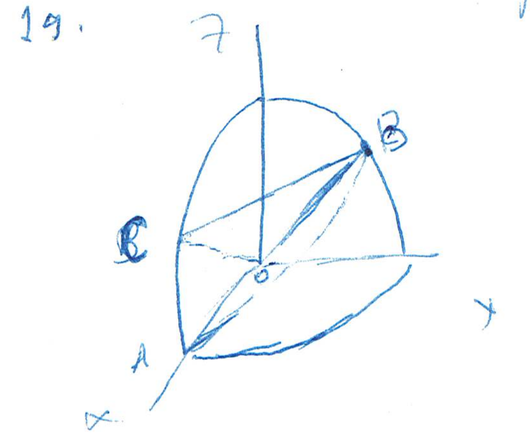
49. a, prove that $\oint d\vec{s} = 0$ for a closed surface

b, Show that $\frac{1}{3}\oint_S \vec{r}\cdot d\vec{s} = V$, enclosing the volume V

c, If $\vec{B} = \nabla\times\vec{A}$, show that $\oint \vec{B}\cdot d\vec{s} = 0$

d, prove that $\oint \nabla\times\vec{A}\cdot d\vec{s} = 0$ for any closed surface S, Solution to problems of Assignment XI for Phys. 531

19.



Since the vectors start from the origin

$\vec{l} = \vec{OA} = (1,0,0)$

$\vec{m} = \vec{OC} = (1/\sqrt{2}, 0, 1/\sqrt{2})$

$\vec{n} = \vec{OB} = (0, 1/\sqrt{2}, 1/\sqrt{2})$

The sides of the triangle are given then by the three vectors

$\vec{OB}-\vec{OA}$ & $\vec{OC}-\vec{OA}$, $\vec{OC}-\vec{OB}$.

But $\vec{OB}-\vec{OA}$ is a vector directed from point A to point B, $\vec{OC}-\vec{OA}$ is a vector directed from point A to point C and $\vec{OC}-\vec{OB}$ is a vector directed from point B to point C. Actually it is also possible to consider the vectors directed opposite to the above vectors i.e.,

$\vec{OA}-\vec{OB}$, $\vec{OA}-\vec{OC}$ & $\vec{OB}-\vec{OC}$. Here there is only a change in direction the magnitudes are the same for two corresponding vectors.

$\vec{OB}-\vec{OA} = (-1, +1/\sqrt{2}, +1/\sqrt{2})$

$\vec{OC}-\vec{OA} = (1/\sqrt{2}-1, 0, 1/\sqrt{2}) = (\frac{1-\sqrt{2}}{2}, 0, 1/\sqrt{2})$

$\vec{OC}-\vec{OB} = (1/\sqrt{2}, -1/\sqrt{2}, 0)$

And the negatives

$\vec{OA}-\vec{OB} = (1, -1/\sqrt{2}, -1/\sqrt{2})$

$\vec{OA}-\vec{OC} = (\frac{\sqrt{2}-1}{\sqrt{2}}, 0, -1/\sqrt{2})$

$\vec{OB}-\vec{OC} = (-1/\sqrt{2}, 1/\sqrt{2}, 0)$

Then

$(\vec{OA}-\vec{OB})\cdot(\vec{OA}-\vec{OC}) = |\vec{A}||\vec{b}|$

$= |\vec{OA}-\vec{OB}||\vec{OA}-\vec{OC}| \cos \measuredangle A$

$|\vec{OB}-\vec{OA}| = |\vec{OA}-\vec{OB}| = 2, |\vec{OB}-\vec{OC}| = |\vec{OC}-\vec{OB}| = 1$

$|\vec{OA}-\vec{OC}| = |\vec{OC}-\vec{OA}| = 2-\sqrt{2}$

* Also find the dot products

$\measuredangle A = \arccos\left\{\frac{(\vec{OA}-\vec{OB})\cdot(\vec{OA}-\vec{OC})}{|\vec{OA}-\vec{OB}||\vec{OA}-\vec{OC}|}\right\}$

$= \arccos\left\{\frac{\frac{\sqrt{2}-1}{\sqrt{2}} + 0 + 1/2}{(1+1/2+1/2)\left(\frac{(\sqrt{2}-1)^2}{2}+0+1/2\right)}\right\}$

$= \arccos\left\{\frac{(2\sqrt{2}-2+\sqrt{2})/2\sqrt{2}}{2\left(\frac{2-2\sqrt{2}+1+1}{2}\right)}\right\}$

$= \arccos\left(\frac{3\sqrt{2}-2}{4(2\sqrt{2}-2)}\right)$

$= \arccos\left(\frac{3\sqrt{2}-2}{8(\sqrt{2}-1)}\right)$

$= 47.41$

$(\vec{OB}-\vec{OA})\cdot(\vec{OB}-\vec{OC}) = |\vec{OB}-\vec{OA}||\vec{OB}-\vec{OC}| \cdot \cos \measuredangle B$

$\measuredangle B = \arccos\left\{\frac{(\vec{OB}-\vec{OA})\cdot(\vec{OB}-\vec{OC})}{|\vec{OB}-\vec{OA}||\vec{OB}-\vec{OC}|}\right\}$

$= \arccos\left\{\frac{1/\sqrt{2} + 1/2 + 0}{2\cdot 1}\right\}$

$= \arccos\left(\frac{2+\sqrt{2}}{4\sqrt{2}}\right)$

$= 52.875^0$

$(\vec{OC}-\vec{OA})\cdot(\vec{OC}-\vec{OB}) = |\vec{OC}-\vec{OA}||\vec{OC}-\vec{OB}| \cos \measuredangle C$

$\measuredangle C = \arccos\left\{\frac{(\vec{OC}-\vec{OA})\cdot(\vec{OC}-\vec{OB})}{|\vec{OC}-\vec{OA}||\vec{OC}-\vec{OB}|}\right\}$

$= \arccos\left\{\frac{\frac{1-\sqrt{2}}{2}+0+0}{(2-\sqrt{2})(1)}\right\}$

$= \arccos\left(\frac{1-\sqrt{2}}{2(2-\sqrt{2})}\right)$

$= 110.704^0$

20. Prove $\text{grad } f(r) = \hat{r}\,\frac{df}{dr}$

$r = \sqrt{x^2+y^2+z^2}$

f depends on x, y, z through

Φ

$$\text{grad } f(r) = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) f(r)$$

$$= \vec{i}\frac{\partial f(r)}{\partial x} + \vec{j}\frac{\partial f(r)}{\partial y} + \vec{k}\frac{\partial f(r)}{\partial z}$$

$$\frac{\partial f(r)}{\partial x} = \frac{d f(r)}{dr}\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2+z^2}}\frac{df(r)}{dr} = \frac{x}{r}\frac{df(r)}{dr}$$

$$\frac{\partial f(r)}{\partial y} = \frac{d f(r)}{dr}\frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2+y^2+z^2}}\frac{df(r)}{dr} = \frac{y}{r}\frac{df(r)}{dr}$$

$$\frac{\partial f(r)}{\partial z} = \frac{d f(r)}{dr}\frac{\partial r}{\partial z} = \frac{z}{\sqrt{x^2+y^2+z^2}}\frac{df(r)}{dr} = \frac{z}{r}\frac{df}{dr}$$

$$\Rightarrow \text{grad } f(r) = \left\{\frac{\vec{i}x}{\sqrt{\;}}\frac{df(r)}{dr} + \vec{j}\frac{y}{r}\frac{df(r)}{dr} + \vec{k}\frac{z}{r}\frac{df(r)}{dr}\right\}$$

$$= \left(\frac{\vec{i}x + \vec{j}y + \vec{k}z}{r}\right)\frac{df(r)}{dr}$$

$$= \frac{\vec{r}}{r}\frac{df(r)}{dr}$$

$$= \vec{r}^{\,0}\frac{df(r)}{dr}$$

21. To show $\nabla(uv) = u\nabla v + v\nabla u$
We have to consider the vector and operator nature del "∇".

$$\nabla(uv) = \nabla(u\overset{\downarrow}{v}) + \nabla(\overset{\downarrow}{u}v)$$

Since u and v are scalars ∇ can act on them in the form of operator.

$$\therefore \nabla(uv) = u\nabla v + v\nabla u$$

22. $s = (x^2+y^2+z^2)^{3/2}$
if we let $r = \sqrt{x^2+y^2+z^2}$, $s = r^3$

$$\nabla s = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)s$$

$$= \vec{i}\frac{\partial s}{\partial x} + \vec{j}\frac{\partial s}{\partial y} + \vec{k}\frac{\partial s}{\partial z}$$

$$\frac{\partial s}{\partial x} = \frac{ds}{dr}\frac{\partial r}{\partial x} = 3r^2\frac{x}{\sqrt{\;}} = 3rx$$

$$\frac{\partial s}{\partial y} = \frac{ds}{dr}\frac{\partial r}{\partial y} = 3r^2\frac{y}{r} = 3ry$$

$$\frac{\partial s}{\partial z} = \frac{ds}{dr}\frac{\partial r}{\partial z} = 3r^2\frac{z}{r} = 3rz$$

$$\Rightarrow \nabla s = (\vec{i}\,3rx + \vec{j}\,3ry + \vec{k}\,3rz)$$

$$= 3r(\vec{i}x + \vec{j}y + \vec{k}z)$$

$$\nabla s = 3r\vec{r}$$

$$\text{or } \nabla s = 3r^2\frac{\vec{r}}{r}$$

$$= 3r^2\vec{r}^{\,0}$$

Now at $(1,2,3)$
Magnitude

$$\nabla s = 3(x^2+y^2+z^2)\vec{r}^{\,0}\Big|_{1,2,3}$$

$$= 3(1^2+2^2+3^2)\vec{r}_0$$

$$= 42\vec{r}_0$$

$$|\nabla s| = 42|\vec{r}_0^{\,0}| = 42$$

Direction cosines at $(1,2,3)$:

$$\cos\alpha = \frac{1}{42}$$

$$\cos\beta = \frac{2}{42} = \frac{1}{21}$$

$$\cos\gamma = \frac{3}{42} = \frac{1}{14}$$

are the direction cosines where α, β and γ are measured wrt the x, y, and z-axes resp.

23.
(i) $\nabla u \times \nabla v = 0$

$$\Rightarrow \nabla u \parallel \nabla v$$

This ~~intern~~ means either ∇u or ∇v can be expressed in terms of one another. This in turn implies u and v are related, i.e.,

$$f(u,v) = 0.$$

But ii, Suppose there exists a function $f(u,v)=0$. From this we must arrive at indeed $\nabla u \times \nabla v = 0$, if $f(u,v)$ is to exist.

Now, $f(u,v) = 0$

$$\Rightarrow \text{grad } f(u,v) = \nabla f = 0$$

$$\nabla f = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) f$$

$$= \vec{i}\frac{\partial f}{\partial x} + \vec{j}\frac{\partial f}{\partial y} + \vec{k}\frac{\partial f}{\partial z}.$$

But f depends on x, y and z through u and v, (i.e.)

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x}$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y}$$

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial z}$$

$$\Rightarrow \nabla f = \vec{i}\left(\frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x}\right) + \vec{j}\left(\frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y}\right) + \vec{k}\left(\frac{\partial f}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial z}\right)$$

$$= \left\{\vec{i}\frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \vec{j}\frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \vec{k}\frac{\partial f}{\partial u}\frac{\partial u}{\partial z}\right\} + \left\{\vec{i}\frac{\partial f}{\partial v}\frac{\partial v}{\partial x} + \vec{j}\frac{\partial f}{\partial v}\frac{\partial v}{\partial y} + \vec{k}\frac{\partial f}{\partial v}\frac{\partial v}{\partial z}\right\}$$

$$= \frac{\partial f}{\partial u}\left\{\vec{i}\frac{\partial u}{\partial x} + \vec{j}\frac{\partial u}{\partial y} + \vec{k}\frac{\partial u}{\partial z}\right\} + \frac{\partial f}{\partial v}\left\{\vec{i}\frac{\partial v}{\partial x} + \vec{j}\frac{\partial v}{\partial y} + \vec{k}\frac{\partial v}{\partial z}\right\}$$

$$= \frac{\partial f}{\partial u}\nabla u + \frac{\partial f}{\partial v}\nabla v$$

(i.e.)

$$\nabla f = \frac{\partial f}{\partial u}\nabla u + \frac{\partial f}{\partial v}\nabla v = 0$$

or

$$\frac{\partial f}{\partial u}\nabla u + \frac{\partial f}{\partial v}\nabla v = 0$$

Take the cross product of this eq with ∇u.

$$\nabla u \times \left(\frac{\partial f}{\partial u}\nabla u\right) + \nabla u \times \left(\frac{\partial f}{\partial v}\nabla v\right) = 0$$

$$\frac{\partial f}{\partial u}(\nabla u \times \nabla u) + \frac{\partial f}{\partial v}(\nabla u \times \nabla v) = 0$$

$$\Downarrow$$

For two identical vectors the cross product vanishes.

$$\nabla u \times \nabla u = 0$$

$$\Rightarrow \frac{\partial f}{\partial v}\nabla u \times \nabla v = 0$$

But $\frac{\partial f}{\partial v} \neq 0$ as f is a function of v. It follows

$$\nabla u \times \nabla v = 0$$

Therefore since $f(u, v) = 0$ leads us to $\nabla u \times \nabla v = 0$, the necessary and sufficient condition for $u(x, y, z)$ and $v(x, y, z)$ to be related by some function $f(u, v) = 0$ is $\nabla u \times \nabla v = 0$.

24.

$$\vec{A} = \vec{A}(x, y, z; t)$$

$$d\vec{A} = \frac{\partial \vec{A}}{\partial x}dx + \frac{\partial \vec{A}}{\partial y}dy + \frac{\partial \vec{A}}{\partial z}dz + \frac{\partial \vec{A}}{\partial t}dt$$

we can write

$$\frac{\partial \vec{A}}{\partial x}dx \equiv dx\frac{\partial \vec{A}}{\partial x}$$

$$\frac{\partial \vec{A}}{\partial y}dy \equiv dy\frac{\partial \vec{A}}{\partial y}$$

$$\frac{\partial \vec{A}}{\partial z}dz \equiv dz\frac{\partial \vec{A}}{\partial z}$$

(i.e.)

$$d\vec{A} = dx\frac{\partial \vec{A}}{\partial x} + dy\frac{\partial \vec{A}}{\partial y} + dz\frac{\partial \vec{A}}{\partial z} + \frac{\partial \vec{A}}{\partial t}dt$$

$$= \left\{\vec{i}dx + \vec{j}dy + \vec{k}dz\right\} \cdot \left\{\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right\}\vec{A} + \frac{\partial \vec{A}}{\partial t}dt$$

$$= (d\vec{r} \cdot \nabla)\vec{A} + \frac{\partial \vec{A}}{\partial t}dt$$

25. $u = 3x^2y$, $v = xz^2 - 2y$

$\text{grad}\, u = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(3x^2y)$

$= \vec{i}\,6xy + \vec{j}\,3x^2 + 0$

$\text{grad}\, v = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(xz^2 - 2y)$

$= \vec{i}z^2 - \vec{j}2 + \vec{k}\,2xz$

$\text{grad}\, u \cdot \text{grad}\, v$

$= 6xy \cdot z^2 - 3x^2 \cdot 2 + 0 \cdot 2xz$

$= 6xyz^2 - 6x^2$

$\Rightarrow \text{grad}\,[\text{grad}\, u \cdot \text{grad}\, v]$

$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(6xyz^2 - 6x^2)$

$= \vec{i}(6yz^2 - 12x) + \vec{j}\,6xz^2 + \vec{k}\,12xyz$

26. $\nabla \cdot \vec{r}$

$= \nabla \cdot (\vec{i}x + \vec{j}y + \vec{k}z)$

$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot (\vec{i}x + \vec{j}y + \vec{k}z)$

$= \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}$

$= 1 + 1 + 1$

$= 3$

27. $\nabla \cdot (\vec{r} f(r))$

$= \nabla \cdot (\overset{\downarrow}{\vec{r}} f(r))$

$+ \nabla \cdot (\vec{r} \overset{\downarrow}{f(r)})$

$= f(r)\nabla \cdot \vec{r} + \vec{r} \cdot \nabla f(r)$

From (26) $\nabla \cdot \vec{r} = 3$

From (20) $\nabla f(r) = \vec{r}_0 \frac{df(r)}{dr}$

$\therefore \nabla \cdot (\vec{r} f(r))$

$= 3f(r) + \vec{r} \cdot \vec{r}_0 \frac{df(r)}{dr}$

$= 3f(r) + r\frac{df(r)}{dr}$

or

$\nabla \cdot \vec{r} f(r)$

$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(\vec{i}x f(r) + \vec{j}y f(r) + \vec{k}z f(r))$

$= \frac{\partial}{\partial x}(x f(r)) + \frac{\partial}{\partial y}(y f(r))$

$+ \frac{\partial}{\partial z}(z f(r))$

$= \frac{\partial x}{\partial x} f(r) + \frac{\partial y}{\partial y} f(r) + \frac{\partial z}{\partial z} f(r)$

$+ x\frac{\partial f(r)}{\partial x} + y\frac{\partial f(r)}{\partial y} + z\frac{\partial f(r)}{\partial z}$

$= 3f(r)$

$+ x\frac{df(r)}{dr}\frac{\partial r}{\partial x} + y\frac{df(r)}{dr}\frac{\partial r}{\partial y}$

$+ z\frac{df(r)}{dr}\frac{\partial r}{\partial z}$

$= 3f(r)$

$+ x \cdot \frac{x}{r}\frac{df(r)}{dr} + y \cdot \frac{y}{r}\frac{df(r)}{dr}$

$+ z\frac{z}{r}\frac{df(r)}{dr}$

$= 3f(r) + \left(\frac{x^2+y^2+z^2}{r}\right)\frac{df(r)}{dr}$

$= 3f(r) + \frac{r^2}{r}\frac{df(r)}{dr}$

$= 3f(r) + r\frac{df(r)}{dr}$

28. Show that

$\nabla \cdot \vec{r}\, r^{n-1} = (n+2)r^{n-1}$

From (27) if we let

$f(r) = r^{n-1}$

$\nabla \cdot \vec{r}\, r^{n-1} = \nabla \cdot \vec{r} f(r)$

$= 3f(r) + r\frac{df(r)}{dr}$

$= 3r^{n-1} + r(n-1)r^{n-2}$

$= 3r^{n-1} + (n-1)r^{n-1}$

$= (3+n-1)r^{n-1}$

$= (2+n)r^{n-1}$

$= (n+2)r^{n-1}$

29. $\nabla\cdot(f\vec{v})$

Let $\vec{v} = v_x\vec{i} + v_y\vec{j} + v_z\vec{k}$

$f\vec{v} = fv_x\vec{i} + fv_y\vec{j} + fv_z\vec{k}$

$\nabla\cdot f(\vec{v})$

$= (\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z})\cdot(\vec{i}fv_x + \vec{j}fv_y + \vec{k}fv_z)$

$= \frac{\partial}{\partial x}(fv_x) + \frac{\partial(fv_y)}{\partial y} + \frac{\partial(fv_z)}{\partial z})$

$= f\frac{\partial v_x}{\partial x} + v_x\frac{\partial f}{\partial x}$
$+ f\frac{\partial v_y}{\partial y} + v_y\frac{\partial f}{\partial y}$
$+ f\frac{\partial v_z}{\partial z} + v_z\frac{\partial f}{\partial z}$

$= f\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right)$
$+ \left(v_x\frac{\partial f}{\partial x} + v_y\frac{\partial f}{\partial y} + v_z\frac{\partial f}{\partial z}\right)$

$= f\nabla\cdot\vec{v} + \vec{v}\cdot\nabla f$

$\therefore \nabla\cdot(f\vec{v})$
$= \nabla f\cdot\vec{v} + f\nabla\cdot\vec{v}$

30. The linear velocity $\vec{v}$ and the angular velocity ω of a body in rotation are related by

$\vec{v} = \vec{\omega}\times\vec{r}$

Where $\vec{r}$ is the position vector of the rotating body.
Then if $\vec{v}$ is solenoidal $\nabla\cdot\vec{v} = 0$.

Now, $\nabla\cdot\vec{v} = \nabla\cdot(\vec{\omega}\times\hat{r})$

~~[illegible]~~
$= \vec{r}\cdot\nabla\times\vec{\omega}$
$- \vec{\omega}\cdot\nabla\times\vec{r}$

Since $\vec{\omega}$ is constant
$\nabla\times\vec{\omega} = 0$

$\Rightarrow \nabla\cdot\vec{v} = -\vec{\omega}\cdot\nabla\times\vec{r}$

$\nabla\times\vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & y & z \end{vmatrix}$

$\vec{i}\left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right)$
$+ \vec{j}\left(\frac{\partial x}{\partial z} - \frac{\partial z}{\partial x}\right)$
$+ \vec{k}\left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right)$

But x, y, z are independent
(i.e.) $\frac{\partial x}{\partial y} = \frac{\partial x}{\partial z} = \frac{\partial y}{\partial x} = \frac{\partial y}{\partial z} = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial y}$

$\nabla\times\vec{r} = 0$

$\Rightarrow \nabla\cdot\vec{v} = 0$

$\therefore \vec{v}$ is solenoidal.

32. Geometrical meanings:

a, grad φ: it is a vector which has direction the maximum space rate of change of the function φ.

b, div $\vec{A}$: ~~It represents~~ ~~that~~ is associated with the flux of the quantity represented by the vector valued function $\vec{A}$? div $\vec{A}$ for any vector $\vec{A}$, div $\vec{A}$ at a point represents the rate of change of $\vec{A}$ about the point. For example, for a fluid that is in motion if $\vec{v}$ is the velocity of the fluid and ρ is its density $\nabla\cdot(\rho\vec{v})$ represents the rate of the change of mass at a point ~~in the fluid~~. Thus at a point (x, y, z) ~~[illegible]~~ if $\nabla\cdot(\rho\vec{v})$ is greater than zero the mass is flowing out of the point and if $\nabla\cdot(\rho\vec{v})$ is less than zero the mass is flowing into the point.

c, curl $\vec{A}$: this is a vector or it is associated with the rotationality or irrotationality of the vector field $\vec{A}$ i.e. it describes the rotation of the vector field $\vec{A}$ at a point at which the curl of $\vec{A}$ is evaluated

d, $\int_S \vec{A}\cdot d\vec{s}$: It represents the flux of the vector field $\vec{A}$ through the surface S

e, $\int \vec{A}\cdot d\vec{l}$: It represents

f, $\oint_C \vec{A}\cdot d\vec{l}$: It is the circulation of vector $\vec{A}$ over the curve C

33, In spherical polar coordinates the del operator gradient of a scalar function is given by

$$\nabla\varphi = \vec{e}_r \frac{\partial}{\partial r} + \frac{\vec{e}_\theta}{r}\frac{\partial}{\partial\theta} + \frac{\vec{e}_\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}$$

But $\vec{e}_r = \sin\theta\cos\varphi\,\vec{i} + \sin\theta\sin\varphi\,\vec{j} + \cos\theta\,\vec{k}$

$\vec{e}_\theta = \cos\theta\cos\varphi\,\vec{i} + \cos\theta\sin\varphi\,\vec{j} - \sin\theta\,\vec{k}$

$\vec{e}_\varphi = \cos\varphi\,\vec{i} - \sin\varphi\,\vec{j}$

Note: These expressions for $\vec{e}_r, \vec{e}_\theta$ and $\vec{e}_\varphi$ are obtained from the result of prob. (37)

Then,

$$\nabla\varphi = (\sin\theta\cos\varphi\,\vec{i} + \sin\theta\sin\varphi\,\vec{j} + \cos\theta\,\vec{k})\frac{\partial}{\partial r} + (\cos\theta\cos\varphi\,\vec{i} + \cos\theta\sin\varphi\,\vec{j} - \sin\theta\,\vec{k})\frac{1}{r}\frac{\partial}{\partial\theta} + (-\sin\varphi\,\vec{i} + \cos\varphi\,\vec{j})\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}$$

$$= \left(\sin\theta\cos\varphi\frac{\partial}{\partial r} + \cos\theta\cos\varphi\frac{1}{r}\frac{\partial}{\partial\theta} - \frac{\sin\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}\right)\vec{i} + \left(\sin\theta\sin\varphi\frac{\partial}{\partial r} + \cos\theta\sin\varphi\frac{1}{r}\frac{\partial}{\partial\theta} + \frac{\cos\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}\right)\vec{j} + \left(\cos\theta\frac{\partial}{\partial r} - \sin\theta\frac{1}{r}\frac{\partial}{\partial\theta}\right)\vec{k}$$

But in cartesian coordinates

$$\nabla = \vec{i}\,\partial/\partial x + \vec{j}\,\partial/\partial y + \vec{k}\,\partial/\partial z$$

Comparing the two expressions of the operator ∇ we have

$$\frac{\partial}{\partial x} = \sin\theta\cos\varphi\frac{\partial}{\partial r} + \cos\theta\cos\varphi\frac{1}{r}\frac{\partial}{\partial\theta} - \frac{\sin\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}$$

$$\frac{\partial}{\partial y} = \sin\theta\sin\varphi\frac{\partial}{\partial r} + \cos\theta\sin\varphi\frac{1}{r}\frac{\partial}{\partial\theta} + \frac{\cos\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}$$

$$\frac{\partial}{\partial z} = \cos\theta\frac{\partial}{\partial r} - \sin\theta\frac{1}{r}\frac{\partial}{\partial\theta}$$

37

$x = r\sin\theta\cos\varphi$

$y = r\sin\theta\sin\varphi$

$$\Rightarrow x\frac{\partial}{\partial y} = (r\sin\theta\cos\varphi)\cdot\left\{\sin\theta\sin\varphi\frac{\partial}{\partial r} + \cos\theta\sin\varphi\frac{1}{r}\frac{\partial}{\partial\theta} + \frac{\cos\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}\right\}$$

$$= r\sin^2\theta\sin\varphi\cos\varphi\frac{\partial}{\partial r} + r\sin\theta\cos\varphi\cos\theta\sin\varphi\frac{1}{r}\frac{\partial}{\partial\theta} + \cos^2\varphi\frac{\partial}{\partial\varphi}$$

$$y\frac{\partial}{\partial x} = (r\sin\theta\sin\varphi)\cdot\left(\sin\theta\cos\varphi\frac{\partial}{\partial r} + \cos\theta\cos\varphi\frac{1}{r}\frac{\partial}{\partial\theta} - \frac{\sin\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}\right)$$

$$= r\sin^2\theta\sin\varphi\cos\varphi\frac{\partial}{\partial r} + r\sin\theta\sin\varphi\cos\theta\cos\varphi\frac{1}{r}\frac{\partial}{\partial\theta} - \sin^2\varphi\frac{\partial}{\partial\varphi}$$

$$\Rightarrow x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} = \cos^2\varphi\frac{\partial}{\partial\varphi} + \sin^2\varphi\frac{\partial}{\partial\varphi} = (\cos^2\varphi + \sin^2\varphi)\partial/\partial\varphi = \partial/\partial\varphi$$

$$\therefore -i\hbar\frac{\partial}{\partial\varphi} = -i\hbar\left(x\frac{\partial}{\partial y} - y\,\partial/\partial x\right)$$

This is actually the quantum mechanical operator corresponding to the z-component of the angular mom.

35: a) $\nabla = \vec{e}_r(\vec{e}_r\cdot\nabla) - \vec{e}_r\times(\vec{e}_r\times\nabla)$

R.H.S $= \vec{e}_r(\vec{e}_r\cdot\nabla) - \vec{e}_r\times(\vec{e}_r\times\nabla)$

From scalar vector product we have

$$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$$

So since ∇ is not acting on any of the unit vectors we have

$$\vec{e}_r\times(\vec{e}_r\times\nabla) = \vec{e}_r(\vec{e}_r\cdot\nabla) - (\vec{e}_r\cdot\vec{e}_r)\nabla$$

It follows (whence)

$$\vec{e}_r(\vec{e}_r\cdot\nabla) - \vec{e}_r\times(\vec{e}_r\times\nabla)$$

$$= \vec{e}_r(\vec{e}_r\cdot\nabla) - \{\vec{e}_r(\vec{e}_r\cdot\nabla) - (\vec{e}_r\cdot\vec{e}_r)\nabla\}$$

$$= \vec{e}_r(\vec{e}_r\cdot\nabla) - \{\vec{e}_r(\vec{e}_r\cdot\nabla) - \nabla\}$$

$$= \vec{e}_r(\vec{e}_r\cdot\nabla) - \vec{e}_r(\vec{e}_r\cdot\nabla) + \nabla$$

$$= \nabla$$

b) $\nabla\times(\vec{r}\times\nabla)$

From vector triple product

$$\nabla\times(\vec{r}\times\nabla)$$

$$= \vec{r}(\nabla\cdot\nabla) - \nabla(\nabla\cdot\vec{r})$$

$$= \vec{r}\,\nabla^2 - \nabla(\nabla\cdot\vec{r})$$

To evaluate $\nabla\cdot\vec{r}$ let us consider ∇ in spherical polar coordinates

$$\nabla = \vec{e}_r\frac{\partial}{\partial r} + \frac{\vec{e}_\theta}{r}\frac{\partial}{\partial\theta} + \frac{\vec{e}_\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}$$

$$\nabla\cdot\vec{r} = \nabla\cdot r\vec{e}_r$$

$$= \vec{e}_r\frac{\partial}{\partial r}\cdot(r\vec{e}_r), \text{ all others vanish}$$

$$= \vec{e}_r\cdot\left\{\vec{e}_r\frac{\partial r}{\partial r} + r\frac{\partial\vec{e}_r}{\partial r}\right\}$$

$$= \vec{e}_r\cdot\left\{\vec{e}_r + r\frac{\partial\vec{e}_r}{\partial r}\right\}$$

$$= \left\{\vec{e}_r\cdot\vec{e}_r + \vec{e}_r\cdot\left(r\frac{\partial\vec{e}_r}{\partial r}\right)\right\}$$

$$= \left\{1 + \vec{e}_r\cdot\left(r\frac{\partial\vec{e}_r}{\partial r}\right)\right\}$$

The vector $\frac{\partial\vec{e}_r}{\partial r}$ does not have a change of direction when r changes, i.e., $\vec{e}_r$ is parallel to $\partial\vec{e}_r/\partial r$ whence

and ~~∇~~ ~~{1}~~

$$\vec{e}_r\cdot\frac{\partial\vec{e}_r}{\partial r} = \frac{\partial}{\partial r}$$

whence,

$$\nabla\cdot\vec{r} = \left\{1 + r\frac{\partial}{\partial r}\right\}$$

$$\therefore\ \nabla\times(\vec{r}\times\nabla)$$

$$= \vec{r}\,\nabla^2 - \nabla\left(1 + r\frac{\partial}{\partial r}\right)$$

36. $\vec{F} = \vec{e}_r\frac{2p\cos\theta}{r^3} + \vec{e}_\theta\frac{p\sin\theta}{r^3}$

$r \geq p/2$

a) $$\nabla\times\vec{F} = \begin{vmatrix} \vec{e}_r & r\vec{e}_\theta & r\sin\theta\,\vec{e}_\varphi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\varphi} \\ \frac{2p\cos\theta}{r^3} & \frac{p\sin\theta}{r^2} & 0 \end{vmatrix}$$

$$= -\vec{e}_r\frac{\partial}{\partial\varphi}\left(\frac{p\sin\theta}{r^2}\right)$$

$$+ r\vec{e}_\theta\frac{\partial}{\partial\varphi}\left(\frac{2p\cos\theta}{r^3}\right)$$

$$+ r\sin\theta\,\vec{e}_\varphi\left\{\frac{\partial}{\partial r}\left(\frac{p\sin\theta}{r^2}\right) - \frac{\partial}{\partial\theta}\left(\frac{2p\cos\theta}{r^3}\right)\right\}$$

$$= \vec{e}_r\cdot 0 + r\vec{e}_\theta\cdot 0$$

$$+ r\sin\theta\,\vec{e}_\varphi\left\{-\frac{2p\sin\theta}{r^3} + \frac{2p\sin\theta}{r^3}\right\}$$

$$= 0 + 0 + r\sin\theta\,\vec{e}_\varphi\cdot 0$$

$$= 0$$

Since the curl vanishes there exists a potential.

b) $\oint\vec{F}\cdot d\vec{l}$

$\theta = \pi/2$ in the xy plane i.e., the plane $z = 0$. Then,

$$d\vec{l} = d\vec{r} = \vec{i}\,dx + \vec{j}\,dy$$

The unit vectors in spherical coordinates can be expressed in terms of their Cartesian counterparts or the vice versa.

$$\vec{i} = \sin\theta\cos\varphi\,\vec{e}_r - \sin\varphi\,\vec{e}_\varphi + \cos\theta\cos\varphi\,\vec{e}_\theta$$

$$\vec{j} = \sin\theta\sin\varphi\,\vec{e}_r + \cos\varphi\,\vec{e}_\varphi + \cos\theta\sin\varphi\,\vec{e}_\theta$$

$$x = r\sin\theta\cos\varphi$$

$$\Rightarrow dx = \frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial \theta}d\theta + \frac{\partial x}{\partial \varphi}d\varphi$$

$$= \sin\theta\cos\varphi\,dr + r\cos\theta\cos\varphi\,d\theta - r\sin\theta\sin\varphi\,d\varphi$$

$$y = r\sin\theta\sin\varphi$$

$$dy = \sin\theta\sin\varphi\,dr + r\cos\theta\sin\varphi\,d\theta + r\sin\theta\cos\varphi\,d\varphi$$

Condition: $\theta = \pi/2 = \text{const}$

$r = \text{const} = \text{radius of circle}$

$$\Rightarrow \vec{i} = \cos\varphi\,\vec{e}_r - \sin\varphi\,\vec{e}_\varphi$$
$$\vec{j} = \sin\varphi\,\vec{e}_r + \cos\varphi\,\vec{e}_\varphi$$
$$dx = -r\sin\varphi\,d\varphi$$
$$dy = r\cos\varphi\,d\varphi$$

which is the case of planar polar coordinates.

Then $d\vec{r} = (\cos\varphi\,\vec{e}_r - \sin\varphi\,\vec{e}_\varphi)(-r\sin\varphi\,d\varphi) + (\sin\varphi\,\vec{e}_r + \cos\varphi\,\vec{e}_\varphi)(r\cos\varphi\,d\varphi)$

$$= (-r\sin\varphi\cos\varphi\,\vec{e}_r + r\sin^2\varphi\,\vec{e}_\varphi)\,d\varphi + (r\sin\varphi\cos\varphi\,\vec{e}_r + r\cos^2\varphi\,\vec{e}_\varphi)\,d\varphi$$

$$\Rightarrow |d\vec{l}| = \vec{e}_\varphi(r\sin^2\varphi + r\cos^2\varphi)\,d\varphi = r\,d\varphi\,\vec{e}_\varphi$$

$$\vec{F}\cdot d\vec{r} = \left(\vec{e}_r\frac{2p\cos\theta}{r^3} + \vec{e}_\theta\frac{p\sin\theta}{r^3}\right)\cdot(r\,d\varphi)\,\vec{e}_\varphi$$

$$= (\;)\,\vec{e}_r\cdot\vec{e}_\varphi + (\;)\,\vec{e}_\theta\cdot\vec{e}_\varphi = 0$$

$$\vec{r} = r\,\vec{e}_r$$

$$d\vec{r} = \vec{e}_r\,dr + r\,d\vec{e}_r$$

But $d\vec{e}_r = \sin\theta\,d\varphi\,\vec{e}_\varphi + d\theta\,\vec{e}_\theta$

$$\Rightarrow d\vec{r} = \vec{e}_r\,dr + \vec{e}_\theta\,d\theta + \vec{e}_\varphi\sin\theta\,d\varphi$$

Now,

$$\vec{F}\cdot d\vec{r} = \left(\vec{e}_r\frac{2p\cos\theta}{r^3} + \vec{e}_\theta\frac{p\sin\theta}{r^3}\right)\cdot(\vec{e}_r\,dr + \vec{e}_\theta\,d\theta + \vec{e}_\varphi\sin\theta\,d\varphi)$$

$$= \frac{2p\cos\theta}{r^3}dr + \frac{p\sin\theta}{r^3}d\theta$$

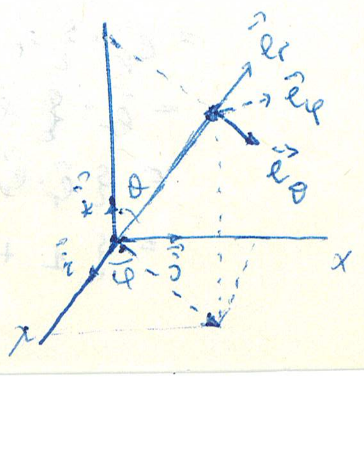
$$\oint \vec{F}\cdot d\vec{r} = \oint\left(\frac{2p\cos\theta}{r^3}dr + \frac{p\sin\theta}{r^3}d\theta\right)$$

$$= \int_0^{?} \frac{2p\cos\theta}{r^3}dr\Big|_{\theta=\pi/2} \text{ to}, \quad d\theta = 0 \text{ as } \theta = \tfrac{\pi}{2} \text{ const}$$

$= 0$, as $\cos\pi/2 = 0$.

This tells that the force field is a conservative one.

37.



Use the $\vec{e}_\theta$-method

$$\vec{e}_r = (\vec{e}_r \cdot \vec{i})\vec{i} + (\vec{e}_r \cdot \vec{j})\vec{j} + (\vec{e}_r \cdot \vec{k})\vec{k}$$

The projection of $\vec{e}_r$ on the xy plane has magnitude

$$\text{proj}_{xy\text{-plane}}\, \vec{e}_r \quad \cos(90-\theta) = \sin\theta$$

But this is further decomposed into x- and y- comps, i.e.,

$$\vec{e}_{r,x} = \sin\theta\cos\varphi\,\vec{i}$$
$$\vec{e}_{r,y} = \sin\theta\sin\varphi\,\vec{j}$$

or
$$\vec{e}_r = \sin\theta\cos\varphi\,\vec{i} + \sin\theta\sin\varphi\,\vec{j} + (\vec{e}_r \cdot \vec{k})\vec{k}$$

But $\vec{e}_r \cdot \vec{k} = \cos\theta$

$$\therefore \vec{e}_r = \sin\theta\cos\varphi\,\vec{i} + \sin\theta\sin\varphi\,\vec{j} + \cos\theta\,\vec{k}$$

– $$\vec{e}_\theta = (\vec{e}_\theta \cdot \vec{i})\vec{i} + (\vec{e}_\theta \cdot \vec{j})\vec{j} + (\vec{e}_\theta \cdot \vec{k})\vec{k}$$

The projection of $\vec{e}_\theta$ on the xy plane has magnitude

$$\text{proj}_{xy\text{-plane}}\, \vec{e}_\theta = \cos\theta$$

Decomposing into the x- and y-comps

$$\text{proj}_{\vec{i}}\, \vec{e}_\theta = \cos\theta\cos\varphi$$
$$\text{proj}_{\vec{j}}\, \vec{e}_\theta = \cos\theta\sin\varphi$$

Also
$$\text{proj}_{\vec{k}}\, \vec{e}_\theta = \cos(90+\theta) = -\sin\theta$$

$$\therefore \vec{e}_\theta = \cos\theta\cos\varphi\,\vec{i} + \cos\theta\sin\varphi\,\vec{j} - \sin\theta\,\vec{k}$$

– $$\vec{e}_\varphi = (\vec{e}_\varphi \cdot \vec{i})\vec{i} + (\vec{e}_\varphi \cdot \vec{j})\vec{j} + (\vec{e}_\varphi \cdot \vec{k})\vec{k}$$
$$= \cos(90+\varphi)\vec{i} + \cos\varphi\,\vec{j} + \cos 90\,\vec{k}$$
$$= -\sin\varphi\,\vec{i} + \cos\varphi\,\vec{j}$$

38. From (37)

$$\sin\theta\cos\varphi\,\vec{i} + \sin\theta\sin\varphi\,\vec{j} + \cos\theta\,\vec{k} = \vec{e}_r$$
$$\cos\theta\cos\varphi\,\vec{i} + \cos\theta\sin\varphi\,\vec{j} - \sin\theta\,\vec{k} = \vec{e}_\theta$$
$$-\sin\varphi\,\vec{i} + \cos\varphi\,\vec{j} = \vec{e}_\varphi$$

Using a transpose matrix we have

$$\begin{bmatrix} \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ \cos\theta\cos\varphi & \cos\theta\sin\varphi & -\sin\theta \\ -\sin\varphi & \cos\varphi & 0 \end{bmatrix} \begin{bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{bmatrix} = \begin{bmatrix} \vec{e}_r \\ \vec{e}_\theta \\ \vec{e}_\varphi \end{bmatrix}$$

$$\begin{bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ \cos\theta\cos\varphi & \cos\theta\sin\varphi & -\sin\theta \\ -\sin\varphi & \cos\varphi & 0 \end{bmatrix}^{-1} \begin{bmatrix} \vec{e}_r \\ \vec{e}_\theta \\ \vec{e}_\varphi \end{bmatrix}$$

i.e.,
$$[A] = [B]^{-1}[C]$$

where $[A] = \begin{bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{bmatrix}$.

$$[B] = \begin{bmatrix} \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ \cos\theta\cos\varphi & \cos\theta\sin\varphi & -\sin\theta \\ -\sin\varphi & \cos\varphi & 0 \end{bmatrix}$$

$$[C] = \begin{bmatrix} \vec{e}_r \\ \vec{e}_\theta \\ \vec{e}_\varphi \end{bmatrix}$$

$$[B]^{-1} = \frac{(\text{Adj } B)^T}{\det B}$$

Expanding about the 3rd raw

$$\det B = 0 + (-1)^{3+2}\cos\varphi(\cos\varphi) + (-1)^{3+1}(-\sin\varphi)(-\sin\varphi)$$
$$= \cos^2\varphi + \sin^2\varphi$$
$$= 1$$

$$(\text{Adj } B)^T = \begin{bmatrix} \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ \cos\theta\cos\varphi & \cos\theta\sin\varphi & -\sin\theta \\ -\sin\varphi & \cos\varphi & 0 \end{bmatrix}^T$$

$$= \begin{bmatrix} \sin\theta\cos\varphi & \cos\theta\cos\varphi & -\sin\varphi \\ \sin\theta\sin\varphi & \cos\theta\sin\varphi & \cos\varphi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix}$$

i.e.,

$$\begin{bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\varphi & \cos\theta\cos\varphi & -\sin\varphi \\ \sin\theta\sin\varphi & \cos\theta\sin\varphi & \cos\varphi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} \vec{e}_r \\ \vec{e}_\theta \\ \vec{e}_\varphi \end{bmatrix}$$

$$= \begin{bmatrix} \sin\theta\cos\varphi\,\vec{e}_r + \cos\theta\cos\varphi\,\vec{e}_\theta - \sin\varphi\,\vec{e}_\varphi \\ \sin\theta\sin\varphi\,\vec{e}_r + \cos\theta\sin\varphi\,\vec{e}_\theta + \cos\varphi\,\vec{e}_\varphi \\ \cos\theta\,\vec{e}_r - \sin\theta\,\vec{e}_\theta \end{bmatrix}$$

Thus the equality of the two matrices gives

$$\vec{i} = \sin\theta\cos\varphi\,\vec{e}_r + \cos\theta\cos\varphi\,\vec{e}_\theta - \sin\varphi\,\vec{e}_\varphi$$

$$\vec{j} = \sin\theta\sin\varphi\,\vec{e}_r + \cos\theta\sin\varphi\,\vec{e}_\theta + \cos\varphi\,\vec{e}_\varphi$$

$$\vec{k} = \cos\theta\,\vec{e}_r - \sin\theta\,\vec{e}_\theta$$

39.

40. $A = \begin{pmatrix} y^2 & -xy \\ -xy & x^2 \end{pmatrix} \equiv \begin{pmatrix} T^{11} & T^{12} \\ T^{21} & T^{22} \end{pmatrix}$

under rotation of the coordinate system the new elements become

$$x' = x\cos\alpha + y\sin\alpha$$
$$y' = -x\sin\alpha + y\cos\alpha$$

And in the new (primed) coordinate system we must have

$$A' = \begin{pmatrix} y'^2 & -x'y' \\ -x'y' & x'^2 \end{pmatrix} \equiv \begin{pmatrix} T'^{11} & T'^{12} \\ T'^{21} & T'^{22} \end{pmatrix}$$

Now, $T'^{11} = y'^2 = (-x\sin\alpha + y\cos\alpha)$

$= \sin^2\alpha\, x^2$

$- 2\sin\alpha\cos\alpha\,(xy)$

$+ \cos^2\alpha\, y^2$ ---- (1)

But the transformation rule for a 2nd rank tensor is

$$T'^{ij} = \sum_{k,l} \frac{\partial x_i'}{\partial x_k} \frac{\partial x_j'}{\partial x_l} T^{kl} \quad \text{---- } (**)$$

$\Rightarrow T'^{11} = \sum_{k,l} \frac{\partial x_1'}{\partial x_k} \frac{\partial x_1'}{\partial x_l} T^{kl}$

$= \cos^2\alpha\, T^{11} + \cos\alpha\sin\alpha\, T^{12}$

$+ \sin\alpha\cos\alpha\, T^{21} + \sin^2\alpha\, T^{22}$

$= \cos^2\alpha\, y^2 + \cos\alpha\sin\alpha(-xy)$

$+ \sin\alpha\cos\alpha(-xy) + \sin^2\alpha\, x^2$

$= \sin^2\alpha\, x^2 - 2\sin\alpha\cos\alpha(xy)$

$+ \cos^2\alpha$ ---- (2)

So from (1) and (2) T^{11} transforms as a comp. of a tensor of 2nd rank

$T'^{12} = -x'y' = -(x\cos\alpha + y\sin\alpha)$

$\cdot(-x\sin\alpha + y\cos\alpha)$

$= \cos\alpha\sin\alpha\,(x^2)$

$- \cos^2\alpha\,(xy) + \sin^2\alpha\,(xy)$

$- \sin\alpha\cos\alpha\,(y^2)$ ---- (3)

Using the transformation rule (*)

$T'^{12} = \sum_{kl} \frac{\partial x_1'}{\partial x_k} \frac{\partial x_2'}{\partial x_l} T^{kl}$

$= -\cos\alpha\sin\alpha\, T^{11}$ ~~+ ...~~

~~- ...~~ $+ \cos^2\alpha\,(T^{12})$

$- \sin^2\alpha\,(T^{21}) + \sin\alpha\cos\alpha\,(T^{22})$

$= -\cos\alpha\sin\alpha\,(y^2) + \cos^2\alpha(-xy)$

$- \sin^2\alpha(-xy) + \sin\alpha\cos\alpha\, x^2$

$= -\cos\alpha\sin\alpha\, y^2 - \cos^2\alpha\,(xy)$

$+ \sin^2\alpha\,(xy) + \sin\alpha\cos\alpha\, x^2$ ---- (4)

from (3) & (4) T^{12} transforms as the comp. of a contravariant tensor

$T'^{21} = -x'y' = -(x\cos\alpha + y\sin\alpha)$

$\cdot(-x\sin\alpha + y\cos\alpha)$

$= \cos\alpha\sin\alpha\, x^2$

$- \cos^2\alpha\,(xy) + \sin^2\alpha\,(xy)$

$- \sin\alpha\cos\alpha\,(y)^2$ ---- (5)

Using the transformation rule

$T'^{21} = \sum_{kl} \frac{\partial x_2'}{\partial x_k} \frac{\partial x_1'}{\partial x_l} T^{kl}$

$= -\sin\alpha\cos\alpha\,(T^{11})$

$- \sin^2\alpha\,(T^{12})$

$+ \cos^2\alpha\,(T^{21}) + \cos\alpha\sin\alpha\,(T^{22})$

$= -\sin\alpha\cos\alpha\, y^2$

$+ \sin^2\alpha\,(xy) - \cos^2\alpha\,(xy)$

$+ \cos\alpha\sin\alpha\, x^2$

---- (6)

from (5) & (6) T^{21} transforms as a comp. of a contravariant tensor.

$T'^{22} = x'^2 = (x\cos\alpha + y\sin\alpha)$

$= \cos^2\alpha\, x^2 + 2\cos\alpha\sin\alpha(xy)$

$+ \sin^2\alpha\, y^2$ ---- (7)

From the transformation rule

$T'^{22} = \sum_{kl} \frac{\partial x_2'}{\partial x_k} \frac{\partial x_2'}{\partial x_l} T^{kl}$

$= +\sin^2\alpha\, T^{11} - \sin\alpha\cos\alpha\,(T^{12})$

$- \cos\alpha\sin\alpha\,(T^{21}) + \cos^2\alpha\, T^{22}$

$$T'^{22} = \sin^2\alpha\,(y^2) + 2\sin\alpha\cos\alpha\,(xy) + \cos^2\alpha\,(x^2) \quad \ldots 8$$

From (7) & (8) T^{22} transforms as the comp. of a contravariant tensor.

It follows from them T transforms as a 2nd rank contravariant tensor. Therefore it is a 2nd rank tensor

41.

$$B = \begin{pmatrix} -xy & x^2 \\ -y^2 & xy \end{pmatrix} \equiv \begin{pmatrix} T^{11} & T^{12} \\ T^{21} & T^{22} \end{pmatrix}$$

Once again under rotation

$$x' = x\cos\alpha + y\sin\alpha$$
$$y' = -x\sin\alpha + y\cos\alpha$$

We must have then

$$B' \equiv \begin{pmatrix} -x'y' & x'^2 \\ -y'^2 & x'y' \end{pmatrix} \equiv \begin{pmatrix} T'^{11} & T'^{12} \\ T'^{21} & T'^{22} \end{pmatrix}$$

Now,

$$T'^{11} = -x'y' = -(x\cos\alpha + y\sin\alpha)\cdot(-x\sin\alpha + y\cos\alpha)$$
$$= \cos\alpha\sin\alpha\,(x^2) - \cos^2\alpha\,(xy) + \sin^2\alpha\,(xy) - \sin\alpha\cos\alpha\,(y^2)$$

But $T'^{11} = \sum_{k,l} \frac{\partial x'_1}{\partial x_k}\frac{\partial x'_1}{\partial x_l} T^{kl}$

$$= \cos^2\alpha\,(T^{11}) + \cos\alpha\sin\alpha\,(T^{12}) + \sin\alpha\cos\alpha\,(T^{21}) + \sin^2\alpha\,(T^{22})$$
$$= -\cos^2\alpha\,(xy) + \cos\alpha\sin\alpha\,(x^2) - \sin\alpha\cos\alpha\,(y^2) + \sin^2\alpha\,(xy)$$

T'^{11} transforms as a comp. of a tensor

$$T'^{12} = x'^2 = (x\cos\alpha + y\sin\alpha)^2$$
$$= \cos^2\alpha\,(x^2) + 2\cos\alpha\sin\alpha\,(xy) + \sin^2\alpha\,(y^2)$$

But $T'^{12} = \sum_{k,l} \frac{\partial x'_1}{\partial x_k}\frac{\partial x'_2}{\partial x_l} T^{kl}$

$$= -\cos\alpha\sin\alpha\,(T^{11}) + \cos^2\alpha\,(T^{12}) - \sin^2\alpha\,(T^{21}) + \sin\alpha\cos\alpha\,(T^{22})$$
$$= 2\cos\alpha\sin\alpha\,(xy) + \cos^2\alpha\, x^2 + \sin^2\alpha\,(y^2)$$

T'^{12} transforms as a comp. of a contravariant tensor

$$T'^{21} = -y'^2 = -(-x\sin\alpha + y\cos\alpha)$$
$$= \sin^2\alpha\,(x^2) + 2\sin\alpha\cos\alpha\,(xy) + \cos^2\alpha\,(y^2)$$

But $T'^{21} = \sum_{k,l} \frac{\partial x'_2}{\partial x_k}\frac{\partial x'_1}{\partial x_l} T^{kl}$

$$= -\sin\alpha\cos\alpha\,(T^{11}) - \sin^2\alpha\,(T^{12}) + \cos^2\alpha\,(T^{21}) + \cos\alpha\sin\alpha\,(T^{22})$$
$$= 2\sin\alpha\cos\alpha\,(xy) - \sin^2\alpha\,(x^2) + \cos^2\alpha\,(y^2)$$

∴ T^{21} transforms as the comp. of a 2nd rank contravariant tensor

$$T'^{22} = x'y' = (x\cos\alpha + y\sin\alpha)(-x\sin\alpha + y\cos\alpha)$$
$$= -\cos\alpha\sin\alpha\,(x^2) + \cos^2\alpha\,(xy) - \sin^2\alpha\,(xy) + \sin\alpha\cos\alpha\, y^2$$

But $T'^{22} = \sum_{k,l} \frac{\partial x'_2}{\partial x_k}\frac{\partial x'_2}{\partial x_l} T^{kl}$

$$= +\sin^2\alpha\, T^{11} - \sin\alpha\cos\alpha\, T^{12} - \cos\alpha\sin\alpha\, T^{21} + \cos^2\alpha\, T^{22}$$
$$= -\sin^2\alpha\,(xy) - \sin\alpha\cos\alpha\,(x^2) + \cos\alpha\sin\alpha\, y^2 + \cos^2\alpha\,(xy)$$

∴ T'^{22} transforms as a comp. of a 2nd rank contravariant tensor.

Therefore B is a tensor.

42.

$$C = \begin{pmatrix} y^2 & xy \\ xy & x^2 \end{pmatrix} \equiv \begin{pmatrix} T^{11} & T^{12} \\ T^{21} & T^{22} \end{pmatrix}$$

As in prob. (40), under rotation

$$x' = x\cos\alpha + y\sin\alpha$$
$$y' = -x\sin\alpha + y\cos\alpha$$

If C is a tensor in the primed frame of reference it must be given by

$$C' = \begin{pmatrix} y'^2 & x'y' \\ x'y' & x'^2 \end{pmatrix} \equiv \begin{pmatrix} T'^{11} & T'^{12} \\ T'^{21} & T'^{22} \end{pmatrix}$$

So now,

$$T'^{11} = y'^2 = (-x\sin\alpha + y\cos\alpha)$$

$$= \sin^2\alpha\,(x^2) - 2\sin\alpha\cos\alpha\,(xy) + \cos^2\alpha\; y^2 \quad \text{---(1)}$$

But from the transformation T'^{11} must be given by

$$T'^{11} = \sum_{k,l} \frac{\partial x_1'}{\partial x_k}\frac{\partial x_1'}{\partial x_l} T^{kl}$$

$$= \cos^2\alpha\; T^{11} + \cos\alpha\sin\alpha\; T^{12} + \sin\alpha\cos\alpha\; T^{21} + \sin^2\alpha\; T^{22}$$

$$= \cos^2\alpha\,(y^2) + 2\cos\alpha\sin\alpha\,(xy) + \sin^2\alpha\,(x^2) \quad \text{---(2)}$$

So the results from (1) and (2) are different i.e., T'^{11} does not obey the transformation rule so that it must not be a tensor ~~the~~ comp. of a tensor.

It follows then C is not a tensor since one of its comps T'^{11} does not obey the transformation rule for a tensor.

$$D = \begin{pmatrix} xy & y^2 \\ x^2 & -xy \end{pmatrix} \equiv \begin{pmatrix} T^{11} & T^{12} \\ T^{21} & T^{22} \end{pmatrix}$$

In the rotated frame we must have

$$D' = \begin{pmatrix} x'y' & y'^2 \\ x'^2 & -x'y' \end{pmatrix} \equiv \begin{pmatrix} T'^{11} & T'^{12} \\ T'^{21} & T'^{22} \end{pmatrix}$$

Now,

$$T'^{11} = x'y' = (x\cos\alpha + y\sin\alpha)\cdot(-x\sin + y\cos\alpha)$$

$$= \cos\alpha\sin\alpha(x^2) + \cos^2\alpha\,(xy) - \sin^2\alpha\,(xy) + \sin\alpha\cos\alpha(y^2) \quad \text{...(3)}$$

Using the transformation rule

$$T'^{11} = \sum_{k,l} \frac{\partial x_1'}{\partial x_k}\frac{\partial x_1'}{\partial x_l} T^{kl}$$

$$T'^{11} = \cos^2\alpha\,(y^2) + 2\cos\alpha\sin\alpha\,(xy) + \sin^2\alpha\,(x^2) \quad \text{---(4)}$$

The results from (3) & (4) are diff. i.e., T'^{11} does not transform as the comp. of a tensor. It follows D is not a tensor since one of its comps T'^{11} does not obey the transformation rule.

4/2. Let us denote the tensor by T

$$T = \begin{pmatrix} -xy & x^2 \\ -y^2 & xy \end{pmatrix} \equiv \begin{pmatrix} T^{11} & T^{12} \\ T^{21} & T^{22} \end{pmatrix}$$

For a 2nd rank tensor the comp. T^{mn} could be resolved into symmetric and antisymmetric parts as

$$T^{mn} = \tfrac{1}{2}(T^{mn} + T^{nm}) + \tfrac{1}{2}(T^{mn} - T^{nm})$$

where

$\tfrac{1}{2}(T^{mn} + T^{nm})$ is the symmetric part and $\tfrac{1}{2}(T^{mn} - T^{nm})$ is the antisymmetric part.

Applying this principle for our case

(i) $T^{11} = -xy = \tfrac{1}{2}(T^{11} + T^{11}) + \tfrac{1}{2}(T^{11} - T^{11})$

$= \tfrac{1}{2}(2T^{11}) + \tfrac{1}{2}\cdot 0$

$= T^{11} + 0$

i.e., the symmetric part is just T^{11} itself and the antisymmetric part is 0.

$T^{11}_{symm.} = T^{11} = -xy$

$T^{11}_{antisymm.} = 0$

(ii) $T^{12} = \tfrac{1}{2}(T^{12} + T^{21}) + \tfrac{1}{2}(T^{12} - T^{21})$

$$T^{12}_{symm.} = \tfrac{1}{2}(T^{12}+T^{21}) = \tfrac{1}{2}(x^2-y^2)$$

$$T^{12}_{antisymm.} = \tfrac{1}{2}(x^2+y^2)$$

(iii) $$T^{21} = \tfrac{1}{2}(T^{21}+T^{12}) + \tfrac{1}{2}(T^{21}-T^{12})$$

$$T^{21}_{symm.} = \tfrac{1}{2}(T^{21}+T^{12}) = \tfrac{1}{2}(-y^2+x^2) = \tfrac{1}{2}(x^2-y^2)$$

$$T^{21}_{antisymm.} = \tfrac{1}{2}(T^{21}-T^{12}) = \tfrac{1}{2}(-y^2-x^2)$$

(iv) $$T^{22} = \tfrac{1}{2}(T^{22}+T^{22}) + \tfrac{1}{2}(T^{22}-T^{22})$$

$$T^{22}_{symm.} = \tfrac{1}{2}(T^{22}+T^{22}) = T^{22} = xy$$

$$T^{22}_{antisymm.} = \tfrac{1}{2}(T^{22}-T^{22}) = 0$$

so for the tensor

$$T = \begin{pmatrix} -xy & x^2 \\ -y^2 & xy \end{pmatrix}$$

From (i), (ii), (iii) & (iv)

The symmetric part is

$$T_{symm.} = \begin{pmatrix} T^{11}_{symm} & T^{12}_{symm.} \\ T^{21}_{symm} & T^{22}_{symm} \end{pmatrix}$$

$$\text{or } T_{symm.} = \begin{pmatrix} -xy & \frac{x^2-y^2}{2} \\ \frac{(x^2-y^2)}{2} & xy \end{pmatrix}$$

and the antisymmetric part is

$$T_{antisymm.} = \begin{pmatrix} T^{11}_{antisym.} & T^{12}_{antisymm.} \\ T^{21}_{antisymm.} & T^{22}_{antisymm} \end{pmatrix}$$

$$\text{or } T_{antisymm.} = \begin{pmatrix} 0 & \frac{x^2+y^2}{2} \\ -\frac{(x^2+y^2)}{2} & 0 \end{pmatrix}$$

43. As an introductory case consider a 2nd rank tensor in 2-D

$$T = \begin{pmatrix} T^{11} & T^{12} \\ T^{21} & T^{22} \end{pmatrix}$$

From the transformation rule for a 2nd rank tensor in an arbitrary rotated (primed) frame of reference

$$T'^{11} = \sum_{k,l} \frac{\partial x_1'}{\partial x_k}\frac{\partial x_1'}{\partial x_l} T^{kl}$$

$$= \frac{\partial x_1'}{\partial x_1}\frac{\partial x_1'}{\partial x_1}T^{11} + \frac{\partial x_1'}{\partial x_1}\frac{\partial x_1'}{\partial x_2}T^{12} + \frac{\partial x_1'}{\partial x_2}\frac{\partial x_1'}{\partial x_1}T^{21} + \frac{\partial x_1'}{\partial x_2}\frac{\partial x_1'}{\partial x_2}T^{22}$$

(i.e.) the components of the tensor in all coordinate systems are expressed as the linear combination of the components in the old (unprimed) frame of

$$T'^{12} = \sum_{k,l} \frac{\partial x_1'}{\partial x_k}\frac{\partial x_2'}{\partial x_l} T^{kl}$$

$$= \frac{\partial x_1'}{\partial x_1}\frac{\partial x_2'}{\partial x_1}T^{11} + \frac{\partial x_1'}{\partial x_1}\frac{\partial x_2'}{\partial x_2}T^{12} + \frac{\partial x_1'}{\partial x_2}\frac{\partial x_2'}{\partial x_1}T^{21} + \frac{\partial x_1'}{\partial x_2}\frac{\partial x_2'}{\partial x_2}T^{22}$$

$$T'^{21} = \sum_{k,l} \frac{\partial x_2'}{\partial x_k}\frac{\partial x_1'}{\partial x_l} T^{kl}$$

$$= \frac{\partial x_2'}{\partial x_1}\frac{\partial x_1'}{\partial x_1}T^{11} + \frac{\partial x_2'}{\partial x_1}\frac{\partial x_1'}{\partial x_2}T^{12} + \frac{\partial x_2'}{\partial x_2}\frac{\partial x_1'}{\partial x_1}T^{21} + \frac{\partial x_2'}{\partial x_2}\frac{\partial x_1'}{\partial x_2}T^{22}$$

$$T'^{22} = \sum_{k,l} \frac{\partial x_2'}{\partial x_k}\frac{\partial x_2'}{\partial x_l}$$

$$= \frac{\partial x_2'}{\partial x_1}\frac{\partial x_2'}{\partial x_1}T^{11} + \frac{\partial x_2'}{\partial x_1}\frac{\partial x_2'}{\partial x_2}T^{12} + \frac{\partial x_2'}{\partial x_2}\frac{\partial x_2'}{\partial x_1}T^{21} + \frac{\partial x_2'}{\partial x_2}\frac{\partial x_2'}{\partial x_2}T^{22}$$

(i.e.) the components of the tensor in all primed coordinat systems are expressed as the linear combination of the components in the old (unprimed) coordinate system.

Suppose

$$T = \begin{pmatrix} T^{11} & T^{12} \\ T^{21} & T^{22} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

It follows from the previous expressions

$$T'^{11} = 0, \quad T'^{12} = 0, \quad T'^{21} = 0, \quad T'^{22} = 0$$

Thus in the primed coordinate system T' vanishes i.e.,

$$T' = \begin{pmatrix} T'^{11} & T'^{12} \\ T'^{21} & T'^{22} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Now let us generalize our result. Consider a tensor of rank n.

$$T'^{i_1 i_2 \cdots i_n} = \sum_{j_1, j_2 \cdots j_n = 1}^{n} \frac{\partial x'_{i_1}}{\partial x_{j_1}} \frac{\partial x'_{i_2}}{\partial x_{j_2}} \cdots \frac{\partial x'_{i_n}}{\partial x_{j_n}} T^{j_1 j_2 \cdots j_n}$$

↓

If the components in the old frame $T^{j_1 j_2 \cdots j_n}$ vanish, it follows that the components in all rotated (or primed) coordinate system $T'^{i_1 i_2 \cdots i_n}$ vanish as they are given by the linear combination of the $T^{j_1 j_2 \cdots j_n}$ s.

~~44.~~ $|T'^{ip k \cdots l}| = |\Sigma|$

44. A^0_{ij} ; B^0_{ij}

Let us transform these tensors in an arbitrary coordinate system. Now,

$$A_{ij} = \frac{\partial x_k}{\partial x^0_i} \frac{\partial x_l}{\partial x^0_j} A^0_{ij}$$

$$B_{ij} = \frac{\partial x_k}{\partial x^0_i} \frac{\partial x_l}{\partial x^0_j} B^0_{ij}$$

But $A^0_{ij} = B^0_{ij}$ whereby

$$A_{ij} = B_{ij}$$

Since the chosen coordinate system is arbitrary, in all coordinate systems

$$A_{ij} = B_{ij}$$

45. All in all the tensor T_{iklm} has $3^4 = 81$ elements in 3-D. But those on the diagonal i.e., the 27 are zeros and the rest 54 are related through

$$T_{iklm} = -T_{kilm}, \text{ etc.}$$

which reduces the number of independent variables to 27. Any set of corresponding elements are related by given relation.

It follows in 3-D T_{iklm} has 27 independent elements.

46. $a_1 - \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$\sigma_1^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0\cdot0+1\cdot1 & 0\cdot1+1\cdot0 \\ 1\cdot0+0\cdot1 & 1\cdot1+0\cdot0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$= 1$, a unit matrix.

9   7   99   90

$\frac{72}{2} = 36$

$$-\sigma_2 = \begin{pmatrix} 0 & -j \\ j & 0 \end{pmatrix}$$

$$\sigma_2^2 = \begin{pmatrix} 0 & -j \\ j & 0 \end{pmatrix}\begin{pmatrix} 0 & -j \\ j & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0\cdot0 - j\cdot j & -0\cdot j + 0\cdot0 \\ j\cdot0 + 0\cdot j & -j\cdot j + 0\cdot0 \end{pmatrix}$$

$$= \begin{pmatrix} -j^2 & 0 \\ 0 & -j^2 \end{pmatrix}$$

$$= \begin{pmatrix} +1 & 0 \\ 0 & 1 \end{pmatrix}$$

$= 1$, a unit matrix.

$$-\sigma_3^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma_3^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1\cdot1+0\cdot0 & 1\cdot0-0\cdot1 \\ 0\cdot1-1\cdot0 & 0\cdot0+1\cdot1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$= 1$, a unit matrix.

? b, $-\sigma_1\cdot\sigma_2$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -j \\ j & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0\cdot0+1\cdot j & -0\cdot j+1\cdot0 \\ 1\cdot0+0\cdot j & -1\cdot j+0\cdot0 \end{pmatrix}$$

$$= \begin{pmatrix} j & 0 \\ 0 & -j \end{pmatrix}$$

$$= j\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$= j\sigma_3$

$-\ \sigma_2\cdot\sigma_3$

$$= \begin{pmatrix} 0 & -j \\ j & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 0\cdot1+j\cdot0 & 0\cdot0+j\cdot1 \\ j\cdot1+0\cdot0 & j\cdot0-0\cdot1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & +j \\ j & 0 \end{pmatrix}$$

$$= j\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$= j\sigma_1$

$-\ \sigma_3\cdot\sigma_1$

$$= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & +1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1\cdot0+0\cdot1 & -1\cdot1+0\cdot0 \\ 0\cdot0-1\cdot1 & -0\cdot1-1\cdot0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}$$

$$= -j\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$$

$= -j/ \not{j}$

$$= \begin{pmatrix} 0 & -j^2 \\ j^2 & 0 \end{pmatrix}$$

$$= j\begin{pmatrix} 0 & -j \\ j & 0 \end{pmatrix}$$

$= j\sigma_2$

c, $\sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij}E$

$-\ i = 1,\ j = 2$

$\sigma_1\sigma_2 + \sigma_2\sigma_1$

$$\sigma_1\sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -j \\ j & 0 \end{pmatrix}$$

$$= j\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix};\quad \text{from 4b (b)}$$

$$\sigma_2\sigma_1 = \begin{pmatrix} 0 & -j \\ j & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0\cdot0-j\cdot1 & 0\cdot1-j\cdot0 \\ j\cdot0+0\cdot1 & j\cdot1+0\cdot0 \end{pmatrix}$$

$$= \begin{pmatrix} -j & 0 \\ 0 & j \end{pmatrix}$$

$$= -j\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

then,

$\sigma_1\sigma_2 + \sigma_2\sigma_1$

$$= j\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - j\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= j\begin{pmatrix} 1-1 & 0 \\ 0 & -1+1 \end{pmatrix}$$

$$= j\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$= j\,\delta_{12}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$= j\,\delta_{12}E$

$-\ i = 2,\ j = 3$

$\sigma_2\sigma_3 + \sigma_3\sigma_2$

$$\sigma_2\sigma_3 = \begin{pmatrix} 0 & -j \\ j & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = j\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ from b}$$

$$\sigma_3\sigma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -j \\ j & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1\cdot 0+0\cdot j & -1\cdot j - 1 0\cdot 0 \\ 0\cdot 0 - 1\cdot j & -0\cdot j - 1\cdot 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -j \\ -j & 0 \end{pmatrix}$$

$$= -j\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$\sigma_2\sigma_3 + \sigma_3\sigma_2$

$$= j\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - j\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= j\begin{pmatrix} 0 & 1-1 \\ 1-1 & 0 \end{pmatrix}$$

$$= j\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$= j\delta_{23}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= j\,\delta_{23}\,E$$

$\sigma_3\sigma_1 + \sigma_1\sigma_3$

$$\sigma_3\sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= j\begin{pmatrix} 0 & -j \\ j & 0 \end{pmatrix}, \text{ from b}$$

$$\sigma_1\sigma_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 0\cdot 1+1\cdot 0 & 0\cdot 0 - 1\cdot 1 \\ 1\cdot 1+0\cdot 0 & 1\cdot 0 - 0\cdot 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & j^2 \\ -j^2 & 0 \end{pmatrix}$$

$$= -j\begin{pmatrix} 0 & -j \\ +j & 0 \end{pmatrix}$$

$\sigma_3\sigma_1 + \sigma_1\sigma_3$

$$= j\begin{pmatrix} 0 & -j \\ j & 0 \end{pmatrix} - j\begin{pmatrix} 0 & -j \\ j & 0 \end{pmatrix}$$

$$= j\begin{pmatrix} 0 & -j+j \\ j-j & 0 \end{pmatrix}$$

$$= j\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$= j\delta_{31}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= j\,\delta_{31}\,E$$

In general

$$\sigma_i\sigma_j + \sigma_j\sigma_i = j\,\delta_{ij}\,E$$

47.

$$M_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$M_y = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -j & 0 \\ j & 0 & -j \\ 0 & j & 0 \end{pmatrix}$$

$$M_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

a, $[M_x, M_y] \equiv M_xM_y - M_yM_z$

Now, $M_xM_y = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -j & 0 \\ j & 0 & -j \\ 0 & j & 0 \end{pmatrix}$

$$= \frac{1}{2}\begin{pmatrix} 0\cdot 0+1\cdot j+0\cdot 0 & -0\cdot j+1\cdot 0+0\cdot j & 0\cdot 0-1\cdot j+0\cdot 0 \\ 1\cdot 0+0\cdot j+1\cdot 0 & -1\cdot j+0\cdot 0+1\cdot j & 1\cdot 0-0\cdot j+1\cdot 0 \\ 0\cdot 0+1\cdot j+0\cdot 0 & -0\cdot j+1\cdot 0+0\cdot j & 0\cdot 0-1\cdot j+0\cdot 0 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} j & 0 & -j \\ 0 & 0 & 0 \\ j & 0 & -j \end{pmatrix}$$

$$M_yM_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -j & 0 \\ j & 0 & -j \\ 0 & j & 0 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 0\cdot 0-j\cdot 1+0\cdot 0 & 0\cdot 1-j\cdot 0+0\cdot 1 & 0\cdot 0-j\cdot 1+0\cdot 0 \\ j\cdot 0+0\cdot 1-j\cdot 0 & j\cdot 1+0\cdot 0-j\cdot 1 & j\cdot 0+0\cdot 1-j\cdot 0 \\ 0\cdot 0+j\cdot 1+0\cdot 0 & 0\cdot 1+j\cdot 0+0\cdot 1 & 0\cdot 0+j\cdot 1+0\cdot 0 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} -j & 0 & -j \\ 0 & 0 & 0 \\ j & 0 & j \end{pmatrix}$$

$[M_x, M_y]$

$$= M_x M_y - M_y M_x$$

$$= \frac{1}{2}\begin{vmatrix} j & 0 & -j \\ 0 & 0 & 0 \\ j & 0 & -j \end{vmatrix} - \frac{1}{2}\begin{pmatrix} -j & 0 & -j \\ 0 & 0 & 0 \\ j & 0 & j \end{pmatrix}$$

$$= \frac{1}{2}\begin{vmatrix} j+j & 0 & -j+j \\ 0 & 0 & 0 \\ j-j & 0 & -j-j \end{vmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 2j & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2j \end{pmatrix}$$

$$= \frac{1}{2} \cdot 2j \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$= j \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}}_{M_z}$$

$$= j M_z$$

Let us consider the cyclic permutation

− $[M_y, M_z]$

$$= M_y M_z - M_z M_y$$

$$M_y M_z = \frac{1}{\sqrt{2}}\begin{vmatrix} 0 & -j & 0 \\ j & 0 & -j \\ 0 & j & 0 \end{vmatrix}\begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{vmatrix}\frac{1}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}}\begin{vmatrix} 0\cdot1-j\cdot0+0\cdot0 & 0\cdot0-j\cdot0+0\cdot0 & 0\cdot0-j\cdot0-0\cdot1 \\ j\cdot1+0\cdot0-j\cdot0 & j\cdot0+0\cdot0-j\cdot0 & j\cdot0+0\cdot0+j\cdot1 \\ 0\cdot1+j\cdot0+0\cdot0 & 0\cdot0+j\cdot0+0\cdot0 & 0\cdot0+j\cdot0-0\cdot1 \end{vmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{vmatrix} 0 & 0 & 0 \\ j & 0 & j \\ 0 & 0 & 0 \end{vmatrix}$$

$$M_z M_y = \frac{1}{\sqrt{2}}\begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{vmatrix}\frac{1}{\sqrt{2}}\begin{vmatrix} 0 & -j & 0 \\ j & 0 & -j \\ 0 & j & 0 \end{vmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{vmatrix} 1\cdot0+0\cdot j+0\cdot0 & -1\cdot j+0\cdot0+0\cdot j & 1\cdot0-0\cdot j+0\cdot0 \\ 0\cdot0+0\cdot j+0\cdot0 & -0\cdot j+0\cdot0+0\cdot j & 0\cdot0-0\cdot j+0\cdot0 \\ 0\cdot0+0\cdot j-1\cdot0 & -0\cdot j+0\cdot0-1\cdot j & 0\cdot0-0\cdot j-1\cdot0 \end{vmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -j & 0 \\ 0 & 0 & 0 \\ 0 & -j & 0 \end{pmatrix}$$

Now, $[M_y, M_z]$

$$= M_y M_z - M_z M_y$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 \\ j & 0 & j \\ 0 & 0 & 0 \end{pmatrix} - \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -j & 0 \\ 0 & 0 & 0 \\ 0 & -j & 0 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{vmatrix} 0 & j & 0 \\ j & 0 & j \\ 0 & j & 0 \end{vmatrix}$$

$$= \frac{j}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = j M_x$$

$$= \frac{j}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= j M_x E$$

− $[M_z, M_x]$

$$= M_z M_x - M_x M_z$$

$$M_z \cdot M_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{vmatrix} 1\cdot0+0\cdot1+0\cdot0 & 1\cdot1+0\cdot0+0\cdot1 & 1\cdot0+0\cdot1+0\cdot0 \\ 0\cdot0+0\cdot1+0\cdot0 & 0\cdot1+0\cdot0+0\cdot1 & 0\cdot0+0\cdot1+0\cdot0 \\ 0\cdot0+0\cdot1-1\cdot0 & 0\cdot1+0\cdot0-1\cdot1 & 0\cdot0+0\cdot1-1\cdot0 \end{vmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$$

$$M_x M_z = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{vmatrix} 0\cdot1+1\cdot0+0\cdot0 & 0\cdot0+1\cdot0+0\cdot0 & 0\cdot0+1\cdot0-0\cdot1 \\ 1\cdot1+0\cdot0+1\cdot0 & 1\cdot0+0\cdot0+1\cdot0 & 1\cdot0+0\cdot0-1\cdot1 \\ 0\cdot1+1\cdot0+0\cdot0 & 0\cdot0+1\cdot0+0\cdot0 & 0\cdot0+1\cdot0-0\cdot1 \end{vmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$[M_z, M_x]$

$= M_z M_x - M_x M_z$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} - \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & +1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -j^2 & 0 \\ j^2 & 0 & -j^2 \\ 0 & j^2 & 0 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -j & 0 \\ j & 0 & -j \\ 0 & j & 0 \end{pmatrix}$$

$$= \frac{j}{\sqrt{2}}\begin{pmatrix} 0 & -j & 0 \\ j & 0 & -j \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$= j M_y E$

$= j M_y$

b, $M^2 = M_x^2 + M_y^2 + M_z^2 = 2E$

$$M_x^2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$M_y^2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -j & 0 \\ j & 0 & -j \\ 0 & j & 0 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -j & 0 \\ j & 0 & -j \\ 0 & j & 0 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

$$M_z^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$M^2 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 6 & 2 \end{pmatrix}$$

$$= 2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$= 2E$

c, $M^2 = 2E$

$[M^2, M_i] = M^2 M_i - M_i M^2$

- $[M^2, M_x] = M^2 M_x - M_x M^2$

$= 2E M_x - M_x 2E$

$= 2(E M_x - M_x E)$

$= 2(M_x - M_x)$, E is

E is identity in multiplication since it is a unit matrix.

$\Rightarrow [M^2, M_x] = 2 \cdot 0 = 0$

- $[M^2, M_y] = M^2 M_y - M_y M^2$

$= 2E M_y - M_y 2E$

$= 2(E M_y - M_y E)$

$= 2(M_y - M_y)$

$= 2 \cdot 0$

$= 0$

- $[M^2, M_z] = M^2 M_z - M_z M^2$

$= 2E M_z - M_z 2E$

$= 2(E M_z - M_z E)$

$= 2(M_z - M_z)$

$= 2 \cdot 0$

$= 0$

d) $L^+ = M_x + jM_y$

$$\Rightarrow L^+ = \frac{1}{\sqrt{2}}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix} + j\frac{1}{\sqrt{2}}\begin{pmatrix}0&-j&0\\j&0&-j\\0&j&0\end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix}0&1&0\\-1&0&1\\0&-1&0\end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix}0&2&0\\0&0&2\\0&0&0\end{pmatrix}$$

Now, $[M_z, L^+]$

$$= M_zL^+ - L^+M_z$$

$$M_zL^+ = \begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}0&2&0\\0&0&2\\0&0&0\end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix}0&2&0\\0&0&0\\0&0&0\end{pmatrix}$$

$$L^+M_z = \frac{1}{\sqrt{2}}\begin{pmatrix}0&2&0\\0&0&2\\0&0&0\end{pmatrix}\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix}0&0&0\\0&0&-2\\0&0&0\end{pmatrix}$$

Then $[M_z, L^+]$

$$= M_zL^+ - L^+M_z$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix}0&2&0\\0&0&0\\0&0&0\end{pmatrix} - \frac{1}{\sqrt{2}}\begin{pmatrix}0&0&0\\0&0&-2\\0&0&0\end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix}0&2&0\\0&0&2\\0&0&0\end{pmatrix}$$

$$= L^+$$

e) $L^- = M_x - jM_y$

$$\Rightarrow L^- = \frac{1}{\sqrt{2}}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix} - \frac{j}{\sqrt{2}}\begin{pmatrix}0&-j&0\\j&0&-j\\0&j&0\end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix} - \frac{1}{\sqrt{2}}\begin{pmatrix}0&+1&0\\-1&0&+1\\0&-1&0\end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix}0&0&0\\2&0&0\\0&2&0\end{pmatrix}$$

Now, $[L^+, L^-]$

$$= L^+L^- - L^-L^+$$

$$L^+L^- = \frac{1}{\sqrt{2}}\begin{pmatrix}0&2&0\\0&0&2\\0&0&0\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}0&0&0\\2&0&0\\0&2&0\end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix}4&0&0\\0&4&0\\0&0&0\end{pmatrix}$$

$$= 2\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}$$

$$L^-L^+ = \frac{1}{\sqrt{2}}\begin{pmatrix}0&0&0\\2&0&0\\0&2&0\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}0&2&0\\0&0&2\\0&0&0\end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix}0&0&0\\0&4&0\\0&0&4\end{pmatrix}$$

$$= 2\begin{pmatrix}0&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

Then, $[L^+, L^-]$

$$= L^+L^- - L^-L^+$$

$$= 2\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix} - 2\begin{pmatrix}0&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

$$= 2\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}$$

$$= 2M_z$$

48.

$$M_x = \begin{pmatrix}0&\sqrt{3}&0&0\\\sqrt{3}&0&+2&0\\0&2&0&\sqrt{3}\\0&0&\sqrt{3}&0\end{pmatrix}$$

$$M_y = \frac{j}{\sqrt{2}}\begin{pmatrix}0&-\sqrt{3}&0&0\\\sqrt{3}&0&-2&0\\0&2&0&-\sqrt{3}\\0&0&\sqrt{3}&0\end{pmatrix}$$

$$M_z = \frac{1}{2}\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}$$

a, $[M_x, M_y]$

$= M_x M_y - M_y M_x$

$$M_x M_y = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix} \cdot \frac{j}{\sqrt{2}}\begin{pmatrix} 0 & -\sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & -2 & 0 \\ 0 & 2 & 0 & -\sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}$$

$$= \frac{j}{2}\begin{pmatrix} 3 & 0 & -2\sqrt{3} & 0 \\ 0 & +1 & 0 & -2\sqrt{3} \\ 2\sqrt{3} & 0 & -1 & 0 \\ 0 & 2\sqrt{3} & 0 & -3 \end{pmatrix}$$

$$M_y M_x = \frac{j}{\sqrt{2}}\begin{pmatrix} 0 & -\sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & -2 & 0 \\ 0 & 2 & 0 & -\sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}$$

$$= \frac{j}{2}\begin{pmatrix} -3 & 0 & -2\sqrt{3} & 0 \\ 0 & -1 & 0 & -2\sqrt{3} \\ 2\sqrt{3} & 0 & 1 & 0 \\ 0 & 2\sqrt{3} & 0 & 3 \end{pmatrix}$$

Now, $[M_x, M_y] = M_x M_y - M_y M_x$

$$= \frac{j}{2}\begin{pmatrix} 3 & 0 & -2\sqrt{3} & 0 \\ 0 & 1 & 0 & -2\sqrt{3} \\ 2\sqrt{3} & 0 & -1 & 0 \\ 0 & 2\sqrt{3} & 0 & -3 \end{pmatrix} - \frac{j}{2}\begin{pmatrix} -3 & 0 & -2\sqrt{3} & 0 \\ 0 & -1 & 0 & -2\sqrt{3} \\ 2\sqrt{3} & 0 & 1 & 0 \\ 0 & 2\sqrt{3} & 0 & 3 \end{pmatrix}$$

$$= \frac{j}{2}\begin{pmatrix} 6 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -6 \end{pmatrix}$$

$$= \frac{j}{2}\cdot 2\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}$$

$$= j\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}$$

$= j M_z$

i.e., $[M_x, M_y] = j M_z$

To consider the cyclic permutation

$-\ [M_y, M_z] = M_y M_z - M_z M_y$

$$M_y M_z = \frac{j}{\sqrt{2}}\begin{pmatrix} 0 & -\sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & -2 & 0 \\ 0 & 2 & 0 & -\sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix} \cdot \frac{1}{2}\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}$$

$$= \frac{j}{2\sqrt{2}}\begin{pmatrix} 0 & -\sqrt{3} & 0 & 0 \\ 3\sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & 3\sqrt{3} \\ 0 & 0 & -\sqrt{3} & 0 \end{pmatrix}$$

$$M_z M_y = \frac{1}{2}\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix} \frac{j}{\sqrt{2}}\begin{pmatrix} 0 & -\sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & -2 & 0 \\ 0 & 2 & 0 & -\sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}$$

$$= \frac{j}{2\sqrt{2}}\begin{pmatrix} 0 & -3\sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & -2 & 0 \\ 0 & -2 & 0 & \sqrt{3} \\ 0 & 0 & -3\sqrt{3} & 0 \end{pmatrix}$$

Then $[M_y, M_z] = M_y M_z - M_z M_y$

$$= \frac{j}{2\sqrt{2}}\begin{pmatrix} 0 & -\sqrt{3} & 0 & 0 \\ 3\sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & 3\sqrt{3} \\ 0 & 0 & -\sqrt{3} & 0 \end{pmatrix} - \frac{j}{2\sqrt{2}}\begin{pmatrix} 0 & -3\sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & -2 & 0 \\ 0 & -2 & 0 & \sqrt{3} \\ 0 & 0 & -3\sqrt{3} & 0 \end{pmatrix}$$

$$= \frac{j}{2\sqrt{2}}\begin{pmatrix} 0 & 2\sqrt{3} & 0 & 0 \\ 2\sqrt{3} & 0 & 4 & 0 \\ 0 & 4 & 0 & 2\sqrt{3} \\ 0 & 0 & 2\sqrt{3} & 0 \end{pmatrix}$$

$$= \frac{j}{\sqrt{2}}\begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}$$

$= j M_x$

i.e., $[M_y, M_z] = j M_x$

$[M_z, M_x] = M_z M_x - M_x M_z$

$$M_z M_x = \frac{1}{2}\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}\begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & +2 & 0 \\ 0 & 2 & 0 & +\sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}\frac{1}{\sqrt{2}}$$

$$= \frac{1}{2\sqrt{2}}\begin{pmatrix} 0 & 3\sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & +2 & 0 \\ 0 & -2 & 0 & -\sqrt{3} \\ 0 & 0 & -3\sqrt{3} & 0 \end{pmatrix}$$

$$M_x M_z = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}\frac{1}{2}\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}$$

$$= \frac{1}{2\sqrt{2}}\begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ 3\sqrt{3} & 0 & -2 & 0 \\ 0 & 2 & 0 & -3\sqrt{3} \\ 0 & 0 & -\sqrt{3} & 0 \end{pmatrix}$$

Then, $[M_z, M_x] = M_z M_x - M_x M_z$

$$= \frac{1}{2\sqrt{2}}\begin{pmatrix} 0 & 3\sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & -2 & 0 & -\sqrt{3} \\ 0 & 0 & -3\sqrt{3} & 0 \end{pmatrix} - \frac{1}{2\sqrt{2}}\begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ 3\sqrt{3} & 0 & -2 & 0 \\ 0 & 2 & 0 & -3\sqrt{3} \\ 0 & 0 & -\sqrt{3} & 0 \end{pmatrix}$$

$$= \frac{1}{2\sqrt{2}}\begin{pmatrix} 0 & 2\sqrt{3} & 0 & 0 \\ -2\sqrt{3} & 0 & 4 & 0 \\ 0 & -4 & 0 & 2\sqrt{3} \\ 0 & 0 & -2\sqrt{3} & 0 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ -\sqrt{3} & 0 & 2 & 0 \\ 0 & -2 & 0 & \sqrt{3} \\ 0 & 0 & -\sqrt{3} & 0 \end{pmatrix}$$

$$= -\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -\sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & -2 & 0 \\ 0 & 2 & 0 & -\sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}$$

$= M_y$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -j^2\sqrt{3} & 0 & 0 \\ j^2\sqrt{3} & 0 & -j^2 2 & 0 \\ 0 & j^2 2 & 0 & -j^2\sqrt{3} \\ 0 & 0 & j^2\sqrt{3} & 0 \end{pmatrix}$$

$$= j^2\begin{pmatrix} 0 & -\sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & -2 & 0 \\ 0 & 2 & 0 & -\sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}$$

$= j^2 M_y E$

b) $M^2 = M_x^2 + M_y^2 + M_z^2$

$$M_x^2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 3 & 0 & 2\sqrt{3} & 0 \\ 0 & 7 & 0 & 2\sqrt{3} \\ 2\sqrt{3} & 0 & 7 & 0 \\ 0 & 2\sqrt{3} & 0 & 3 \end{pmatrix}$$

$$M_y^2 = \frac{j}{\sqrt{2}}\begin{pmatrix} 0 & -\sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & -2 & 0 \\ 0 & 2 & 0 & -\sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}\frac{j}{\sqrt{2}}\begin{pmatrix} 0 & -\sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & -2 & 0 \\ 0 & 2 & 0 & -\sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}$$

$$= -\frac{1}{2}\begin{pmatrix} -3 & 0 & 2\sqrt{3} & 0 \\ 0 & -7 & 0 & 2\sqrt{3} \\ 2\sqrt{3} & 0 & -7 & 0 \\ 0 & 2\sqrt{3} & 0 & -3 \end{pmatrix}$$

$$M_z^2 = \frac{1}{2}\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}\frac{1}{2}\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} 9 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 9 \end{pmatrix}$$

Assignment I

S-1. A medium has electromagnetic constants $\sigma = 10^2$ s/m, $\mu_r = 2$, $\epsilon_r = 3$. Find the skin depth at a) 50 Hz b) 2 MHz c) 3 GHz

6-2. At 500 MHz the medium has electromagnetic constants: $\sigma = 10^3$ s/m, $\mu_r = 5$, $\epsilon_r = 7$. Find a) λ/λ_0 b) c/c_0 c) the skin depth

3. Given a plane wave: $\bar{E} = \bar{e}_x E_0 \cos\omega(\sqrt{\epsilon\mu}\, z - t) + \bar{e}_y E_0 \sin\omega(\sqrt{\epsilon\mu}\, z - t)$ Determine the corresponding magnetic field $\bar{B}$ and the Poynting vector $\bar{\pi}$

4. Given a plane wave characterized by E_x, B_y propagating in the positive z-direction: $\bar{E} = \bar{e}_x E_0 \sin\frac{2\pi}{\lambda}(z - ct)$ show that it is possible to take the scalar potential $\varphi = 0$ and find a possible vector potential $\bar{A}$ for which the Lorentz condition is satisfied.

5. Show that in free space ($\rho = 0$, $\bar{J} = 0$) the Maxwell's eqs are correctly obtained from a single vector function $\bar{A}$ satisfying

$$\nabla\cdot\bar{A} = 0, \quad \nabla^2\bar{A} - \frac{1}{c^2}\frac{\partial^2\bar{A}}{\partial t^2} = 0$$

(The condition $\nabla\cdot\bar{A} = 0$ is called the coulomb gauge)

6. Show that in a linear conducting medium a suitable gauge can be chosen so that $\bar{A}$ and φ each satisfy the wave equation

$$\nabla^2\bar{E} - \epsilon\mu\frac{\partial^2\bar{E}}{\partial t^2} - \sigma\mu\frac{\partial\bar{E}}{\partial t} = 0, \text{ assume } \rho = 0$$

7. Given a medium in which $\rho = 0$, $\bar{J} = 0$, $\mu = \mu_0$ but where the polarization vector $\bar{P}$ is a given function of position and time $\bar{P} = \bar{P}(\bar{r}, t)$. Show that the Maxwell's equations are correctly obtained from a single vector function $\bar{Z}$ which satisfies the eq.

$$\nabla^2\bar{Z} - \epsilon_0\mu_0\frac{\partial^2\bar{Z}}{\partial t^2} = -\bar{P}/\epsilon_0 \quad \text{and}$$

$$\bar{E} = \nabla\times(\nabla\times\bar{Z}) - \frac{\bar{P}}{\epsilon_0}, \quad \bar{B} = \epsilon_0\mu_0\nabla\times\frac{\partial\bar{Z}}{\partial t}$$

$n = \frac{c_0}{c_\varphi}$ $\quad \sqrt{\frac{\epsilon}{\epsilon_0}} = \frac{\sqrt{\epsilon\mu}}{\sqrt{\epsilon_0\mu_0}} = n$

8. Given a medium in which $\rho = 0$, $\bar{J} = 0$, $\epsilon = \epsilon_0$ but where the magnetization vector $\bar{M}(\bar{r}, t)$ is a given function. Show that the Maxwell eqs are correctly obtained from a single wave function $\bar{Z}'$, which satisfies the eq.

$$\nabla^2\bar{Z}' - \epsilon_0\mu_0\frac{\partial^2\bar{Z}'}{\partial t^2} = \mu_0\bar{M} \quad \text{and where} \quad \bar{B} = \nabla\times(\nabla\times\bar{Z}'), \quad \bar{E} = -\nabla\times\partial\bar{Z}'/\partial t$$

9. A beam of monochromatic light is incident normally on a dielectric film of refractive index $n = \sqrt{\epsilon/\epsilon_0}$. The thickness of the film is d. Calculate the reflection coeff. for the reflected wave as a function of d and n. [Hint: assum two waves traveling in opposite directions inside the film].

Result: $$R = \frac{2r[1 + \cos(2\omega n d/c)]}{1 + r^2 + 2r\cos(2\omega n d/c)}; \quad r = \left(\frac{n-1}{n+1}\right)^2$$

10. Show that for a vacuum-conductor interface the reflection coefficient R may be written as

$$R = 1 - 4\pi\frac{\mu}{\mu_0}\frac{\delta}{\lambda_0}$$ (where δ is the skind depth.

11. Introducing a new variable $x_4 = ict$, prove that the in homogeneous wave equations $\nabla^2 \bar{A} - \frac{1}{c^2}\frac{\partial^2 \bar{A}}{\partial t^2} = -\mu \bar{J}$ --- (*)

and $\nabla^2 \varphi - \frac{1}{c^2}\frac{\partial^2 \varphi}{\partial t^2} = -\frac{\rho}{\epsilon}$ --- (**) ~~Can~~

Can be written as follows:

$$\sum_{\beta=1}^{4} \frac{\partial^2 \bar{A}}{\partial x_\beta^2} = \Box^2 \bar{A} = -\mu \bar{J} \quad (*)$$

$$\sum_{\beta=1}^{4} \frac{\partial^2 \varphi}{\partial x_\beta^2} = \Box^2 \varphi = -\frac{\rho}{\epsilon} \quad (**)$$

where $x_1 = x$, $x_2 = y$, $x_3 = z$, $x_4 = ict$

$\Box^2$ is d alembertian operator

12. Show that eqs (*), (**) from the preceeding probs can be written as a single relation

$\Box^2 A_\alpha = -\mu J_\alpha$, where A_α are comps of the four-vector potential

$A_\alpha = (A_x, A_y, A_z, i\varphi/c)$ and J_α are comp of the four-vector current

$J_\alpha = (J_x, J_y, J_z, ic\rho)$

$$\Box^2 \bar{A} = -\mu \bar{J} \quad \text{---} (1)$$

$$\Box^2 \varphi = -\frac{\rho}{\epsilon}$$

$$\Box^2 \frac{i\varphi}{c} = -\frac{i}{c}\frac{\rho}{\epsilon} = -i\mu c\rho \quad \text{--} (2)$$

(1) + (2) = $\Box^2 \bar{A} + \Box^2\left(\frac{i\varphi}{c}\right) = -\mu(\bar{J} + ic\rho)$

$$\Box^2\left(\bar{A} + \frac{i\varphi}{c}\right) = -\mu(\bar{J} + ic\rho)$$

Let $A_\alpha = \bar{A} + \frac{i\varphi}{c}$, $J_\alpha = \bar{J} + ic\rho$

i.e., $\Box^2 A_\alpha = -\mu J_\alpha$

Solution to problems on Assignment I

1. The skin depth δ is given by

$$\delta = 1/\beta = \sqrt{\frac{2}{\sigma\mu\omega}} = \sqrt{\frac{2}{\sigma(\mu_r\mu_0)\omega}}$$

a, at $f = 50\ Hz$, $\omega = 2\pi f = 2\pi \cdot 50\ s^{-1} = 100\pi\ s^{-1}$

$$\delta = \sqrt{\frac{2}{10^{2}\,S/m \cdot 2\times 4\pi\times10^{-7}\,\frac{Wb}{A\cdot m}\times 100\pi\, s^{-1}}}$$

b, $f = 2\ MHz$, $\omega = 2\pi \cdot 2\times10^{6}\ Hz = 4\pi\times10^{6}\ s^{-1}$

$$\delta = \sqrt{\frac{2}{10^{2}\,S/m\times 2\times4\pi\times10^{-7}\,\frac{Wb}{A\cdot m}\times 4\pi\times10^{6}\, s^{-1}}}$$

c, $f = 3\ GHz$, $\omega = 2\pi\cdot 3\times10^{9}\ Hz = 6\pi\times10^{9}\ s^{-1}$

$$\delta = \sqrt{\frac{2}{10^{2}\,S/m\times2\times4\pi\times10^{-7}\,\frac{Wb}{A\cdot m}\times6\pi\times10^{9}\, s^{-1}}}$$

2. a, λ_0, the free space wavelength of EM wave is

$$\lambda_0 = \frac{2\pi}{k_0} = \frac{2\pi}{\omega\sqrt{\epsilon_0\mu_0}}$$

and that in ~~dielectric~~ conducting medium is $\lambda = 2\pi\delta = 2\pi\sqrt{\frac{2}{\sigma\mu\omega}}$

i.e., $$\frac{\lambda}{\lambda_0} = 2\pi\sqrt{\frac{2}{\sigma\mu\omega}} \Big/ \frac{2\pi}{\omega\sqrt{\epsilon_0\mu_0}} = \sqrt{\frac{2}{\sigma\mu\omega}}\cdot\omega\sqrt{\epsilon_0\mu_0} = \sqrt{\frac{2\omega\epsilon_0\mu_0}{\sigma\mu}}$$

or $$\lambda/\lambda_0 = \frac{1}{\lambda_0}\sqrt{\frac{2\omega\epsilon_0\mu_0}{\sigma\mu_r\mu_0}} = \sqrt{\frac{2\omega\epsilon_0}{\sigma\mu_r}}$$

$$\therefore \frac{\lambda}{\lambda_0} = \sqrt{\frac{2\times2\pi\times5\times10^{8}\,s^{-1}\times8.85\times10^{-12}\,F/m}{10^{3}\,S/m\times5}}$$

b, $$c/c_0 = \frac{1/\sqrt{\epsilon\mu}}{1/\sqrt{\epsilon_0\mu_0}} = \sqrt{\frac{\epsilon_0\mu_0}{\epsilon\mu}} = \frac{1}{\sqrt{\epsilon_r\mu_r}} = \frac{1}{\sqrt{5\times7}} = \frac{1}{\sqrt{35}}$$

c, $$\delta = \sqrt{\frac{2}{\sigma\mu\omega}} = \sqrt{\frac{2}{10^{3}\,S/m\times5\times4\pi\times10^{-7}\frac{F}{m}\times2\pi\times5\times10^{8}\,s^{-1}}}$$

3. Here Maxwell's eq. must be satisfied

$$\nabla\times\bar{E} = -\frac{\partial\bar{B}}{\partial t} = i\omega\bar{B}_s e^{i\omega t}$$

$E_x = E_0\cos\omega(\sqrt{\epsilon\mu}\,z - t)$, $E_y = E_0\sin\omega(\sqrt{\epsilon\mu}\,z - t)$

$$\nabla\times\bar{E} = \begin{vmatrix} \bar{e}_x & \bar{e}_y & \bar{e}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \bar{E}_x & \bar{E}_y & 0 \end{vmatrix} = -\frac{\partial\bar{B}}{\partial t}$$

$$-\bar{e}_x\frac{\partial\bar{E}_y}{\partial z} + \bar{e}_y\frac{\partial\bar{E}_x}{\partial z} + \bar{e}_z\left(\frac{\partial\bar{E}_y}{\partial x} - \frac{\partial E_x}{\partial y}\right) = -\bar{e}_x\frac{\partial B_x}{\partial t} - \bar{e}_y\frac{\partial B_y}{\partial t} - \bar{e}_z\frac{\partial B_z}{\partial t}$$

Comparison gives

$$\frac{\partial E_y}{\partial z} = \frac{\partial B_x}{\partial t}, \quad \frac{\partial E_x}{\partial z} = -\frac{\partial B_y}{\partial t}$$

or

$$\frac{\partial B_x}{\partial t} = \omega\sqrt{\epsilon\mu}\, E_0 \cos\omega(\sqrt{\epsilon\mu}\, z - t)$$

$$\frac{\partial B_y}{\partial t} = +\omega\sqrt{\epsilon\mu}\, E_0 \sin\omega(\sqrt{\epsilon\mu}\, z - t)$$

Integrating w.r.t. time (over a finite period of time of course)

$$B_x = -\sqrt{\epsilon\mu}\, E_0 \sin\omega(\sqrt{\epsilon\mu}\, z - t) + \text{const } c(z)$$

$$B_y = +\sqrt{\epsilon\mu}\, E_0 \cos\omega(\sqrt{\epsilon\mu}\, z - t) + \text{const } c(z)$$

Since a const cannot affect the nature of a function we may assume that the constants are zero (i.e.)

$$B_x = -\frac{K}{\omega} E_0 \sin\omega(\sqrt{\epsilon\mu}\, z - t) = -\frac{K}{\omega} E_y$$

$$B_y = +\frac{K}{\omega} E_0 \cos\omega(\sqrt{\epsilon\mu}\, z - t) = +\frac{K}{\omega} E_x$$

(i.e.)

$$\bar{B} = +\bar{e}_x \frac{K}{\omega} E_y + \bar{e}_y \frac{K}{\omega} E_x$$

$$= -\frac{K}{\omega}(\bar{e}_x E_y + \bar{e}_y E_x)$$

$$= -\frac{K}{\omega} E_0 [\bar{e}_x \sin\omega(\sqrt{\epsilon\mu}\, z - t) + \bar{e}_y \cos\omega(\sqrt{\epsilon\mu}\, z - t)]$$

4.

The poyinting vector $\bar{\pi}$ is given by

$$\bar{\pi} = \bar{E} \times \bar{H}$$

or

$$\bar{\pi} = \begin{vmatrix} \bar{e}_x & \bar{e}_y & \bar{e}_z \\ E_x & E_y & E_z \\ H_x & H_y & H_z \end{vmatrix}$$

$$= \bar{e}_x(E_y H_z - E_z H_y) + \bar{e}_y(E_z H_x - E_x H_z) + \bar{e}_z(E_x H_y - E_y H_x)$$

Since $E_z = 0$, $H_z = 0$ we have

$$\bar{\pi} = (E_x H_y - E_y H_x)\bar{e}_z$$

$$= [+E_0 \cos\omega(\sqrt{\epsilon\mu}\, z - t)\cdot\sqrt{\tfrac{\epsilon}{\mu}}\, E_0 \cos\omega(\sqrt{\epsilon\mu}\, z - t) + E_0 \sin\omega(\sqrt{\epsilon\mu}\, z - t)\cdot\sqrt{\tfrac{\epsilon}{\mu}}\, E_0 \sin\omega(\sqrt{\epsilon\mu}\, z - t)]\,\bar{e}_z$$

~~= 0 assuming a linear medium.~~

$$E_0^2 \sqrt{\tfrac{\epsilon}{\mu}}\, \bar{e}_z$$

4.
$$\bar{E} = \bar{e}_x E_0 \sin \frac{2\pi}{\lambda}(z-ct)$$

In general $\bar{E} = -\nabla\varphi - \partial\bar{A}/\partial t$

Suppose $\varphi = 0$. To show that under this condition the Lorentz condition is satisfied. then,

$$\bar{E} = -\frac{\partial\bar{A}}{\partial t} = \bar{e}_x E_0 \sin \frac{2\pi}{\lambda}(z-ct)$$

$$\frac{\partial\bar{A}}{\partial t} = -\bar{e}_x E_0 \sin \frac{2\pi}{\lambda}(z-ct)$$

Integrating over a finite period of time

$$\bar{A} = -\bar{e}_x \frac{\lambda}{2\pi c} \cos \frac{2\pi}{\lambda}(z-ct)$$

From the Lorentz condition

$$\nabla\cdot\bar{A} = -\epsilon\mu \frac{\partial\varphi}{\partial t}$$

Since $\varphi = 0$, the right hand is zero (i.e., $\epsilon\mu \frac{\partial\varphi}{\partial t} = 0$)

$$\nabla\cdot\bar{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$= \frac{\partial A_x}{\partial x}, \quad A_y = 0, A_z = 0$$

$$= \frac{\partial}{\partial x}\left\{-\frac{\lambda}{2\pi c}\cos\frac{2\pi}{\lambda}(z-ct)\right\}$$

$$= 0$$

$$\therefore \nabla\cdot\bar{A} = -\epsilon\mu\frac{\partial\varphi}{\partial t} = 0.$$

So it is possible to take $\varphi = 0$ and find a vector potential $\bar{A}$ for which the Lorentz condition is fulfilled.

5.
Maxwell's eqs for free space become then

$$\nabla\cdot\bar{D} = 0 \qquad \nabla\times\bar{H} - \frac{\partial\bar{D}}{\partial t} = 0$$
$$\nabla\cdot\bar{B} = 0 \qquad \nabla\times\bar{E} + \partial\bar{B}/\partial t = 0$$

$$\nabla\cdot\bar{B} = 0 \Rightarrow \bar{B} = \text{curl}\,\bar{A} = \nabla\times\bar{A}$$

$$\nabla\times\bar{E} + \frac{\partial\bar{B}}{\partial t} = 0 \Rightarrow \nabla\times\bar{E} + \frac{\partial}{\partial t}(\nabla\times\bar{A}) = 0$$

$$\nabla\times\left(\bar{E} + \frac{\partial\bar{A}}{\partial t}\right) = 0$$

$$\Rightarrow E + \partial\bar{A}/\partial t = -\nabla\varphi$$

or $\quad \bar{E} = -\nabla\varphi - \partial\bar{A}/\partial t$

$$\nabla\cdot\bar{D} = 0 \Rightarrow \nabla\cdot\epsilon_0\bar{E} = 0, \text{ free space } \bar{P} = 0$$

$$\nabla\cdot\bar{E} = 0$$

$$\nabla\cdot\left(-\nabla\varphi - \frac{\partial\bar{A}}{\partial t}\right) = 0$$

$$\nabla^2\varphi + \frac{\partial}{\partial t}(\nabla\cdot\bar{A}) = 0$$

Choose it/s first $\varphi = 0$. then $\nabla\cdot\bar{A} = 0$.
This is true if $\varphi \neq$ constant, in particular if $\varphi \neq 0$.

$$\nabla\times\bar{H} - \partial\bar{D}/\partial t = 0$$

$$\Rightarrow \nabla\times\left(\frac{\nabla\times\bar{A}}{\mu_0}\right) + \epsilon_0\frac{\partial}{\partial t}\left(\nabla\varphi + \frac{\partial\bar{A}}{\partial t}\right) = 0$$

$$\nabla(\nabla\cdot\bar{A}) - \nabla^2\bar{A} + \epsilon_0\mu_0\frac{\partial^2\bar{A}}{\partial t^2} = 0$$

$$-\nabla^2\bar{A} + \epsilon_0\mu_0\frac{\partial^2\bar{A}}{\partial t^2} = 0$$

or $$\nabla^2\bar{A} - \epsilon_0\mu_0\frac{\partial^2\bar{A}}{\partial t^2} = 0$$

$$\nabla^2\bar{A} - \frac{1}{c_0^2}\frac{\partial^2\bar{A}}{\partial t^2} = 0$$

i.e., in free space Maxwell's eqs are correctly obtained from a single vector potential $\bar{A}$.

6.

$$\nabla\cdot\bar{D} = 0 \qquad \nabla\times\bar{H} - \frac{\partial\bar{D}}{\partial t} = \bar{J}$$

$$\nabla\cdot\bar{B} = 0 \qquad \nabla\times\bar{E} + \frac{\partial\bar{B}}{\partial t} = 0$$

$$\nabla\cdot\bar{B} = 0 \Rightarrow \bar{B} = \nabla\times\bar{A}$$

$$\nabla\times\bar{E} + \frac{\partial\bar{B}}{\partial t} = 0 \Rightarrow \nabla\times\bar{E} + \frac{\partial}{\partial t}(\nabla\times\bar{A}) = 0$$

$$\nabla\times\left(\bar{E} + \frac{\partial\bar{A}}{\partial t}\right) = 0$$

$$\bar{E} + \partial\bar{A}/\partial t = -\nabla\varphi$$

i.e., $$\bar{E} = -\nabla\varphi - \partial\bar{A}/\partial t$$

$$\nabla\times\bar{H} - \frac{\partial\bar{D}}{\partial t} = \bar{J} \Rightarrow \nabla\times\left(\frac{1}{\mu}\nabla\times\bar{A}\right) + \epsilon\frac{\partial}{\partial t}\left(\nabla\varphi + \frac{\partial\bar{A}}{\partial t}\right) = \sigma\left(\nabla\varphi + \frac{\partial\bar{A}}{\partial t}\right)$$

... for a linear medium.

$$\nabla\times(\nabla\times\bar{A}) + \epsilon\mu\frac{\partial}{\partial t}\left(\nabla\varphi+\frac{\partial\bar{A}}{\partial t}\right) = -\sigma\mu\left(\nabla\varphi+\frac{\partial\bar{A}}{\partial t}\right)$$

$$\nabla(\nabla\cdot\bar{A}) - \nabla^2\bar{A} + \epsilon\mu\frac{\partial}{\partial t}(\nabla\varphi) + \epsilon\mu\frac{\partial^2\bar{A}}{\partial t^2} = -\sigma\mu\nabla\varphi - \sigma\mu\frac{\partial\bar{A}}{\partial t}$$

$$-\nabla(\nabla\cdot\bar{A}) + \nabla^2\bar{A} - \epsilon\mu\frac{\partial}{\partial t}(\nabla\varphi) - \epsilon\mu\frac{\partial^2\bar{A}}{\partial t^2} = \sigma\mu\nabla\varphi + \sigma\mu\frac{\partial\bar{A}}{\partial t}$$

$$\nabla^2\bar{A} - \epsilon\mu\frac{\partial^2\bar{A}}{\partial t^2} - \sigma\mu\frac{\partial\bar{A}}{\partial t} = \nabla(\nabla\cdot\bar{A}) + \epsilon\mu\frac{\partial}{\partial t}(\nabla\varphi) + \sigma\mu\nabla\varphi$$

$$\nabla^2\bar{A} - \epsilon\mu\frac{\partial^2\bar{A}}{\partial t^2} - \sigma\mu\frac{\partial\bar{A}}{\partial t} = \nabla\left\{\nabla\cdot\bar{A} + \epsilon\mu\frac{\partial\varphi}{\partial t} + \sigma\mu\varphi\right\} \quad \ldots(1)$$

$$\nabla\cdot\bar{D} = 0 \Rightarrow \nabla\cdot\epsilon\left(-\nabla\varphi - \frac{\partial\bar{A}}{\partial t}\right) = 0$$

$$\nabla^2\varphi + \frac{\partial}{\partial t}(\nabla\cdot\bar{A}) = 0 \quad \ldots\ldots(2)$$

From (1) suppose $\nabla\cdot\bar{A} + \epsilon\mu\frac{\partial\varphi}{\partial t} + \sigma\mu\varphi = 0$

$$\Rightarrow \nabla\cdot\bar{A} = -\left(\epsilon\mu\frac{\partial\varphi}{\partial t} + \sigma\mu\varphi\right)$$

Using this result in (2)

$$\nabla^2\varphi - \frac{\partial}{\partial t}\left(\epsilon\mu\frac{\partial\varphi}{\partial t} + \sigma\mu\varphi\right) = 0$$

$$\nabla^2\varphi - \epsilon\mu\frac{\partial^2\varphi}{\partial t^2} - \sigma\mu\frac{\partial\varphi}{\partial t} = 0$$

So then in a linear medium $\bar{A}$ and φ satisfy the W.E.

$$\nabla^2\bar{A} - \epsilon\mu\frac{\partial^2\bar{A}}{\partial t^2} - \sigma\mu\frac{\partial\bar{A}}{\partial t} = 0$$

$$\nabla^2\bar{A} - \epsilon\mu\frac{\partial^2\varphi}{\partial t^2} - \sigma\mu\frac{\partial\varphi}{\partial t} = 0$$

With a condition (guage)

$$\nabla\cdot\bar{A} = -\left(\epsilon\mu\frac{\partial\varphi}{\partial t} + \sigma\mu\varphi\right).$$

7. For $\rho = 0$, $\bar{J} = 0$, $\mu = \mu_0$

$\nabla\cdot\bar{D} = 0$ $\quad \nabla\times\bar{H} - \frac{\partial\bar{D}}{\partial t} = 0$

$\nabla\cdot\bar{B} = 0$ $\quad \nabla\times\bar{E} + \frac{\partial\bar{B}}{\partial t} = 0$

$\nabla\cdot\bar{B} = 0 \Rightarrow \bar{B} = \text{curl}\,\bar{A} = \nabla\times\bar{A}$

$\nabla\times\bar{E} + \frac{\partial\bar{B}}{\partial t} = 0 \Rightarrow \nabla\times\bar{E} + \frac{\partial}{\partial t}(\nabla\times\bar{A}) = 0$

$$\nabla \times \left(\bar{E} + \frac{\partial \bar{A}}{\partial t}\right) = 0$$

$$\Rightarrow \bar{E} + \frac{\partial \bar{A}}{\partial t} = -\nabla\varphi \quad \text{or} \quad \bar{E} = -\nabla\varphi - \frac{\partial \bar{A}}{\partial t}$$

Since φ is scalar and $\bar{A}$ a vector we may choose

$$\varphi = -\nabla\cdot\bar{Z}, \quad \bar{A} = \frac{1}{c^2}\frac{\partial \bar{Z}}{\partial t} = \epsilon_0\mu_0\frac{\partial \bar{Z}}{\partial t}$$

$$\nabla\times\bar{H} - \partial\bar{D}/\partial t = 0$$

~~$\nabla\cdot\bar{D} = 0 \Rightarrow \nabla\cdot(\epsilon_0\bar{E} + \bar{P}) = 0$~~

$$\Rightarrow \nabla\times\left(\frac{1}{\mu}\nabla\times\bar{A}\right) - \frac{\partial}{\partial t}(\epsilon_0\bar{E} + \bar{P}) = 0$$

$$\nabla\times(\nabla\times\bar{A}) - \mu\frac{\partial}{\partial t}(\epsilon_0\bar{E}) = \mu\frac{\partial\bar{P}}{\partial t}$$

$$\nabla\times(\nabla\times\bar{A}) - \epsilon_0\mu_0\frac{\partial\bar{E}}{\partial t} = \mu_0\frac{\partial\bar{P}}{\partial t}, \quad \text{since } \mu = \mu_0.$$

$$\nabla\times(\nabla\times\bar{A}) + \epsilon_0\mu_0\frac{\partial}{\partial t}\left(\nabla\varphi + \frac{\partial\bar{A}}{\partial t}\right) = \mu_0\frac{\partial\bar{P}}{\partial t}$$

$$\nabla(\nabla\cdot\bar{A}) - \nabla^2\bar{A} + \epsilon_0\mu_0\frac{\partial}{\partial t}(\nabla\varphi) + \epsilon_0\mu_0\frac{\partial^2\bar{A}}{\partial t^2} = \mu_0\frac{\partial\bar{P}}{\partial t}$$

$$\Rightarrow \nabla\left(\nabla\cdot\bar{A} + \epsilon_0\mu_0\frac{\partial\varphi}{\partial t}\right) - \nabla^2\bar{A} + \epsilon_0\mu_0\frac{\partial^2\bar{A}}{\partial t^2} = \mu_0\frac{\partial\bar{P}}{\partial t}$$

$$\nabla\left\{\nabla\cdot\epsilon_0\mu_0\frac{\partial\bar{Z}}{\partial t} - \epsilon_0\mu_0\frac{\partial}{\partial t}(\nabla\cdot\bar{Z})\right\} - \nabla^2\left\{\epsilon_0\mu_0\frac{\partial\bar{Z}}{\partial t}\right\}$$

$$+ \epsilon_0\mu_0\frac{\partial^2}{\partial t^2}\left(\epsilon_0\mu_0\frac{\partial\bar{Z}}{\partial t}\right) = \mu_0\frac{\partial\bar{P}}{\partial t}$$

$$\nabla\left\{\epsilon_0\mu_0\frac{\partial}{\partial t}(\nabla\cdot\bar{Z} - \nabla\cdot\bar{Z})\right\} - \epsilon_0\mu_0\frac{\partial}{\partial t}(\nabla^2\bar{Z})$$

$$+ (\epsilon_0\mu_0)^2\frac{\partial^2\bar{Z}}{\partial t^2} = \mu_0\frac{\partial\bar{P}}{\partial t}$$

$$\Rightarrow \frac{\partial}{\partial t}\left\{-\nabla^2\bar{Z} + \epsilon_0\mu_0\frac{\partial^2\bar{Z}}{\partial t^2}\right\} = \frac{1}{\epsilon_0}\frac{\partial\bar{P}}{\partial t}$$

$$-\nabla^2\bar{Z} + \epsilon_0\mu_0\frac{\partial^2\bar{Z}}{\partial t^2} = \frac{1}{\epsilon_0}\bar{P}$$ (integrating over finite time interval)

$$\text{or} \quad \nabla^2\bar{Z} - \epsilon_0\mu_0\frac{\partial^2\bar{Z}}{\partial t^2} = -\frac{\bar{P}}{\epsilon_0}$$

$$\nabla\cdot\bar{D} = 0$$

$$\Rightarrow \nabla\cdot(\epsilon_0\bar{E} + \bar{P}) = 0$$

$\Rightarrow$ ~~$\nabla\cdot\bar{E} = -\nabla\cdot\frac{\bar{P}}{\epsilon_0}$~~

~~$\nabla\cdot\left(-\nabla\varphi + \frac{\partial\bar{A}}{\partial t}\right) = -\nabla\cdot\frac{\bar{P}}{\epsilon_0}$~~

~~$\nabla\cdot\left(\nabla(\nabla\cdot\bar{Z}) - \epsilon_0\mu_0\frac{\partial^2\bar{Z}}{\partial t^2}\right) = -\nabla\cdot\frac{\bar{P}}{\epsilon_0}$~~

$$\bar{E} = -\nabla\varphi - \frac{\partial\bar{A}}{\partial t}$$

$$= \nabla(\nabla\cdot\bar{Z}) + \epsilon_0\mu_0\frac{\partial^2\bar{Z}}{\partial t^2}$$

But $\nabla\times(\nabla\times\bar{Z}) = \nabla(\nabla\cdot\bar{Z}) - \nabla^2\bar{Z}$

$\Rightarrow \nabla(\nabla\cdot\bar{Z}) = \nabla\times(\nabla\times\bar{Z}) + \nabla^2\bar{Z}$

i.e., $\bar{E} = \nabla\times(\nabla\times\bar{Z}) + \underbrace{\nabla^2\bar{Z} - \epsilon_0\mu_0\frac{\partial^2\bar{Z}}{\partial t^2}}_{-\bar{P}/\epsilon_0}$

$\therefore \bar{E} = \nabla\times(\nabla\times\bar{Z}) - \bar{P}/\epsilon_0$

So we obtain

$$\bar{B} = \nabla\times\bar{A} = \epsilon_0\mu_0\,\nabla\times\frac{\partial\bar{Z}}{\partial t}$$

$$\bar{E} = \nabla\times(\nabla\times\bar{Z}) - \bar{P}/\epsilon_0$$

and $\nabla^2\bar{Z} - \epsilon_0\mu_0\frac{\partial^2\bar{Z}}{\partial t^2} = -\bar{P}/\epsilon_0$, from Maxwell's eqs.

$\therefore$ Maxwells are correctly obtained from a single potential $\bar{Z}$ which satisfies the W.E.

8.

$\nabla\cdot\bar{D} = 0$ $\quad$ $\nabla\times\bar{H} - \frac{\partial\bar{D}}{\partial t} = 0$, $\epsilon = \epsilon_0$

$\nabla\cdot\bar{B} = 0$ $\quad$ $\nabla\times\bar{E} + \frac{\partial\bar{B}}{\partial t} = 0$

$\nabla\cdot\bar{D} = 0 \Rightarrow \bar{D} = -\text{curl}\,\bar{A} = -\nabla\times\bar{A}^{\circ}$

or $\bar{E} = -\frac{1}{\epsilon_0}\nabla\times\bar{A}$, since $\epsilon = \epsilon_0$.

$\nabla\times\bar{H} - \frac{\partial\bar{D}}{\partial t} = 0 \Rightarrow \nabla\times\bar{H} + \frac{\partial}{\partial t}(\nabla\times\bar{A}^{\circ}) = 0$

$$\nabla\times\left(\bar{H} + \frac{\partial\bar{A}^{\circ}}{\partial t}\right) = 0$$

or $\bar{H} + \frac{\partial\bar{A}^{\circ}}{\partial t} = -\nabla\varphi^{\circ}$

i.e., $\bar{H} = -\nabla\varphi^{\circ} - \frac{\partial\bar{A}^{\circ}}{\partial t}$

Since φ is scalar and $\bar{A}$ is vector we choose $\varphi = -\nabla\cdot\bar{Z}'$, $\bar{A} = \epsilon_0\mu_0 \frac{\partial \bar{Z}'}{\partial t}$

$$\nabla\times\bar{E} + \frac{\partial \bar{B}}{\partial t} = 0$$

$$\nabla\times\left(-\frac{1}{\epsilon_0}\nabla\times\bar{A}\right) + \frac{\partial}{\partial t}(\mu_0\bar{H} + \mu_0\bar{M}) = 0$$

$$\nabla\times(\nabla\times\bar{A}) - \epsilon_0\mu_0\frac{\partial \bar{H}}{\partial t} = \epsilon_0\mu_0\frac{\partial \bar{M}}{\partial t}$$

$$\nabla\times(\nabla\times\bar{A}) + \epsilon_0\mu_0\frac{\partial}{\partial t}\left(\nabla\varphi^0 + \frac{\partial \bar{A}^0}{\partial t}\right) = \epsilon_0\mu_0\frac{\partial \bar{M}}{\partial t}$$

$$\nabla(\nabla\cdot\bar{A}) - \nabla^2\bar{A} + \epsilon_0\mu_0\frac{\partial}{\partial t}\left(\nabla\varphi^0 + \frac{\partial \bar{A}^0}{\partial t}\right) = \epsilon_0\mu_0\frac{\partial \bar{M}}{\partial t}$$

$$\nabla\left(\nabla\cdot\epsilon_0\mu_0\frac{\partial \bar{Z}'}{\partial t}\right) - \nabla^2\left(\epsilon_0\mu_0\frac{\partial \bar{Z}'}{\partial t}\right) - (\epsilon_0\mu_0)\frac{\partial}{\partial t}\nabla(\nabla\cdot\bar{Z}')$$

$$+ (\epsilon_0\mu_0)^2\frac{\partial^3 \bar{Z}'}{\partial t^3} = \epsilon_0\mu_0\frac{\partial \bar{M}}{\partial t}$$

$$\cancel{\epsilon_0\mu_0\nabla\left(\nabla\cdot\frac{\partial \bar{Z}'}{\partial t}\right)} - \epsilon_0\mu_0\nabla^2\left(\frac{\partial \bar{Z}'}{\partial t}\right) - \cancel{\epsilon_0\mu_0\nabla\left(\nabla\cdot\frac{\partial \bar{Z}'}{\partial t}\right)}$$

$$+ (\epsilon_0\mu_0)^2\frac{\partial^3 \bar{Z}'}{\partial t^3} = \epsilon_0\mu_0\frac{\partial \bar{M}}{\partial t}$$

$$\frac{\partial}{\partial t}\left(-\nabla^2\bar{Z}' + \epsilon_0\mu_0\frac{\partial^2 \bar{Z}'}{\partial t^2}\right) = \frac{\partial \bar{M}}{\partial t}$$

$$-\nabla^2\bar{Z}' + \epsilon_0\mu_0\frac{\partial^2 \bar{Z}'}{\partial t^2} = \bar{M}$$, integrating over finite period of time

$$\nabla^2\bar{Z}' - \epsilon_0\mu_0\frac{\partial^2 \bar{Z}'}{\partial t^2} = -\bar{M}$$

$$\bar{B} = \mu_0(\bar{H} + \bar{M})$$

$$= \mu_0\left(-\nabla\varphi^0 - \frac{\partial \bar{A}^0}{\partial t}\right) + \mu_0\bar{M}$$

$$= \mu_0\left(\nabla(\nabla\cdot\bar{Z}') - \epsilon_0\mu_0\frac{\partial^2 \bar{Z}'}{\partial t^2}\right) + \mu_0\bar{M}$$

$$= \mu_0\left(\nabla\times(\nabla\times\bar{Z}') + \nabla^2\bar{Z}' - \epsilon_0\mu_0\frac{\partial^2 \bar{Z}'}{\partial t^2}\right) + \mu_0\bar{M}$$

$$= \mu_0\left(\nabla\times(\nabla\times\bar{Z}') - \bar{M}\right) + \mu_0\bar{M}$$, from the W.E.

$$= \mu_0\nabla\times(\nabla\times\bar{Z}') - \mu_0\bar{M} + \mu_0\bar{M}$$

$$= \mu_0\nabla\times(\nabla\times\bar{Z}')$$

$$\bar{E} = -\frac{1}{\epsilon_0}\nabla\times\bar{A}^0 = -\frac{1}{\epsilon_0}\nabla\times\left(\epsilon_0\mu_0\frac{\partial \bar{Z}'}{\partial t}\right)$$

or $$\bar{E} = -\mu_0\nabla\times\frac{\partial \bar{Z}'}{\partial t}$$

Assignment II

17. Show that the spherical Bessel functions satisfy the following condition

$$\int_{-\infty}^{\infty} j_n(\rho)\, j_m(\rho)\, d\rho = \begin{cases} 0 & \text{when } m \neq n \\ \frac{\pi}{2}\frac{1}{2n+1} & \text{when } m = n. \end{cases}$$

m, n are positive ~~no~~ integers. For the application of such integrals to the expansion of functions in Bessel series see Watson - Bessel Functions.

18. a Determine, as a function of the angles θ and φ, the average power density radiated into vacuum by an oscillating electric dipole. (i.e.) $P = \oint \vec{\Pi}\cdot d\vec{S}$ and find $\bar{p}$) : $P \cdot \frac{R^2}{\mu_0}$) to be sinodoidal

~~19~~. b, Calculate the total power radiated by a dipole of length 3m at a frequency of 500 kHz. Consider the rms value of the current in the dipole of 2A.

c, What is, the radiation resistance of the dipole oscillator in prob b?

19. A circular loop of wire carrying the current $I = I_0 \cos\omega t$ constitutes an oscillating magnetic dipole. Determine the radiation fields $\vec{E}$ and $\vec{B}$ for this oscillator and total power radiated.

Result :

$$B_\theta = \frac{\mu_0 I_0}{4\pi} A \frac{\omega^2}{c^2 r} \sin\theta \cos\omega(t - r/c)$$

$$E_\varphi = -\frac{I_0 A}{4\pi\epsilon_0} \frac{\omega^2}{c^3 r} \sin\theta \cos\omega(t - r/c)$$

$$P = \frac{\mu_0}{6\pi} I_0^2 A^2 \frac{\omega^4}{c^3} \cos^2\omega(t - r/c)$$

A is the loop area.

20. Determine the relative efficiency of an electric dipole of length 2m compared with a magnetic dipole of the same diameter at frequency of 1 MHz.

21. Suppose a spherically symmetric charge distribution is oscillating purely in the radial direction so that it remains spherically symmetric at every instant. Prove that no radiation is emitted.

22. An electric dipole rotates with a constant angular velocity ω about an axis perpendicular to the dipole moment. Find the radiation fields and the poyinting vector.

[Hint : Treat the dipole as the superposition of two sinusoidally varying dipoles at right angles to each other.)

23. For metals in the infrared region, it happens that $Re\,\hat{\varepsilon}_r = -Im\,\hat{\varepsilon}_r$ at a frequency $\omega \approx 10^{14}$ rad/sec. Calculate the optical constants $Re\,\hat{n}$; $Im\,\hat{n}$ for this case, in terms of $Im\,\hat{\varepsilon}_r$.

24. For a dielectric which becomes absorbing at high frequencies or for semiconductors, it happens that $Re\,\hat{\varepsilon}_r = Im\,\hat{\varepsilon}_r$. Calculate $Re\,\hat{n}$ and $Im\,\hat{n}$ for this case, in terms of $Re\,\hat{\varepsilon}_r$. Find δ/λ, the ratio of the skin depth to the wavelength.

25. The density and refractive index of liquid benzene at 20°C are 879 kg/m³ and 1.50 resp. From the Clausius-Mossotti eq. compute the refractive index of benzene vapour at 20°C, where its pressure is 0.1 atm; also at the boiling point 80°C.

Name : Biru Tsegaye

solution to problems for Assignment III

23. Consider harmonic plane waves propagating within metals

$$\bar{B} = \bar{B}_0 e^{i\bar{g}\cdot\bar{r}} e^{-i\omega t}$$

$$\bar{D} = \bar{D}_0 e^{i\bar{g}\cdot\bar{r}} e^{-i\omega t}$$

, where $g^2 = \omega^2 \epsilon \mu + i\sigma\mu\epsilon$

from one of Maxwell's equations

$$\nabla \times \bar{B} = \mu \frac{\partial \bar{D}}{\partial t} + \mu\sigma\bar{E} \quad (\text{for a linear medium})$$

$$\Rightarrow i\bar{g} \times \bar{B} = -i\omega\mu\epsilon\bar{E} + \mu\sigma\bar{E}$$

$$\bar{g} \times \bar{B} = -\frac{\omega}{C_0^2}\left(\epsilon_r + i\frac{\sigma}{\omega\epsilon_0}\right)\bar{E}; \quad \text{for which } \mu \approx \mu_0$$

The term in the brackets may be viewed as a complex permitivity $\hat{\epsilon}_r$.

So now, $\hat{n}^2 = \hat{\epsilon}_r = \epsilon_r + i\frac{\sigma}{\omega\epsilon_0}$

(i.e., the index of refraction is also complex.

We then write

$$\hat{n} = Re\,\hat{n} + i\,Im\,\hat{n}$$

$$\Rightarrow \hat{n}^2 = (Re\,\hat{n} + i\,Im\,\hat{n})^2 = \epsilon_r + i\frac{\sigma}{\omega\epsilon_0}$$

$$(Re\,\hat{n})^2 - (Im\,\hat{n})^2 + i\,2\,Re\,\hat{n}\,Im\,\hat{n}$$

$$= \epsilon_r + i\frac{\sigma}{\omega\epsilon_0}$$

Equating real and imaginary parts

$$(Re\,\hat{n})^2 - (Im\,\hat{n})^2 = \epsilon_r = Re\,\hat{\epsilon}_r \quad \text{--------(1)}$$

$$2\,Re\,\hat{n}\cdot Im\,\hat{n} = \sigma/\omega\epsilon_0 = Im\,\hat{\epsilon}_r \quad \text{--------(2)}$$

From eq. 2, $Im\,\hat{n} = Im\,\hat{\epsilon}_r / 2Re\,\hat{n}$

$$Re^4\hat{n} - (Re\,\hat{\epsilon}_r)\,Re^2\hat{n} - \frac{Im^2\hat{\epsilon}_r}{4} = 0$$

$$\Rightarrow \quad Re\,\hat{n} = \left[\tfrac{1}{2}\left[Re\,\hat{\epsilon}_r + \left\{Re^2\hat{\epsilon}_r + Im^2\hat{\epsilon}_r\right\}^{1/2}\right]\right]^{1/2} \quad \text{---(3)}$$

Again from eq. (2), $Re\,\hat{n} = Im\,\hat{\epsilon}_r / 2\,Im\,\hat{n}$

Using this in (1)

$$\frac{Im^2\,\hat{\epsilon}_r}{4\,Im^2\hat{n}} - Im^2\,\hat{n} = Re\,\hat{\epsilon}_r$$

$$\Rightarrow Im^4\hat{n} + (Re\,\hat{\epsilon}_r)\,Im^2\hat{n} - \frac{Im^2\,\hat{\epsilon}_r}{4} = 0$$

$$\Rightarrow Im\,\hat{n} = \left[\tfrac{1}{2}\left[-Re\,\hat{\epsilon}_r + \left\{Re^2\hat{\epsilon}_r + Im^2\hat{\epsilon}_r\right\}^{1/2}\right]\right]^{1/2} \quad \text{---4}$$

□ To find $Re\,\hat{n}$ and $Im\,\hat{n}$ in terms of $Im\,\hat{\epsilon}_r$ when $Re\,\hat{\epsilon}_r = -Im\,\hat{\epsilon}_r$.

- Using eq. (3) and the above condition we find

$$Re\,\hat{n} = \left[\tfrac{1}{2}\left[-Im\,\hat{\epsilon}_r + \left\{Im^2\hat{\epsilon}_r + Im^2\hat{\epsilon}_r\right\}^{1/2}\right]\right]^{1/2}$$

$$= \left[\tfrac{1}{2}\left(-Im\,\hat{\epsilon}_r + \sqrt{2}\,Im\,\hat{\epsilon}_r\right)\right]^{1/2}$$

$$= \left[\tfrac{1}{2}\left(-Im\,\hat{\epsilon}_r + 1.41\,Im\,\hat{\epsilon}_r\right)\right]^{1/2}$$

$$= \left[\tfrac{1}{2}\left(0.41\,Im\,\hat{\epsilon}_r\right)\right]^{1/2}$$

$$= \left(0.205\,Im\,\hat{\epsilon}_r\right)^{1/2}$$

$$= 0.45\sqrt{Im\,\hat{\epsilon}_r}$$

$$\therefore \boxed{Re\,\hat{n} = 0.45\sqrt{Im\,\hat{\epsilon}_r} = 0.45\sqrt{\sigma/\omega\epsilon_0}}$$

- Using eq. (4) and the condition $Re\,\hat{\epsilon}_r = -Im\,\hat{\epsilon}_r$ we find

$$Im\,\hat{n} = \left[\tfrac{1}{2}\left[Im\,\hat{\epsilon}_r + \left\{Im^2\hat{\epsilon}_r + Im^2\hat{\epsilon}_r\right\}^{1/2}\right]\right]^{1/2}$$

$$= \left[\tfrac{1}{2}\left(Im\,\hat{\epsilon}_r + \sqrt{2}\,Im\,\hat{\epsilon}_r\right)\right]^{1/2}$$

$$= \left[\tfrac{1}{2}\left(2.41\,Im\,\hat{\epsilon}_r\right)\right]^{1/2}$$

$$\therefore \boxed{Im\,\hat{n} = 1.10\sqrt{Im\,\hat{\epsilon}_r} = 1.10\sqrt{\sigma/\omega\epsilon_0}}$$

24. To find $Re\,\hat{n}$ and $Im\,\hat{n}$ in terms of $Re\,\hat{\epsilon}_r$ when $Re\,\hat{\epsilon}_r = Im\,\hat{\epsilon}_r$ and the ratio δ/λ.

- Using eq. (3) and the condition $Re\,\hat{\epsilon}_r = Im\,\hat{\epsilon}_r$, we find

$$Re\,\hat{n} = [\tfrac{1}{2}[Re\,\hat{\epsilon}_r + \{Re^2\,\hat{\epsilon}_r + Re^2\,\hat{\epsilon}_r\}^{1/2}]]^{1/2}$$

$$= [\tfrac{1}{2}(Re\,\hat{\epsilon}_r + \sqrt{2}\,Re\,\hat{\epsilon}_r)]^{1/2}$$

$$= [\tfrac{1}{2}(2.41\,Re\,\hat{\epsilon}_r)]^{1/2}$$

$$\therefore \boxed{Re\,\hat{n} = 1.10\sqrt{Re\,\hat{\epsilon}_r} = 1.10\sqrt{\epsilon_r}}$$

- Using eq. (4) from prob. (23) and the condition above

$$Im\,\hat{n} = [\tfrac{1}{2}[-Re\,\hat{\epsilon}_r + \{Re^2\,\hat{\epsilon}_r + Re^2\,\hat{\epsilon}_r\}^{1/2}]]^{1/2}$$

$$= [\tfrac{1}{2}(-Re\,\hat{\epsilon}_r + \sqrt{2}\,Re\,\hat{\epsilon}_r)]^{1/2}$$

$$= [\tfrac{1}{2}(0.41\,Re\,\hat{\epsilon}_r)]^{1/2}$$

$$\therefore \boxed{Im\,\hat{n} = 0.45\sqrt{Re\,\hat{\epsilon}_r} = 0.45\sqrt{\epsilon_r}}$$

(i.e.) $\alpha = \frac{3 \times 8.85 \times 10^{-12} c^2/N\cdot m^2 (1.5^2 - 1)}{6.78 \times 10^{27} m^{-3} (1.5^2 + 2)}$

$= 1.51 \times 10^{-39} \frac{c^2 \cdot m}{N}$

To find the refractive index of benzene vapour at 20°C and its boiling point 80°C where the pressure is 0.1 atm.

In these cases, since the benzene is in the vapour state we may treat it as an ideal gas and apply the equation of state

$$PV = nRT$$

But for our case

$$PV = RT$$

Since we have one mole of Benzene.

$T = 20°C = 293°K$, $P = 0.1\ atm = 10^4\ N/m^2$

$\Rightarrow V = \frac{RT}{P} = \frac{8.31\ J \cdot mole^{-1} k^{-1} \times 293 K}{10^4\ N\ m^{-2}}$

or $V = V_g = 0.24\ m^3\ mole^{-1}$

Now, $N_g = \frac{N_A \rho_g}{M} = \frac{N_A\ M/V_g}{M} = \frac{N_A}{V_g}$, for the gas

From the Clausius-Mossotti eq.

$$n = \left(- \frac{(3\epsilon_0 + N\alpha)}{N\alpha - 3\epsilon_0} \right)^{1/2}$$

$$= \sqrt{\frac{3\epsilon_0 + N\alpha}{-N\alpha + 3\epsilon_0}}$$

For $N = N_g = N_A/V_g = \frac{6.02 \times 10^{23}\ mole^{-1}}{0.24\ m^3\ mole^{-1}}$

$= 2.51 \times 10^{24}\ m^{-3}$ (molecules/volume)

$$n = \sqrt{\frac{3 \times 8.85 \times 10^{-12} c^2/N\cdot m^2 + 2.51 \times 10^{24} m^{-3} \times 1.51 \times 10^{-39} \frac{c^2 \cdot m}{N}}{-2.51 \times 10^{24} m^{-3} \times 1.51 \times 10^{-39} \frac{c^2 \cdot m}{N} + 3 \times 8.85 \times 10^{-12} c^2/N\cdot m^2}}$$

or $\boxed{n = 1.00029}$ at 20°C

Assignment IV

1. By making two consequetive Lorentz transformations first to Σ'-system moving $\bar{c}$ velocity u relative to the Σ-system, and then to the Σ''-system moving $\bar{c}$ velocity u' relative to the Σ'-system, prove the relativistic addition theorem for velocities:

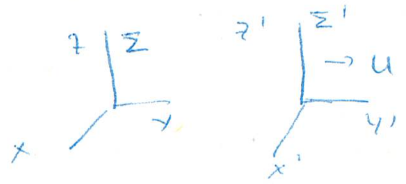
$$u'' = \frac{u+u'}{1+\frac{uu'}{c^2}}$$

2. Given uniform electric & M: field $\bar{E}, \bar{B}$, find a Lorentz transformation which will make $\bar{E}$ & $\bar{B}$ parallel to each other. (Hint: choose the velocity u of the Σ'-system in a direction $\perp$ to both $\bar{E}$ & $\bar{B}$) and determine the magnitude of $\bar{u}/(1+\beta^2)$ in terms of E^2, B^2 and $\bar{E}\times\bar{B}$.

$$\ldots \quad \frac{\bar{u}}{1+\beta^2} = \frac{\bar{E}\times\bar{B}}{E^2/c^2+B^2}$$

3. Show that the scalar product $\bar{E}\cdot\bar{B}$ is unchanged by a Lorentz transformation. Show the same for $E^2-c^2B^2$.

1.



For an object c̄ x, y, z & t being its spatial and time coordinates wrt to Σ and x', y', z' & t' being its spatial and time coordinates wrt Σ' we have the transformation of coordinates

$$x' = \frac{x - ut}{\sqrt{1-u^2/c^2}}$$

$$y' = y$$

$$z' = z$$

and time

$$t' = \frac{t - \frac{ux}{c^2}}{\sqrt{1-u^2/c^2}}$$

$$\dot{x}' = \frac{dx'}{dt'}$$

$$dx' = \frac{dx - u\,dt}{\sqrt{1-u^2/c^2}}$$

$$dt' = \frac{dt - \frac{u\,dx}{c^2}}{\sqrt{1-u^2/c^2}}$$

$$\Rightarrow \dot{x}' = \frac{dx - u\,dt}{dt - \frac{u\,dx}{c^2}}$$

$$= \frac{\frac{dx}{dt} - u}{1 - \frac{u}{c^2}\frac{dx}{dt}}$$

$$= \frac{\dot{x} - u}{1 - \frac{u\dot{x}}{c^2}}$$

i.e.) $\boxed{\dot{x}' = \dfrac{\dot{x} - u}{1 - \frac{u\dot{x}}{c^2}}}$ *

Consider then our three frames of reference Σ, Σ' & Σ'' with Σ' moving at a velocity u relative to Σ & Σ'' moving at u' relative Σ'. Now we wish to find the velocity of Σ'' wrt Σ.

From *

$$u' = \frac{\dot{x} - u}{1 - \frac{u\dot{x}}{c^2}}$$

Where $\dot{x}$ is the velocity of Σ'' wrt Σ. Let $\dot{x} = u''$

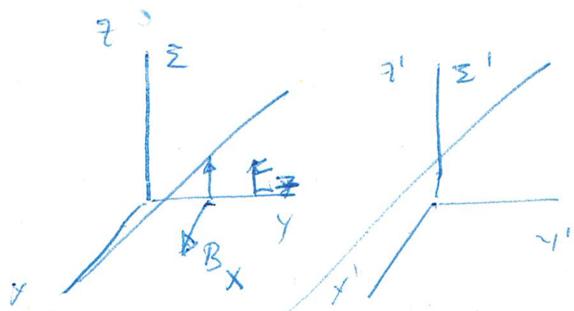
Then $$u' = \frac{u'' - u}{1 - \frac{uu''}{c^2}}$$

$$u'\left(1 - \frac{uu''}{c^2}\right) = u'' - u$$

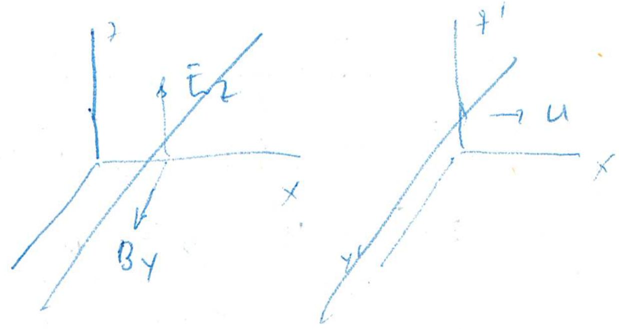
$$u' + u = \left(\frac{uu'}{c^2} + 1\right)u''$$

i.e.) $$u'' = \frac{u' + u}{1 + \frac{uu'}{c^2}}$$

2.



or



The general transformations for the frame Σ' moving along the x-axis is

$E_x' = E_x$

$E_y' = \frac{1}{\sqrt{1-\beta^2}}(E_y - c\beta B_z)$

$E_z' = \frac{1}{\sqrt{1-\beta^2}}(E_z + c\beta B_y)$

$B_x' = B_x$

$B_y' = \frac{1}{\sqrt{1-\beta^2}}(B_y + \frac{u}{c^2}E_z)$

$B_z' = \frac{1}{\sqrt{1-\beta^2}}(B_z - \frac{u}{c^2}E_y)$

$\bar{E}_\parallel' = \bar{E}_\parallel$

$\bar{B}_\parallel' = \bar{B}_\parallel$

$\bar{E}_\perp' = \frac{1}{\sqrt{1-\beta^2}} \cdot (\bar{E}_\perp + \bar{u}\times\bar{B})$

$\bar{B}_\perp' = \frac{1}{\sqrt{1-\beta^2}} \cdot (\bar{B}_\perp - \frac{1}{c^2}\bar{u}\times\bar{B})$

Suppose then $\bar{E}$ and $\bar{B}$ are perpendicular to the direction of motion of the frame Σ' i.e., $\perp \bar{u}$.

$\Rightarrow$ $E_x = 0$, $B_x = 0$ and $\bar{E}_\parallel' = \bar{E}_\parallel = 0$, $\bar{B}_\parallel' = \bar{B}_\parallel = 0$

Also, $\bar{E}_\perp = \bar{E}$, $\bar{B}_\perp = \bar{B}$, $\bar{E}_\perp' = \bar{E}'$, $\bar{B}_\perp' = \bar{B}'$

$\bar{E}_\perp' = \bar{E}' = \frac{1}{\sqrt{1-\beta^2}}(\bar{E} + \bar{u}\times\bar{B})$

$B_\perp' = \bar{B}' = \frac{1}{\sqrt{1-\beta^2}}(\bar{B} - \frac{1}{c^2}\bar{u}\times\bar{E})$

In the Σ' if $\bar{E}' \parallel \bar{B}'$, then

$\bar{E}' \times \bar{B}' = 0$.

$$\Rightarrow \frac{1}{\sqrt{1-\beta^2}}(\bar{E} + \bar{u}\times\bar{B}) \times \frac{1}{\sqrt{1-\beta^2}}(\bar{B} - \frac{1}{c^2}\bar{u}\times\bar{E}) = 0$$

$$\Rightarrow \bar{E}\times\bar{B} - \frac{1}{c^2}\bar{E}\times(\bar{u}\times\bar{E}) + (\bar{u}\times\bar{B})\times\bar{B} - \frac{1}{c^2}(\bar{u}\times\bar{B})\times(\bar{u}\times\bar{E}) = 0$$

$$\bar{E}\times\bar{B} - \frac{\bar{u}}{c^2}E^2 - \bar{u}B^2 - \frac{\bar{u}}{c^2}(\bar{u}\times\bar{B})\cdot\bar{E} = 0$$

$$\bar{E}\times\bar{B} - \bar{u}(\frac{E^2}{c^2} + B^2) - \frac{\bar{u}}{c^2}\bar{u}\cdot\bar{B}\times\bar{E} = 0$$

$$\bar{E}\times\bar{B} - \bar{u}(\frac{E^2}{c^2} + B^2) - \frac{u^2}{c^2}\bar{B}\times\bar{E} = 0$$

$$\bar{E}\times\bar{B} + \frac{u^2}{c^2}\bar{E}\times\bar{B} - \bar{u}(\frac{E^2}{c^2} + B^2) = 0$$

$$(\bar{E}\times\bar{B})(1 + \frac{u^2}{c^2}) = \bar{u}(\frac{E^2}{c^2} + B^2)$$

$$\frac{\bar{u}}{1 + u^2/c^2} = \frac{\bar{E}\times\bar{B}}{E^2/c^2 + B^2}$$

or $$\frac{\bar{u}}{1+\beta^2} = \frac{\bar{E}\times\bar{B}}{E^2/c^2 + B^2}$$

which is the required transformation for which $\bar{E}'$ is parallel to $\bar{B}'$.

3. In the Σ'-system $\bar{E} \to \bar{E}'$, $\bar{B} \to \bar{B}'$

$$\bar{E}'\cdot\bar{B}' = E_x'B_x' + E_y'B_y' + E_z'B_z'$$

$$= E_xB_x + \frac{1}{1-\beta^2}\{(E_y - c\beta B_z)(B_y + \frac{u}{c^2}E_z) + (E_z + c\beta B_y)(B_z - \frac{u}{c^2}E_y)\}$$

$$= E_xB_x + \frac{1}{1-\beta^2}\{E_yB_y + \frac{u}{c^2}E_yE_z - c\beta B_yB_z - \beta^2E_zB_z + E_zB_z - \frac{u}{c^2}E_yE_z + c\beta B_yB_z - \beta^2E_yB_y\}$$

$$= E_xB_x + \frac{1}{1-\beta^2}\{E_yB_y - \beta^2E_zB_z + E_zB_z - \beta^2E_yB_y\}$$

$$= E_xB_x + \frac{1}{1-\beta^2}\{(1-\beta^2)E_yB_y + (1-\beta^2)E_zB_z\}$$

$$= E_xB_x + E_yB_y + E_zB_z$$

$$= \bar{E}\cdot\bar{B}$$

or $\bar{E}'\cdot\bar{B}' = \bar{E}\cdot\bar{B}$

Thus the scalar product $\bar{E}\cdot\bar{B}$ is invariant under a Lorentz transformation.

To show $E^2 - c^2 B^2$ is invariant. Consider the dot product $(\vec{E}' + c\vec{B}') \cdot (\vec{E}' - c\vec{B}')$

$$= \vec{E}' \cdot \vec{E}' - c\vec{E}' \cdot \vec{B}' + c\vec{B}' \cdot \vec{E}' - c^2 \vec{B}' \cdot \vec{B}'$$

$$= E'^2 - c^2 B'^2$$

$$= E_x'^2 + E_y'^2 + E_z'^2 - c^2 (B_x'^2 + B_y'^2 + B_z'^2)$$

Using the expression of E_x', E_y', E_z', B_x', B_y' & B_z' we obtain

$E'^2 - c^2 B'^2$

$$= E_x^2 + \frac{1}{1-\beta^2}\{(E_y - c\beta B_z)^2 + (E_z + c\beta B_y)^2\}$$

$$- c^2 \{B_x^2 + \frac{1}{1-\beta^2}[(B_y + \frac{u}{c^2}E_z)^2 + (B_z - \frac{u}{c^2}E_y)^2]\}$$

$$= E_x^2 + \frac{1}{1-\beta^2}\{E_y^2 + c^2\beta^2 B_z^2 - 2c\beta E_y B_z + E_z^2 + c^2\beta^2 B_y^2 + 2c\beta E_z B_y\}$$

$$- c^2 B_x^2 - \frac{c^2}{1-\beta^2}\{B_y^2 + \frac{u^2}{c^4}E_z^2 + \frac{2u}{c^2}E_z B_y + B_z^2 + \frac{u^2}{c^4}E_y^2 - \frac{2u}{c^2}E_y B_z\}$$

$$= E_x^2 - c^2 B_x^2 + \frac{1}{1-\beta^2}\{E_y^2 + c^2\beta^2 B_z^2 - 2c\beta E_y B_z + E_z^2 + c^2\beta^2 B_y^2 + 2c\beta E_z B_y - c^2(B_y^2 + B_z^2) - \beta^2(E_y^2 + E_z^2) - 2c\beta E_z B_y + 2c\beta E_y B_z\}$$

$$= E_x^2 - c^2 B_x^2 + \frac{1}{1-\beta^2}\{E_y^2 + E_z^2 - \beta^2(E_y^2 + E_z^2) + c^2\beta^2(B_y^2 + B_z^2) - c^2(B_y^2 + B_z^2)\}$$

$$= E_x^2 - c^2 B_x^2 + \frac{1}{1-\beta^2}\{(1-\beta^2)(E_y^2 + E_z^2) - c^2(1-\beta^2)(B_y^2 + B_z^2)\}$$

$$= E_x^2 - c^2 B_x^2 + \frac{1}{1-\beta^2}(1-\beta^2)\{E_y^2 + E_z^2 - c^2(B_y^2 + B_z^2)\}$$

$$= E_x^2 - c^2 B_x^2 + E_y^2 + E_z^2 - c^2(B_y^2 + B_z^2)$$

$$= E_x^2 + E_y^2 + E_z^2 - c^2(B_x^2 + B_y^2 + B_z^2)$$

$$= E^2 - c^2 B^2$$

or $E'^2 - c^2 B'^2 = E^2 - c^2 B^2$

Therefore $E^2 - c^2 B$ is invariant under a Lorentz transformation.