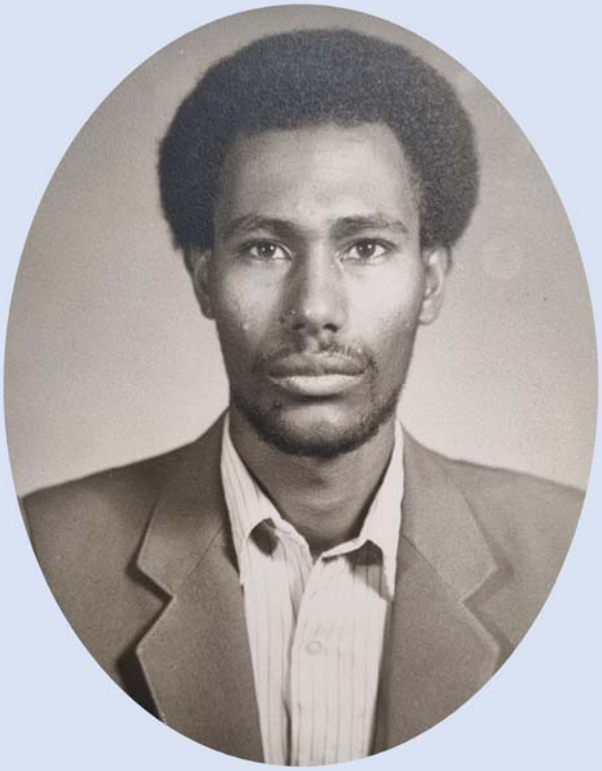

BIRU TSEGAYE'S LECTURE NOTES

On Newtonian Dynamics, Electromagnetism and Thermodynamics

This document is the first part of the release of Biru Tsegaye's handwritten lecture notes, providing a comprehensive overview of fundamental topics in Physics. Biru Tsegaye, who taught various subjects of physics at Kotebe Teacher's Education College and Addis Ababa University before his passing in 1989 EC, covers key concepts such as vectors, kinematics, Newton's Laws of Motion, work and energy, momentum, and rotational dynamics. Additional topics include oscillatory motion, planetary motion, electromagnetism, electric circuits, and Maxwell's equations. The document concludes with sections on thermodynamics, heat, and the kinetic theory of gases, supplemented several assignments and exam questions for practice.

Anteneh Biru Tsegaye

25.09.2024

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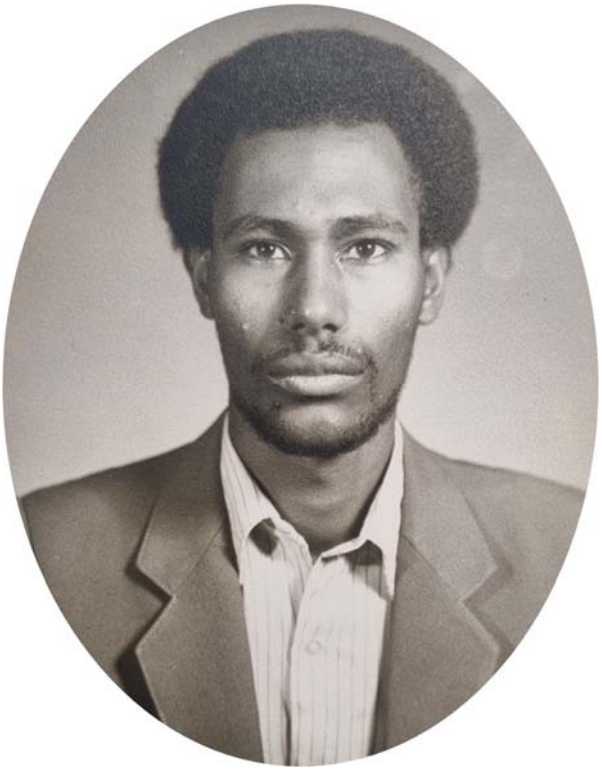
Foreword

This foreword accompanies the release of the first part of **Biru Tsegaye**'s handwritten lecture notes, a project I feel privileged to share as his son. Though his life was cut short, I believe, my father's work made a lasting impression on his students and colleagues. These notes reflect not only his knowledge of physics but also his dedication to education.



Biru Tsegaye was born in 1954 (Ethiopian calendar) in the village of Boren, Wollo Province, to W/ro Tiruwork Asfaw Mashile and Balambaras Tsegaye W/Michael. His parents were deeply committed to ensuring their children received a modern education, something uncommon in rural Ethiopia at the time. Despite the family's frequent moves, their dedication to ensuring Biru and Telele received a modern education remained steadfast.

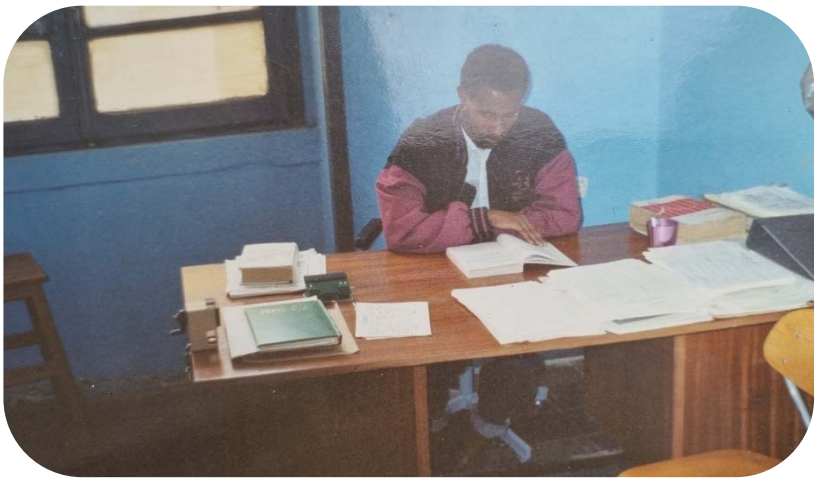
From a young age, Biru and his older brother, Telele, began their education under the guidance of a local priest. While Telele thrived, Biru resisted learning Fidel, the Ethiopian alphabet, often hiding his Fidel book to avoid studying. Despite this, their parents' unwavering commitment to education kept both brothers on their academic path. After the family relocated to Dera in Merhabete Awuraja, Showa Province, they continued their studies and eventually enrolled at Sheh Redi School, a private institution with a modern curriculum.

Two years later, the family moved again, this time to Bita Bilew in Shilafaf Woreda, where there were no nearby schools. After two years, the family arranged for Telele to move to Degolo to live with their uncle and continue his education. A year later, he rejoined the family, now relocated to Woremo, Wajitu Woreda. Both brothers progressed rapidly, skipping grades. In 1965 EC, the family moved to Meragna, where Biru completed the seventh grade and Telele the eighth.

After relocating once more to Bita Bilew, the brothers had to travel to other towns to pursue their education. Telele moved to Alem Ketema, where he completed the ninth grade, while Biru went to Degolo to finish the eighth grade. Following their father's retirement in 1969 EC, the family settled permanently in Boren. Telele and Biru continued their education in Woreilu, with Telele finishing high school in 1971 EC and being admitted to Hawassa College of Agriculture. A year later, Biru completed high school and was admitted to Addis Ababa University to study physics.In 1984 (1977EC), Biru graduated with a bachelor's degree in physics and began teaching at Kotebe Teacher's Education College. In 1988 (1981 EC), Biru completed his master's degree at Addis Ababa University with a thesis on "Path Integrals in Quantum and Statistical Mechanics." He later became head of the physics department at Kotebe and eventually a lecturer at Addis Ababa University. His teaching style, known for its clarity and depth, had a lasting impact on those he taught. His physics courses—covering subjects such as electromagnetism, quantum mechanics, thermodynamics, and Newtonian mechanics—earned him a strong reputation as a dedicated teacher. His lecture notes became widely used among students and I learned that they continued to be a valuable resource even after his time there.

Biru aspired to pursue a PhD in condensed matter physics and was accepted at the University of Cincinnati. However, before he could begin, he fell ill and passed away at the age of 35 (1989EC.)

Though I pursued a career in civil engineering, specializing in geotechnical engineering, I share my father's passion for physics. His handwritten lecture notes have remained with me for years, and I now feel it is the right time to share them with others. This first part of his notes reflects his commitment to teaching and offers insights that I believe are still valuable to students and educators today.

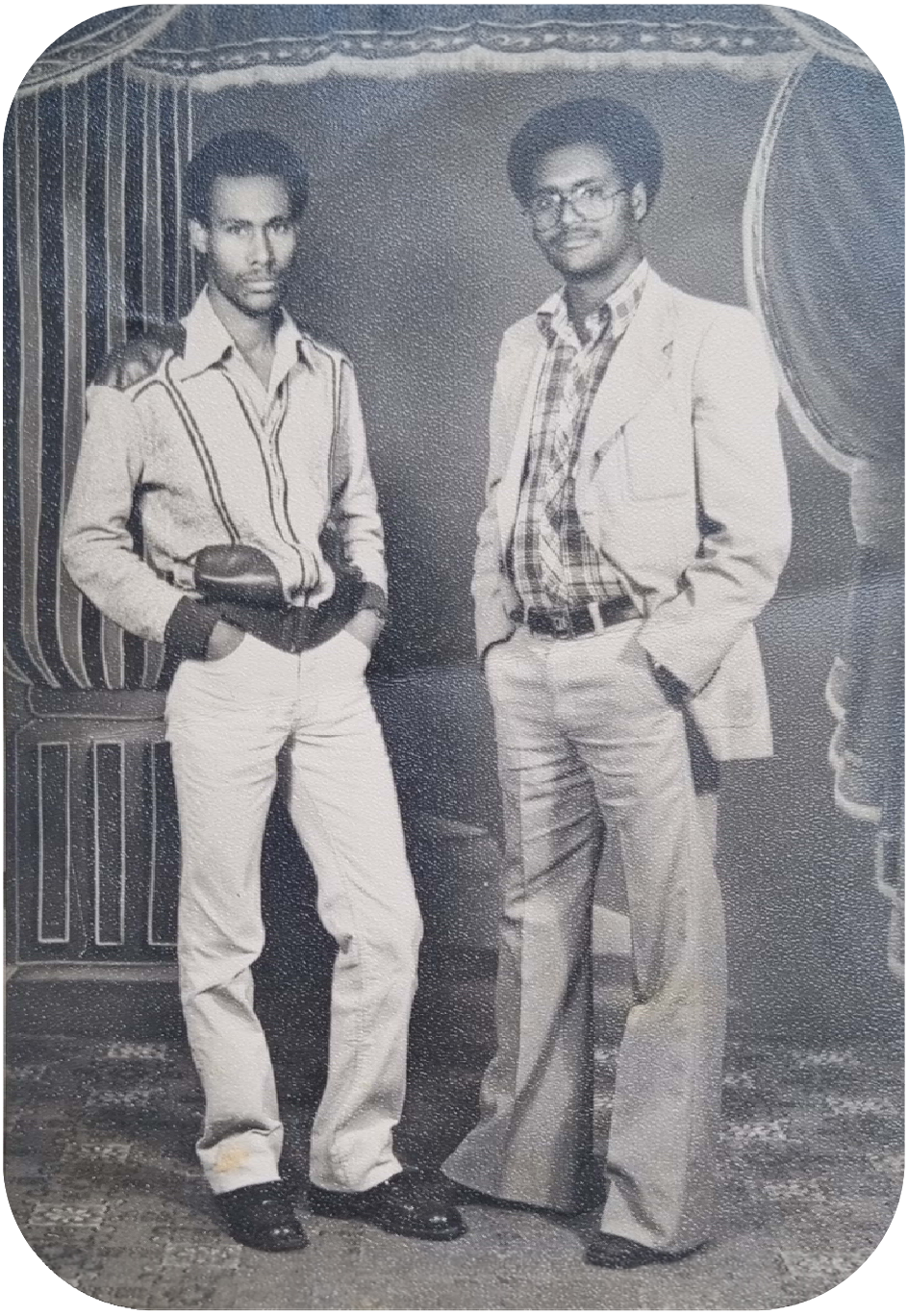


It is my hope that by releasing these notes, I can preserve my father's legacy and continue his contribution to education. Though his life was brief, his work lives on, and I am honoured to play a part in ensuring his lecture notes continue to inspire others.

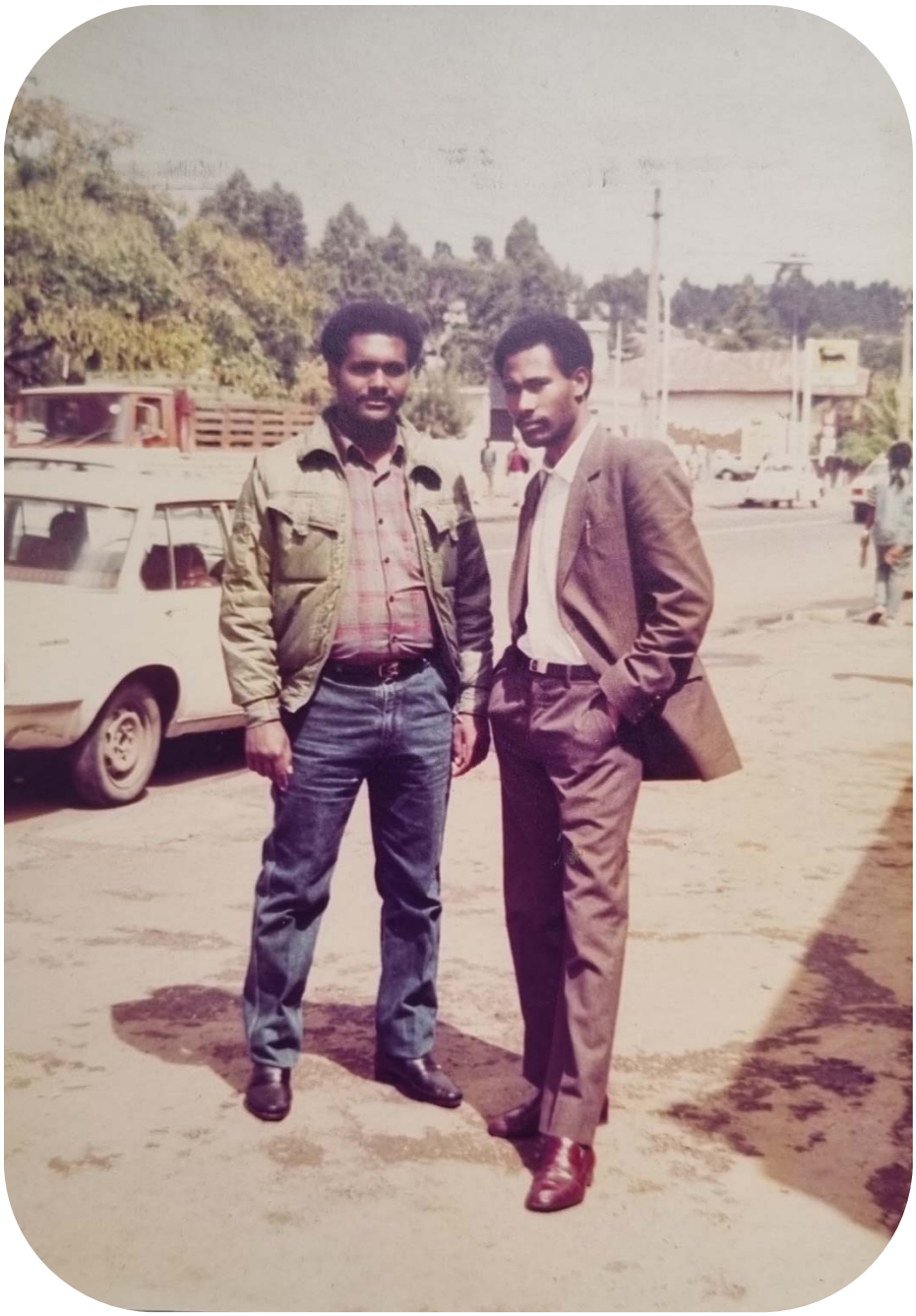
I am grateful to my uncle, Telele Tsegaye, for his support and for sharing valuable information about Biru's childhood and educational journey.

Anteneh Biru Tsegaye

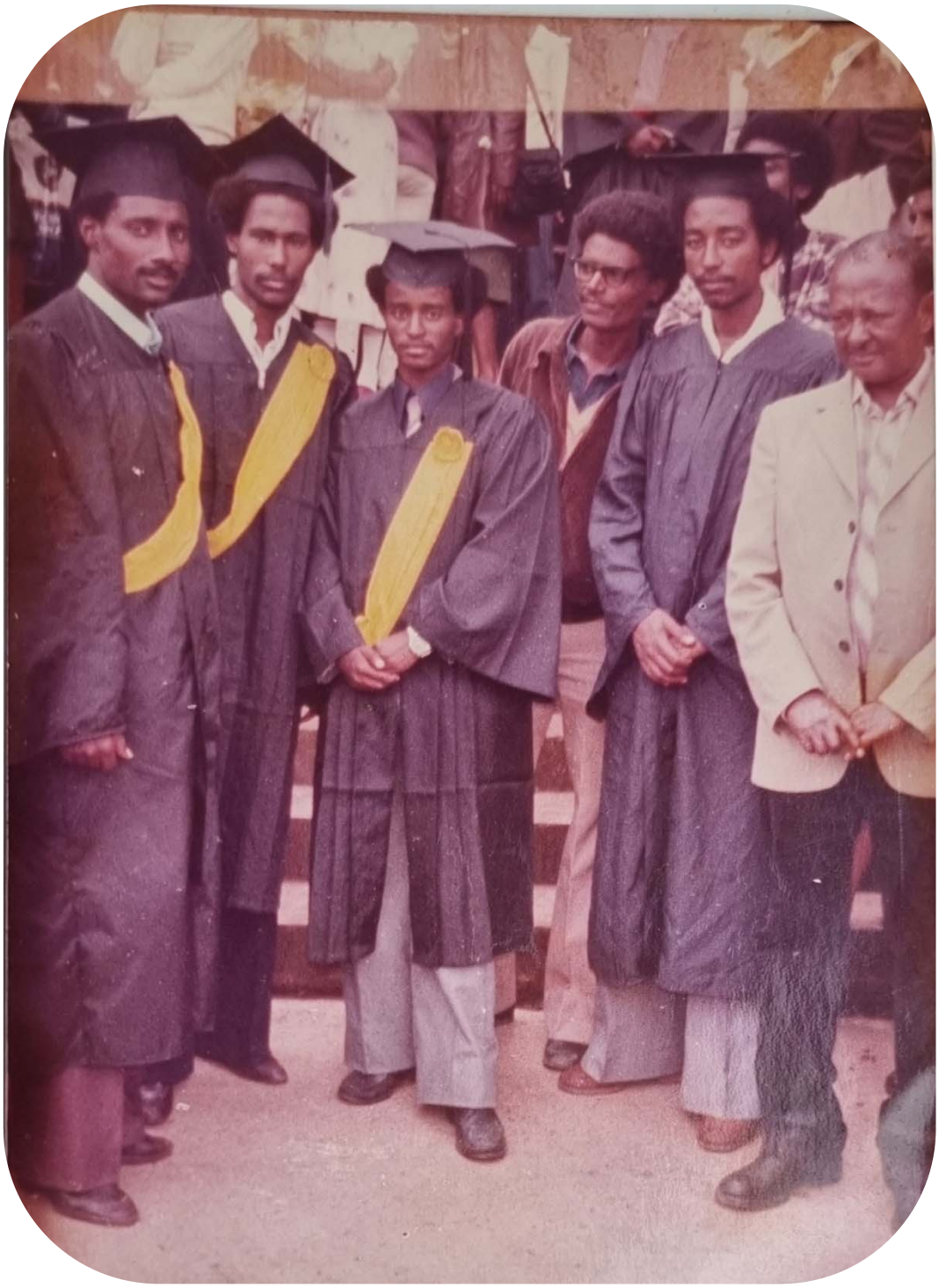
Some selected photos



Studio photo: Biru Tsegaye with his older brother, Telele Tsegaye

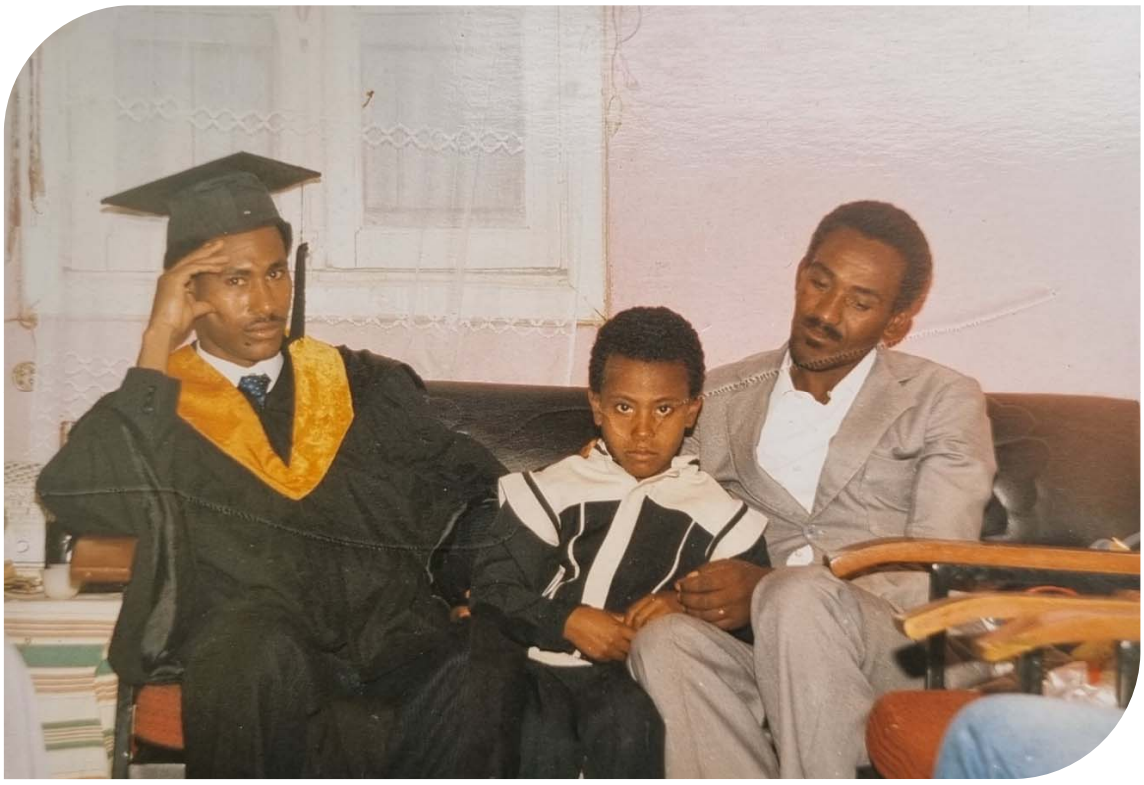


Biru Tsegaye, with his brother, Telele Tsegaye in Gondar



Biru Tsegaye, second from the left, celebrating his graduation with bachelor's degree in physics 1976 EC

Biru Tsegaye with his family, celebrating his graduation with master's degree in physics



Biru Tsegaye with his son, Anteneh Biru, and his uncle Dagne Reta

Biru Tsegaye, fifth from the right, farewell after research visit at Homerton

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Phys. 201 Notes (1982/1988-89)

Chapter 1 Vectors

1.1 Vectors and Scalars

Physical quantities can be of two kinds either scalar or vector quantities. A scalar quantity is characterized only by magnitude which may consist of a no. and a unit. But a vector quantity has an angular orientation (i.e., direction) in addition to magnitude.

Graphically, a vector quantity may be represented by an arrow with the length of the arrow showing the magnitude of the vector and the tip of the arrow showing the direction. Vectors are designated by letters of the alphabet such as $\vec{a}$, $\vec{B}$, $\vec{C}$, $\vec{d}$, etc. Examples of vectors are displacement, velocity, acc., mom. and force. They all have direction and magnitude. The magnitude of vector $\vec{a}$ is represented by $|\vec{a}|$ or simply a.

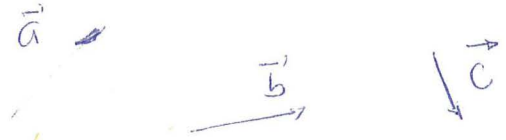
1.2 Addition of Vectors (Graphical Method)

A group of vectors may be represented by a single vector which will have the same effect as all these vectors. This single vector is called the sum (or resultant) of the group of vectors. If there are two or more vectors then the resultant is found as follows.

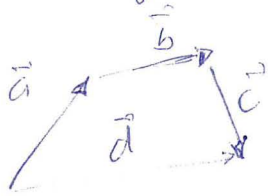
First draw an arrow representing one of the vectors in the group. Then put the arrow of the second vector such that its tail is in contact with the tip of the arrow of the first vector. Now continue the process of joining tail to tip of the arrows of all the vectors. The arrow of the resultant vector is given by the line segment that is directed from the tail of the first arrow to the tip of the last arrow.

On the other hand, if there are only two vectors in the group the so called Parallelogram method may be applied, though the polygon method is applicable. In this method the two arrows form the two adjoining sides of the parallelogram. The other two sides are constructed just by drawing two lines that pass through the tips of these arrows and intersect at a point (one vertex of the parallelogram).

Examples: 1. Find the resultant of the three vectors $\vec{a}, \vec{b}, \vec{c}$ given below.

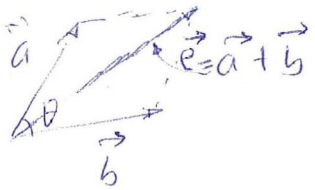


The sum is given below.



$$\vec{d} = \vec{a} + \vec{b} + \vec{c}$$

2. Find the sum of the vectors $\vec{a}$ and $\vec{b}$ in example 1 by applying the parallelogram method.



One can use the polygon method to show that the diagonal is in deed the resultant.
Show that

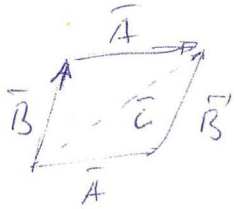
$$|\vec{e}| = e = \sqrt{a^2 + b^2 + 2ab\cos\theta}$$

Where θ is the angle between vectors $\vec{a}$ and $\vec{b}$.

Note: This method is useful if only proper scale is applied.

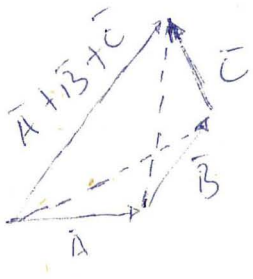
1.2.1 Algebraic properties of Vectors

- Addition of vectors is Commutative.



$$\Rightarrow \quad \vec{A} + \vec{B} = \bar{B} + \bar{A} = \bar{C}$$

- Addition of vectors is Associative



$$\Rightarrow \quad (\vec{A} + \vec{B}) + \bar{C} = \bar{A} + (\bar{B} + \bar{C})$$

- Additive Identity

There is a vector called the zero vector s.t. when it is added to any vector it gives the same vector. Most commonly the zero vector is designated as $\vec{0}$. If $\vec{V}$ is any vector, then

$$\vec{V} + \vec{0} = \vec{0} + \vec{V} = \vec{V}$$

N.B. The direction of the zero vector is not specified and its magnitude is zero.

- Additive Inverse

For any vector $\vec{V}$ there is its own unique inverse vector $-\vec{V}$ s.t.

$$\vec{V} + (-\vec{V}) = (-\vec{V}) + \vec{V} = \vec{0}$$

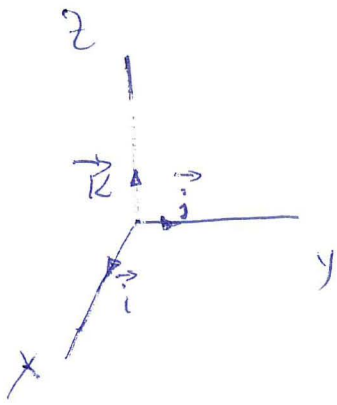
Note that subtraction of vectors is defined in terms of addition i.e.) subtracting one vector from the other is the same as adding the inverse of the second vector, or

$$\vec{a} - \vec{b} = \vec{a} + (-\vec{b})$$

1.3 Resolution and Addition of Vectors

Not only can we add a group of vectors but also we can represent a single vector by several vectors s.t. their sum gives the original vectors. This process is referred to as the resolution of the vector into its components. This is done wrt a particular type of coordinate system (or indicated directions). Here we consider a special type of coordinate system which is referred to as the cartesian or rectangular coordinate system; in particular, the 3 dimensional (3-D) rectangular coordinate system. The 3 axes of the coordinate system point along three mutually perpendicular directions. These directions are (indicated by what are referred to as unit vectors. (A unit vector (to be defined later) has got a magnitude of (unity).

The 3-D cartesian coordinate system is shown in the figure below.



$\vec{i}$ points along the x-axis

$\vec{j}$ " " " y-axis

$\vec{k}$ " " " z-axis

Magnitude:

$$i = |\vec{i}|, \; j = |\vec{j}|, \; k = |\vec{k}| = 1$$

So now in this coordinate system an arbitrary vector $\vec{a}$ may be represented as

$$\vec{a} = a_x \vec{i} + a_y \vec{j} + a_z \vec{k}$$

where a_x is the x-comp. of vector $\vec{a}$, a_y is the y-comp of $\vec{a}$ and a_z is the z-comp. of $\vec{a}$. Then if we are given two vectors $\vec{a}$ and $\vec{b}$ we write

$$\vec{a} + \vec{b} = (a_x + b_x)\vec{i} + (a_y + b_y)\vec{j} + (a_z + b_z)\vec{k}$$

and

$$\vec{a} - \vec{b} = (a_x - b_x)\vec{i} + (a_y - b_y)\vec{j} + (a_z - b_z)\vec{k}$$

The unit vector along a given vector $\vec{a}$ is defined as

$$\vec{e}_a = \vec{a} / |\vec{a}|$$

In 2-D, the 3rd comp. of a vector is zero.

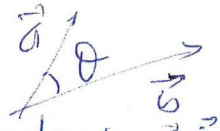
1.4 Multiplication of Vectors

- Scalar Multiplication (or Dot product) of Vectors

The scalar (or dot) product of two vectors $\vec{a}$ and $\vec{b}$ written as $\vec{a} \cdot \vec{b}$, is defined as

$$\vec{a} \cdot \vec{b} = ab \cos\theta$$

Where a is the magnitude of $\vec{a}$, b is the magnitude of vector $\vec{b}$, and θ is the angle between the two vectors.



It can be shown that $\vec{i}\cdot\vec{i} = \vec{j}\cdot\vec{j} = \vec{k}\cdot\vec{k} = 1$; $\vec{i}\cdot\vec{j} = \vec{i}\cdot\vec{k} = \vec{j}\cdot\vec{k} = 0$.

Since a and b are scalars and $\cos\theta$ is a pure no., the scalar product of two vectors is a scalar. One can easily show that the (dot) scalar product of vectors is commutative as well as distributive over addition. Using the comp. notation, and applying the latter property we find that

$$\vec{a}\cdot\vec{b} = a_x b_x + a_y b_y + a_z b_z \text{ ; check.}$$

Also, show that

$$a = |\vec{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2}$$

- Vector (or cross) product of Vectors

The vector product of two vectors $\vec{a}$ and $\vec{b}$ is written as $\vec{a}\times\vec{b}$ and is another vector $\vec{c}$, where $\vec{c} = \vec{a}\times\vec{b}$. The magnitude of $\vec{c}$ is defined as

$$c = ab\sin\theta$$

where θ is the (smaller) angle between $\vec{a}$ and $\vec{b}$.

The direction of the cross product vector $\vec{c}$ is given by the so called right hand rule; (holding) curl your fingers of the right hand from the first vector $\vec{a}$ to the second vector $\vec{b}$. Now the thumb (which is kept) $\perp$ to the other four fingers and also to the plane determined by the two vectors gives the direction of $\vec{c}$. Also, the direction of progress of a right hand screw which is rotated, once again, from the first vector $\vec{a}$ to the second vector $\vec{b}$ in the cross product is the same as the direction of the vector $\vec{c}$. So we can see that the cross product vector is always $\perp$ to the plane of the two vectors and hence to each of the vectors in the cross product.

cross product is not commutative but it is distributive over addition. Then using the comp. representation and the latter algebraic property we will find that

$$\vec{a}\times\vec{b} = \vec{i}(a_y b_z - a_z b_y) + \vec{j}(a_z b_x - a_x b_z) + \vec{k}(a_x b_y - a_y b_x)$$

Ex. Show that $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$

Examples:

1. Find the sum of the vector displacements $\vec{s}$ and $\vec{r}$ whose components in km along three $\perp$ directions are

$r_x = -3, r_y = 4, r_z = 6; \; s_x = 5, s_y = 0, s_z = -2.$

Solution: Let the resultant vector be denoted by $\vec{u}$. Then

$u_x = r_x + s_x = -3 + 5 = 2, \; u_y = r_y + s_y = 4 + 0 = 4, \; u_z = r_z + s_z = 6 - 2 = 4$

(i.e.)

$$\vec{u} = u_x\vec{i} + u_y\vec{j} + u_z\vec{k} = 2\vec{i} + 4\vec{j} + 4\vec{k}, \text{ in km.}$$

2. What is the sum of the two vectors $\vec{a} = 4\vec{i} + 3\vec{j}$ and $\vec{b} = -3\vec{i} + 7\vec{j}$? What are the magnitude and direction of $\vec{a} + \vec{b}$?

Solution: Let $\vec{c} = \vec{a} + \vec{b}$

- $\vec{c} = (a_x + b_x)\vec{i} + (a_y + b_y)\vec{j} + (a_z + b_z)\vec{k} = (4-3)\vec{i} + (3+7)\vec{j}$

or

$$\vec{c} = \vec{i} + 10\vec{j}$$

- $|\vec{c}| = \sqrt{c_x^2 + c_y^2} = \sqrt{1^2 + 10^2} \approx 10$

- The angle $\vec{c}$ makes with the x-axis is

$$\theta_x = \tan^{-1}\left(\frac{c_y}{c_x}\right) = \tan^{-1}\left(\frac{10}{1}\right) \approx 84^\circ.$$

3. Two vectors $\vec{A}$ and $\vec{B}$ have comps. in arbitrary units $A_x = 3.2, A_y = 1.6, B_x = 0.5, B_y = 4.5$. a) Find the angle between $\vec{A}$ and $\vec{B}$. b) Find the comps. of a vector $\vec{c}$ that is $\perp$ to $\vec{A}$, is in the x-y plane, and has a magnitude of 5 units.

Solution:

a) $\theta_{xA} = \tan^{-1}\left(\frac{1.6}{3.2}\right) = 27^\circ, \quad \theta_{xB} = \tan^{-1}\left(\frac{4.5}{0.5}\right) = 84^\circ$

i.e.) the angle between $\vec{A}$ and $\vec{B}$ (since they lie in the same plane) is 57° ($=84^\circ - 27^\circ$).

b, vector $\vec{C}$ being $\perp$ to vector $\vec{A}$ must be oriented at an angle of $27^\circ + 90^\circ = 117^\circ$ wrt the +ve x-axis (ccw). or $(27 - 90^\circ) = -63^\circ$ (cw). Then

$$C_x = 5\cos 117^\circ = -2.3, \quad C_y = 5\sin 117^\circ = 4.41$$

or

$$C_x = 5\cos(-63^\circ) = 2.3, \quad C_y = 5\sin(-63^\circ) = -4.41$$

i.e.)

$$C_x = \mp 2.3, \quad C_y = \pm 4.41$$

4. Two vectors $\vec{R}$ and $\vec{S}$, lie in the x-y plane. Their magnitudes are 4.5 and 7.3 units resp., whereas their directions are 320° and 85° measured ccw from the +ve x-axis. What are the values of a, $\vec{R}\cdot\vec{S}$ and b, $\vec{R}\times\vec{S}$?

Solution:

a, The smaller angle between $\vec{R}$ and $\vec{S}$ is 125° ($360^\circ - 235^\circ$). Then

$$\vec{R}\cdot\vec{S} = 4.5 \times 7.3 \cos 125^\circ = -18.84$$

b, $|\vec{R}\times\vec{S}| = 4.5 \times 7.3 |\sin 125^\circ| = 26.91$

so $\vec{R}\times\vec{S}$ is a vector of magnitude 26.91 units with its direction given by the right hand rule.

Also the comp. rep. of $\vec{R}$ and $\vec{S}$ and their "X" product with ... will give the co... of $\vec{R}\times\vec{S}$ and ... direction. $(\vec{R}\times\vec{S})\cdot\vec{R}=0, (\vec{R}\times\vec{S})\cdot$

5. Find the angle between the two vectors given by $\vec{a} = 3\vec{i} + 3\vec{j} - 3\vec{k}$ and $\vec{b} = 2\vec{i} + \vec{j} + 3\vec{k}$.

solution: We use the def. of the dot product of vectors i.e., $\vec{a}\cdot\vec{b} = ab\cos\theta = \sqrt{27}\sqrt{14}\cos\theta =$ [illegible]. Also, $\vec{a}\cdot\vec{b} = a_x b_x + a_y b_y + a_z b_z = 3\cdot 2 + 3\cdot 1 - 3\cdot 3 = 0$.

$\Rightarrow \sqrt{27}\sqrt{14}\cos\theta = 0$, $\cos\theta = 0$ or $\theta = 90^\circ$.

6. Vector $\vec{A}$ lies in the y-z plane 63° from the +ve y-axis and has magnitude of 3.2. Vector $\vec{B}$ [illegible] lies in the x-z

Plane 48° from the x-axis and has magnitude 1.4. Find $\vec{A}\cdot\vec{B}$, $\vec{A}\times\vec{B}$, and the angle between $\vec{A}$ and $\vec{B}$.

Solution:

- $\vec{A}$ has y- and z-comp. while $\vec{B}$ has x and z-comp.

$$A_y = 3.2\cos 63^\circ = 1.45, \quad A_z = 3.2\sin 63^\circ = 2.85$$
$$B_x = 1.4\cos 48^\circ = 0.94, \quad B_z = 1.4\sin 48^\circ = 1.04$$

- $\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z = 2.85 \times 1.04 = 3.0$

Also,

$$\vec{A}\cdot\vec{B} = AB\cos\theta = 3.2\times 1.4\cos\theta = 4.48\cos\theta$$

$$\Rightarrow \quad \theta = \cos^{-1}(3/4.48) = 48^\circ$$

- $\vec{A}\times\vec{B} = (A_yB_z - A_zB_y)\vec{i} + (A_zB_x - A_xB_z)\vec{j} + (A_xB_y - A_yB_x)\vec{k}$

$$= 1.51\vec{i} + 2.68\vec{j} - 1.36\vec{k}$$

Chapter 2 Kinematics of a Particle

In kinematics the motion of a particle is described in terms of position, velocity and acc. These are vector quantities which have both magnitude and direction.

2.1 Average Velocity and Instantaneous Velocity

In the present chapter we shall emphasize 1-D motion. But the various definitions of velocity and acc. which we present next hold for 1-D motion as well for 3-D motion.

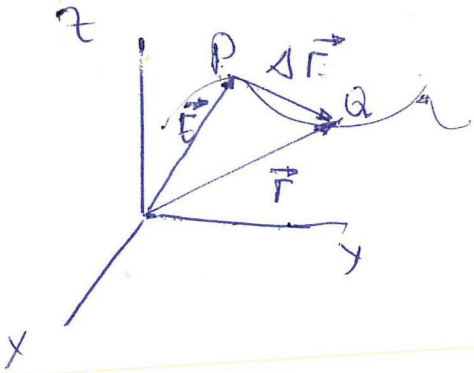
The velocity of a particle is the rate at which its position changes with time. The position of a particle in a particular reference frame is given by a position vector $\vec{r}$ drawn from the origin of that frame to the particle. This vector is a function of time which, in 3-D, may be written as

$$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$$

If at an initial time t_0 the position vector is $\vec{r}_0$ and at a later time t this vector is $\vec{r}$, the average velocity is given as

$$\bar{v} = \frac{\vec{r}-\vec{r}_0}{t-t_0} = \frac{\Delta\vec{r}}{\Delta t} = \frac{\text{displacement}}{\text{time}}$$

So $\bar{v}$ is a vector having the direction of $\Delta\vec{r}$ and the magnitude of $\Delta r/\Delta t$ ($= |\Delta\vec{r}/\Delta t|$). The fig. below illustrates the case discussed here.



At time t_0 the particle is at point P where its p.v. is $\vec{r}_0$ and after a time t it is at Q_0 where the p.v. is $\vec{r}$.

If the average velocity of the particle which is measured for a no. of diff. time intervals is not const., this particle is said to move with a variable velocity. Then we need to determine the velocity of the particle at any given instant of time, called the instantaneous velocity. This is the limiting value of the ratio $\Delta\vec{r}/\Delta t$ as the time interval of the motion Δt approaches zero. or

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t}$$

From the def. of the derivative we write

$$\vec{v} = d\vec{r}/dt$$

The magnitude of the instantaneous velocity is called the speed i.e., the speed is given by

$$v = |\vec{v}| = |d\vec{r}/dt|$$

Finally, using the comp. notation we write

$$\vec{v} = \vec{i}\frac{dx}{dt} + \vec{j}\frac{dy}{dt} + \vec{k}\frac{dz}{dt} \equiv \vec{i}\,v_x + \vec{j}\,v_y + \vec{k}\,v_z .$$

2D Acceleration In 2-D, $\vec{v} = \vec{i}\,v_x + \vec{j}\,v_y$ whereas in 1-D, $\vec{v} = \vec{i}\,v_x$.

2.2 Acceleration

Often the velocity of a moving body changes either in magnitude, in direction or in both as the motion proceeds. The body is then said to have an acc. The acc. of a particle is the rate of change of its velocity with time. The average velo acc. is defined as similar to the average velocity. Thus, if $\vec{v}_0$ is the velocity at an initial time t_0 and $\vec{v}$ is the velocity at a later time t, the average acc.

$\bar{a}$ is given by

$$\bar{a} = \frac{\vec{v} - \vec{v}_0}{t - t_0} = \frac{\Delta \vec{v}}{\Delta t}$$

for the inst. velocity.

and the instantaneous acc., as differential is given by

$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{d\vec{v}}{dt}.$$

Using the Comp. notation

$$\vec{a} = \vec{i}\,\frac{dv_x}{dt} + \vec{j}\,\frac{dv_y}{dt} + \vec{k}\,\frac{dv_z}{dt} = \vec{i}\,a_x + \vec{j}\,a_y + \vec{k}\,a_z$$

In 2-D, $\bar{a} = \vec{i}\,a_x + \vec{j}\,a_y$ and in 1-D, $\vec{a} = \vec{i}\,a_x$.

Examples:

1. a) Write $\vec{r}$ in terms of the integral of $\vec{v}$. b) Write $\vec{v}$ in terms of the integral of $\vec{a}$.

Solution:

a) $\vec{v} = d\vec{r}/dt \Leftrightarrow d\vec{r} = \vec{v}\,dt$

or $\vec{r} = \int \vec{v}\,dt$.

b) $\vec{a} = \frac{d\vec{v}}{dt} \Leftrightarrow d\vec{v} = \vec{a}\,dt$

or $\vec{v} = \int \vec{a}\,dt$.

which can be integrated comp. by comp.

2. For a particle moving with const. acc. $\vec{a}(t) = a_{x0}\vec{i} + a_{y0}\vec{j} + a_{z0}\vec{k}$ where a_{x0}, a_{y0} & a_{z0} are const, find the velocity and position as functions of time. Assume,

$$\vec{v}(0) = \vec{v}_0 = v_{x0}\vec{i} + v_{y0}\vec{j} + v_{z0}\vec{k}$$
$$\vec{r}(0) = \vec{r}_0 = x_0\vec{i} + y_0\vec{j} + z_0\vec{k}$$

Solution:

$$\vec{v} = \int \vec{a}\,dt$$

$$\Rightarrow \quad v_x = \int a_x\,dt = \int a_{x0}\,dt = a_{x0}t + c_1$$
$$v_y = \int a_y\,dt = \int a_{y0}\,dt = a_{y0}t + c_2$$
$$v_z = \int a_z\,dt = \int a_{z0}\,dt = a_{z0}t + c_3$$

$v_x(0) = v_{x0}$, $v_y(0) = v_{y0}$, $v_z(0) = v_{z0}$

$\Rightarrow$ $C_1 = v_{x0}$, $C_2 = v_{y0}$, $C_3 = v_{z0}$. Thus,

$v_x(t) = a_{x0}t + v_{x0}$, $v_y(t) = a_{y0}t + v_{y0}$, $v_z(t) = a_{z0}t + v_{z0}$

$\therefore \vec{v}(t) = (a_{x0}t + v_{x0})\vec{i} + (a_{y0}t + v_{y0})\vec{j} + (a_{z0}t + v_{z0})\vec{k}$

or

$\vec{v}(t) = (a_{x0}\vec{i} + a_{y0}\vec{j} + a_{z0}\vec{k})t + v_{x0}\vec{i} + v_{y0}\vec{j} + v_{z0}\vec{k}$

or $\vec{v}(t) = \vec{a}_0 t + \vec{v}_0$ ..... (*)

Also,

$\vec{r}(t) = \int \vec{v}(t)\, dt$

$\Rightarrow$ $x(t) = \int v_x\, dt = \int (a_{x0}t + v_x)\, dt = \frac{1}{2}a_{x0}t^2 + v_{x0}t + C_1'$

$y(t) = \int v_y\, dt = \int (a_{y0}t + v_{y0})\, dt = \frac{1}{2}a_{y0}t^2 + v_{y0}t + C_2'$

$z(t) = \int v_z\, dt = \int (a_{z0}t + v_{z0})\, dt = \frac{1}{2}a_{z0}t^2 + v_{z0}t + C_3'$

$x(0) = x_0$, $y(0) = y_0$, $z(0) = z_0$

$\Rightarrow$ $C_1' = x_0$, $C_2' = y_0$, $C_3' = z_0$

$\therefore \vec{r}(t) = (\frac{1}{2}a_{x0}t^2 + v_{x0}t + x_0)\vec{i} + (\frac{1}{2}a_{y0}t^2 + v_{y0}t + y_0)\vec{j}$
$+ (\frac{1}{2}a_{z0}t^2 + v_{z0}t + z_0)\vec{k}$

or $\vec{r}(t) = \frac{1}{2}(a_{x0}\vec{i} + a_{y0}\vec{j} + a_{z0}\vec{k})t^2 + (v_{x0}\vec{i} + v_{y0}\vec{j} + v_{z0}\vec{k})t$
$+ (x_0\vec{i} + y_0\vec{j} + z_0\vec{k})$

or

$\vec{r}(t) = \frac{1}{2}\vec{a}_0 t^2 + \vec{v}_0 t + \vec{r}_0$ .... (**)

One can also directly integrate * to obtain (**).

Another useful eq. of motion with const. acc. $\vec{a}_0$ is found as follows.

$\vec{v} = d\vec{r}/dt$, from the def. of instantaneous velocity.

$\Rightarrow$ $d\vec{r} = \vec{v}\, dt$ ... (i)

$\vec{a}_0 = d\vec{v}/dt$, from the def. of instantaneous acc.

$\Rightarrow$ $d\vec{v} = \vec{a}_0\, dt$ ... (ii)

Taking dot product both sides of eq (ii) by $\vec{v}$ we get

$$\vec{v}\cdot d\vec{v} = \vec{v}\cdot\vec{a}_0 dt = \vec{v}\cdot(d\vec{r}/dt)\,\vec{a}_0 = \vec{a}_0\cdot\vec{v}\,dt = \vec{a}_0\cdot d\vec{r}$$

Now integrating both sides

$$\int_{\vec{v}_0}^{\vec{v}} \vec{v}\cdot d\vec{v} = \vec{a}_0\cdot\int_{\vec{r}_0}^{\vec{r}} d\vec{r}$$

$$\frac{1}{2}\int_{\vec{v}_0}^{\vec{v}} d(\vec{v}\cdot\vec{v}) = \vec{a}_0\cdot(\vec{r}-\vec{r}_0)$$

$$\frac{1}{2}\int_{\vec{v}_0}^{\vec{v}} d\vec{v}^{\,2} = \vec{a}_0\,(\vec{r}-\vec{r}_0)$$

$$\tfrac{1}{2}(\vec{v}^{\,2}-\vec{v}_0^{\,2}) = \vec{a}_0\cdot(\vec{r}-\vec{r}_0)$$

or

$$\vec{v}^{\,2}-\vec{v}_0^{\,2} = 2\vec{a}_0\cdot(\vec{r}-\vec{r}_0)$$

So for a body moving with a const. acc. we have

$$\vec{a}(t) = \vec{a}_0$$
$$\vec{v}(t) = \vec{a}_0 t + \vec{v}_0$$
$$\vec{r}(t) = \tfrac{1}{2}\vec{a}_0 t^2 + \vec{v}_0 t + \vec{r}_0$$
$$\vec{v}^{\,2}-\vec{v}_0^{\,2} = 2\vec{a}_0\cdot(\vec{r}-\vec{r}_0)$$

These eqs. can be applied for 3-D, 2-D or 1-D motion with const. acc. as well as the following cases may illustrate.

a, Free fall : This is a simple example of a 1-D motion of a particle with const. acc. For a freely falling particle with zero initial velocity and original height h_0, we have

$$\vec{a}_0 = -g\vec{k},\quad \vec{v}_0 = 0,\quad \vec{r}_0 = h_0\vec{k}$$

whence

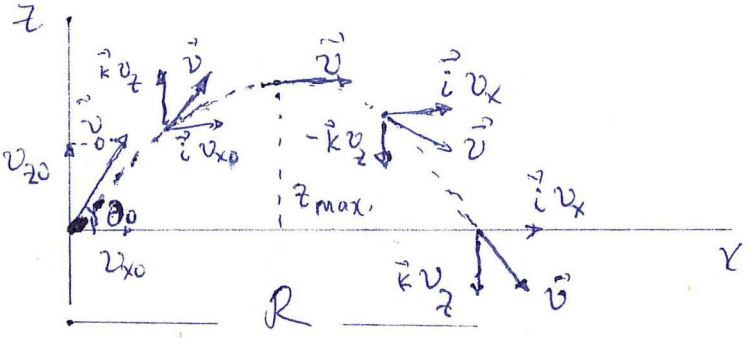
$$\vec{a}(t) = \vec{a}_0 = -g\vec{k}$$
$$\vec{v}(t) = \vec{a}_0 t + \vec{v}_0 = -gt\,\vec{k}$$
$$\vec{r}(t) = \tfrac{1}{2}\vec{a}_0 t^2 + \vec{v}_0 t + \vec{r}_0 = (-\tfrac{1}{2}gt^2 + h_0)\vec{k}$$

and of course

$$v_z^2 = v_{z0}^2 = -2g(z-z_0) = 2g(h_0-z)$$

b, projectile motion: This is an example of 2-D motion with const acc.

Let us find the eq of motion of a projectile.



R is the max. x-dist moved by the projectile.

Now using the eqs for motion with const acc. we have

$$\vec{a}(t) = \vec{a}_0 = 0\vec{i} - g\vec{k} = -g\vec{k}$$

$$\vec{v}(t) = \vec{v}_0 + \vec{a}_0 t = v_{x0}\vec{i} + v_{z0}\vec{k} - gt\vec{k} = v_0\cos\theta_0\,\vec{i} + (v_0\sin\theta_0 - gt)\vec{k}$$

$$\vec{r}(t) = \vec{r}_0 + \vec{v}_0 t + \tfrac{1}{2}\vec{a}_0 t^2$$

$$= (v_0\cos\theta_0)t\,\vec{i} + [(v_0\sin\theta_0)t - \tfrac{1}{2}gt^2]\vec{k}, \text{ assuming}$$

the particle starts from the origin of coordinates. These expressions give

$$a_x(t) = 0, \quad a_z(t) = -g$$

$$v_x(t) = v_{x0} = v_0\cos\theta_0, \quad v_z(t) = v_0\sin\theta_0 - gt$$

$$x(t) = v_0\cos\theta_0 \cdot t, \quad z(t) = v_0\sin\theta_0 - \tfrac{1}{2}gt^2$$

The magnitude of the resultant velocity at any instant is

$$v = \sqrt{v_x^2 + v_z^2}$$

And the angle θ that the velocity vector makes with the horizontal at that instant is given by

$$\tan\theta = v_z / v_x$$

As with any type of motion, the velocity vector is tangent to the path of the particle at every point, as shown in the above figure.

To find an eq. for the trajectory (or path) of the

projectile we proceed as follows.

$$x = v_0 \cos\theta_0 \, t, \quad z = v_0 \sin\theta_0 \cdot t - \tfrac{1}{2} g t^2$$

Solving for t from the first eq. and using the resulting expression in the second eq. we have

$$z(x) = v_0 \sin\theta_0 \frac{x}{v_0 \cos\theta_0} - \frac{1}{2}\left(\frac{g}{v_0^2 \cos^2\theta_0}\right) x^2$$

$$= (\tan\theta_0 \, x) - \left(\frac{g}{2 v_0^2 \cos^2\theta_0}\right) x^2$$

Redefine

$$a = \tan\theta_0, \quad b = g / 2 v_0^2 \cos^2\theta_0$$

which are constants. Then

$$z(x) = ax - bx^2$$

This is the eq. of an inverted parabola. So the path followed by a projectile is parabolic.

Finally, the range of the particle projectile is obtained by finding the x-intercepts (i.e.)

$$z = 0 = \tan\theta_0 \, x - \frac{g}{2 v_0^2 \cos^2\theta_0} x^2 = 0$$

$$(x)\left(\tan\theta_0 - \frac{g x}{2 v_0^2 \cos^2\theta_0}\right) = 0$$

which gives

$$x = 0 \quad \text{or} \quad \tan\theta_0 - \frac{g x}{2 v_0^2 \cos^2\theta_0} = 0$$

The first expression means that the particle has no displacement. Therefore the correct expression is given by the second eq. or,

$$R \equiv x = \frac{2 v_0^2 \cos^2\theta_0 \tan\theta_0}{g} = v_0^2 \frac{2 \cos\theta_0 \sin\theta_0}{g}$$

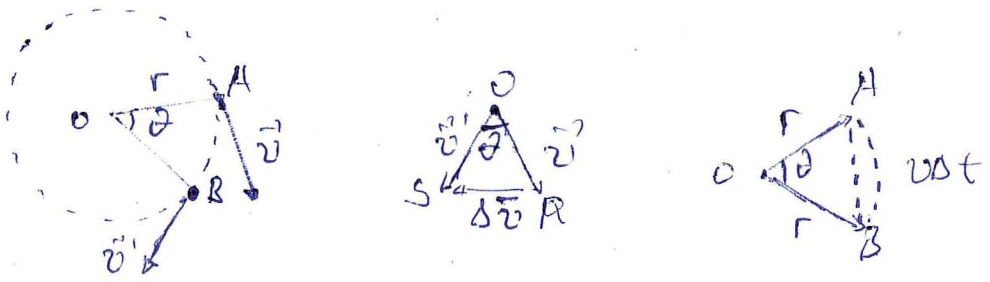
or,

$$R = \frac{v_0^2 \sin 2\theta_0}{g}$$

2.3 Uniform circular motion

In uniform circular motion the velocity of the moving body continuously changes in direction whereas its magnitude remains a const. This is so because at each point on the circle the velocity is a tangent vector acquiring diff. directions. So uniform circular motion is accelerated motion.

To derive an expression for the acc. consider the fig. below in which a particle moves on a circle of radius r.



(a) (b) (c)

In fig (a)
Let A be the position of the particle at the time t and B its position at the time $t+\Delta t$. The velocity at A is $\vec{v}$ and that at B is $\vec{v}'$. Their magnitudes are the same. By making translations the velocity vectors $\vec{v}$ and $\vec{v}'$ are redrawn with their tales at a common origin in fig. (b) shows the change in velocity as the particle moves from A to B. Notice that the velocity change $\Delta\vec{v} = \vec{v}' - \vec{v}$ points inward, approximately toward the center of the circle. On the other hand, fig. (c) shows a triangle formed by the chord AB and the radii OA and OB.

$\triangle ORS \sim \triangle OAB$. they are both isosceles triangles with the same vertex angle θ.

$$\Rightarrow \frac{\Delta v}{v} = \frac{v\Delta t}{r}, \quad \text{approximately since } AB \simeq \overset{\frown}{AB}.$$

This relation becomes more nearly exact as Δt is diminished, since the chord and the arc then approach each other. In this consideration $\Delta\vec{v}$ also approaches closer and closer to a direction $\perp$ to $\vec{v}$ and $\vec{v}'$ and therefore to a direction pointing to the exact center of the circle. Now we write

$$\frac{\Delta v}{\Delta t} = \frac{v^2}{r}$$

and in the limit $\Delta t \to 0$, this expression becomes exact. We therefore obtain

$$a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{v^2}{r}$$

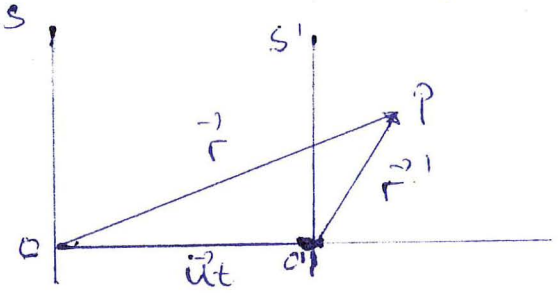
as the magnitude of the acc. The direction of $\vec{a}$ is instantaneously along a radius inward toward the center of the circle. For that reason it is called centripetal acc. or radial acc.

2.4 Relative velocity

The displacements, velocities and accelerations of a particle in motion that are measured by observers in diff. frames of reference are diff. That is, two observers moving ~~moving~~ with respect to each other will generally not agree on the outcome of a measurement.

For example, suppose that a person riding on a moving vehicle throws a ball straight up in the air according to his frame of reference. According to this observer, the ball will move in a vertical path. On the other hand, a stationary observer on the ground will see the path of the ball as a parabola, 2-D motion.

In a more general situation, consider a particle located at the point P in the fig. below. Imagine



that the motion of this particle is being described by two observers, one in reference frame S, fixed wrt the Earth, and another in reference frame S', moving to the right relative to S with a const. velocity $\vec{u}$. (Relative to an observer in S', S moves to the left with a velocity $-\vec{u}$.) Suppose that $\vec{r}$ and $\vec{r}'$ are the position vectors of the particle with respect to the frames S and S' resp.

$$\vec{r} = \vec{r}' + \vec{u}t \quad \cdots (*)$$

or

$$\vec{r}' = \vec{r} - \vec{u}t$$

Differentiating once wrt time we get

$$d\vec{r}'/dt = d\vec{r}/dt - \vec{u}$$

or

$$\vec{v}' = \vec{v} - \vec{u} \quad \ldots (**)$$

where $\vec{v}'$ is the velocity observed in the S' frame and $\vec{v}$ is the velocity of the particle observed in the S frame. Eqs. (*) and (**) are known as Galilean transformations. They relate the coordinates and velocity of a particle in the earths reference frame to those measured in a frame of reference in uniform motion wrt the earth.

If we differentiate (**) once more with time we get

$$\frac{d\vec{v}'}{dt} = \frac{d\vec{v}}{dt}, \quad \vec{u} \text{ is const.}$$

or

$$\vec{a}' = \vec{a}$$

That is, the acc. $\vec{a}$ of the particle measured by an observer in the earths frame of reference will be the same as $\vec{a}$ that is measured by any other observer moving with const. velocity wrt the first observer.

Examples:

1. A boat heading due north crosses a wide river with a speed of 10 km/h relative to the water. The river has a uniform speed of 5 km/h due east. Determine the velocity of the boat wrt a stationary ground observer.

Solution:

The moving reference fram S' is the river, and the observer is in the stationary reference frame S, the earth. The vectors $\vec{u}$, $\vec{v}$ and $\vec{v}'$ are defined as follows.

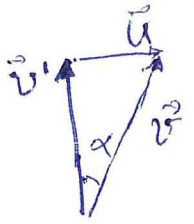
$\vec{u}$ = velocity of the water wrt the earth

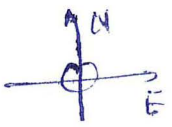
$\vec{v}$ = " " " boat " " "

$\vec{v}'$ = " " " " " " water

In this example, $\vec{u}$ is to the right, $\vec{v}'$ is straight up, and $\vec{v}$ is at an angle α as defined in the fig. below. Since these three vectors form a right triangle, the speed

of the boat wrt the earth is





$$v = \sqrt{v'^2 + u^2} = \sqrt{10^2 + 5^2}\ \text{km/h} = 11.2\ \text{km/h}.$$

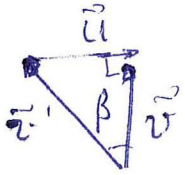
and the direction of $\vec{v}$ is

$$\alpha = \tan^{-1}\left(\frac{u}{v'}\right) = \tan^{-1}(5/10) = 26.6^\circ$$

∴ the boat will be traveling 63.4° north of east wrt the earth.

2. If the boat in example 1 travels with the same speed of 10 km/h relative to the water and wishes to travel due north as in the fig. below, in what direction should it head?

Solution:



Thus the boats speed relative to the earth is

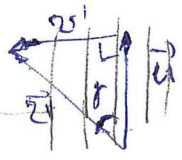
$$v = \sqrt{v'^2 - u^2} = \sqrt{10^2 - 5^2}\ \text{km/h}$$
$$= 8.66\ \text{km/h}.$$

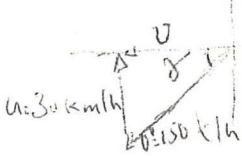
$\beta = \tan^{-1}\left(\frac{u}{v}\right) = \tan^{-1}\left(\frac{5}{8.66}\right) = 30^\circ$, where β is west of north.

3. The pilot of an airplane notes that the compass indicates a heading due west. The airplane's speed relative the air is 150 km/h. If there is a wind of 30 km/h toward the north, find the velocity of the airplane relative to the ground.

Solution

As in the other two examples, we draw a diagram that shows the orientations of the three vectors $\vec{u}$, $\vec{v}$ & $\vec{v}'$.





$$v = \sqrt{v'^2 + u^2} = \sqrt{150^2 + 30^2}\ \text{km/h}$$
$$= 147\ \text{km/h}.$$

and $\gamma = \tan^{-1}(u/v') = \tan^{-1}\left(\frac{30}{150}\right) = 11.54^\circ$

west of ~~north~~ south of west.

Chapter 3 Newton's Laws of Motion

3.1 Introduction:

In chapter 2 we discussed the motion of a particle in terms of the kinematic variables $\vec{r}$, $\vec{v}$ and $\vec{a}$ without considering the causes of motion. Thus our description was largely geometrical. In this chapter we discuss the causes of motion via Newton's three laws and this branch of mechanics is called dynamics.

3.2 Newton's First Law and Inertial Frames

Newton's first law of motion states as follows.

> Everybody continues in its state of rest or uniform motion in a straight line unless it is compelled to change that state by forces impressed on it.

Clearly this statement contains two assertions:

1. It asserts that in the absence of an external (or applied) force a body remains at rest.

This point is actually quite familiar with our daily experience. We know that unless a nonzero force is exerted on a body, the body will not start to move by itself. That means the body will remain in its state of rest which is a case of static equil.

2. In the second place it asserts that in the absence of any applied force a body continues to move uniformly in a straight line.

Unlike the first assertion this one seems to be contradicted by everyday experience. Suppose we exert a force with the hand to move a book along a horizontal table top. After the book left our hand, and we are no longer exerting a force on it, it does not continue to move indefinitely, but slows down and eventually comes to rest. If we wish to keep it moving uniformly we must continue to exert a forward force on it. But this force is required only because a frictional force is exerted on the sliding book by the tabletop, in a direction opposite to the motion of the book. The smoother the surfaces of the book and table, the smaller the

frictional forces and the smaller the force we must exert to keep the book moving. The first law asserts that if the frictional force could be eliminated completely, no forward force at all would be required to keep the book moving, once it had been set in motion. Moreover, the law implies that if the resultant force on the book is zero, as it is when the frictional force is balanced by an equal forward force, the book also continues to move uniformly.

Also, the first law defines by implication what is known as an inertial frame of reference. This is a frame of reference in which an object will move with const. velocity if left undisturbed. Therefore, an inertial frame is one with no acc.

3.3 Inertial Mass

If we attempt to change the state of motion of any body, the body will resist this change. The resistance of a body to a change in its state of motion is called inertia. Mass is a term used to quantify inertia. The greater the mass of a body, the less it will accelerate (change its state of motion) under the action of an applied force. For example, if a given force acting on a 3 kg mass produces an acc. of $4 m/s^2$, the same force when applied to a 6 kg mass will produce an acc. of $2 m/s^2$. Thus a quantitative measurement of mass can be made by comparing the accelerations that a given force will produce on diff. bodies. Suppose a force acting on a body of mass m_1 produces an acc. a_1, and the same force acting on a body of mass m_2 produces an acc. a_2. The ratio of the two masses is defined as the inverse ratio of the magnitudes of the accelerations produced by the same force:

$$\frac{m_1}{m_2} = \frac{a_2}{a_1}$$

If one of these is a standard known mass of, say, 2 kg (SI unit of mass), the mass of an unknown body can be obtained from acc. measurements.

Mass is an inherent property of a body and is independent of the body's surroundings and of the method used to measure the mass.

3.4. Newton's second law

Newton's second law gives us the answer to the question what happens to a body that is acted upon by an external force. To state the law in its most general form, we first define the mom. $\vec{p}$ of a particle as the product of the mass m and the velocity $\vec{v}$:

$$\vec{p} = m\vec{v}.$$

Since the velocity is a vector quantity so is the mom. In SI the unit of mom. is kg·m/s.

Newton's second law of motion states that the time rate of change of mom. of a particle is equal to the resultant external force acting on the particle:

$$\Sigma \vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt}(m\vec{v})$$

This is the most general form of Newton's second law, which is valid in any inertial frame of reference.

If m is treated as a const., then the last eq. above can be expressed as

$$\Sigma \vec{F} = \frac{d}{dt}(m\vec{v}) = m\frac{d\vec{v}}{dt} = m\vec{a}$$

Since this is a vector eq. we may write it in terms of its comps. as

$$\Sigma F_x = ma_x, \quad \Sigma F_y = ma_y, \quad \Sigma F_z = ma_z$$

Therefore, we conclude that the resultant force on a particle is equal to its mass multiplied by its acc. if the mass is const.

If the resultant force is zero, then $\vec{a} = 0$, which corresponds to the equil. situation where $\vec{v}$ is either const. or zero. Hence, the first law of motion is a special case of the second law.

In SI $[F] = kgm/s^2 = \text{Newton (N)}$. Other units are the dyne (g·cm/s²), pound (lb = slug·ft/s²) etc.

3.4.1 Weight

We know that everybody is attracted to the earth. The force exerted by the earth on a body is called the true weight of the body. This force is directed toward the center of the earth.

On the other hand, a freely falling body is accelerated

at g_s which is directed toward the center of the earth. Thus, if W is the ~~for~~ weight and m is the mass of this body, Newtons second law gives

$$W = mg_s$$

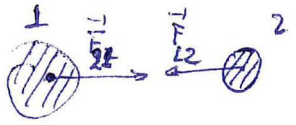
Since g_s varies from place to place depending on altitude so does the weight W.

3.5 Newtons Third Law

This law is stated ~~first~~ as follows.

> If two bodies interact, the force of body 1 on body 2 is equal in magnitude and opposite in direction to the force of body 2 on body 1.

To illustrate these ideas consider the fig. below.



Body 1 exerts a force $\vec{F}_{12}$ on body 2 whereas body 2 exerts a force $\vec{F}_{21}$ on body 1. Then according to Newtons third law we may write

$$\vec{F}_{12} = -\vec{F}_{21}$$

This is equivalent to stating that forces always occur in pairs, or that a single isolated force is an impossibility. The force that body 1 exerts on body 2 is called the ~~reaction~~ action force, while the force of body 2 on body 1 is called the reaction force. However, either force can be ~~called~~ labeled the action or the reaction force. The action force is equal in magnitude and opposite in direction to the reaction force. Note also that the action and reaction forces always act on diff. objects.

3.6 Some Applications of Newtons Laws

In this section we present some simple applications of Newtons laws to bodies that are either in equil. ($\vec{a} = 0$) or moving linearly under the acc. of const. external forces. As a model, we shall assume that the bodies behave as particles so that we need not worry about rotational motions. Here we shall also neglect the effects of friction for those problems invol-

ving motion. This is equivalent to stating that the surfaces are smooth. Finally, we shall usually neglect the mass of any ropes involved in a particular prob. In this approximation, the magnitude of the force exerted at any point along the rope, called the tension, is the same at all points along the rope.

The following procedure may be useful in applying Newton's Second law

1. Draw a neat sketch representing the physical situation in the prob.

2. Isolate the object of interest whose motion is to be discussed. Draw a diagram showing all the forces acting on the body you have isolated. For systems containing more than one object, draw separate diagrams for each object. Do not include forces that the object exerts on its surroundings.

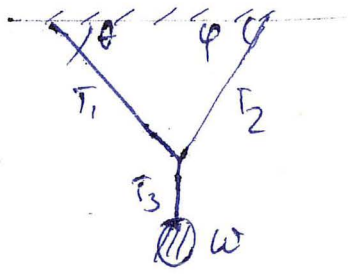
3. Introduce convenient coordinate systems for each body so as to resolve the force vectors into their comps. Now, apply Newton's Second law, $\sum \vec{F} = m\vec{a}$, in the comp. form.

4. Solve the equations of Newton's second law for the unknowns. But remember that the no. of independent eqs. must be the same as the no. of the unknowns in these eqs.

5. Check the solutions you have got for extreme values of the variables. This may help to detect errors in the results.

The following problems are then illustrative examples.

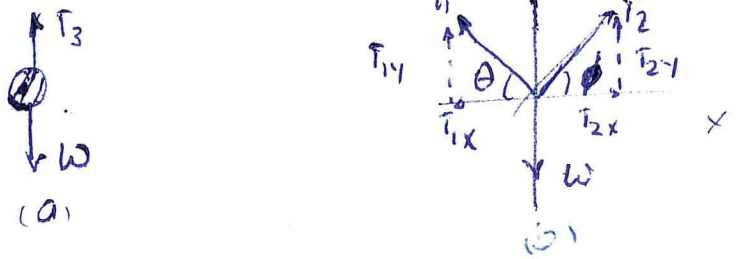
1. A weight w hangs from a rope that is tied to two other cordless ropes that are fastened to the ceiling as in the fig. below. The upper ropes make angles of θ and ϕ with the horizontal. Assuming that θ and ϕ are known find the tensions in the ropes.



Solution:

First the free body diagram of the weight is drawn as in the fig. (a) below. The force of tension in the rope, T_3, supports the weight, so that $T_3 = w$. Now, we construct a free

body diagram for the knot, as in fig. (b) below. This is a convenient point to choose since all forces in question act at this point. We choose the coordinate axes as shown in fig. (b) and resolve the forces into their x and y comps. as follows:



$$T_{1x} = -T_1 \cos\theta, \quad T_{1y} = T_1 \sin\theta$$
$$T_{2x} = T_2 \cos\phi, \quad T_{2y} = T_2 \sin\phi$$
$$T_{3x} = 0, \quad T_{3y} = w$$

Since the system is stationary $\vec{a} = 0$; therefore, Newton's second law applied to the knot gives

1. $\Sigma F_x = T_2 \cos\phi - T_1 \cos\theta = 0$
2. $\Sigma F_y = T_1 \sin\theta + T_2 \sin\phi - w = 0$

The first eq. gives $T_2 = \frac{\cos\theta}{\cos\phi} T_1$ and using this eq. in 2 we get

$$T_1 \sin\theta + \frac{\cos\theta}{\cos\phi} T_1 \sin\phi = w$$

$$\left(\sin\theta \cos\phi + \tan\phi \cos\theta \right) T_1 = w$$

or

$$T_1 = \frac{w \cos\phi}{\sin\theta \cos\phi + \tan\phi \cos\theta} = \frac{w}{\sin\theta + \cos\theta \tan\phi}$$

and

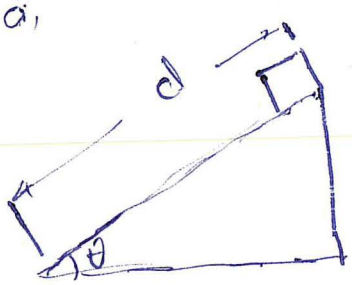
$$T_2 = \frac{w}{\sin\phi + \tan\theta \cos\phi}$$

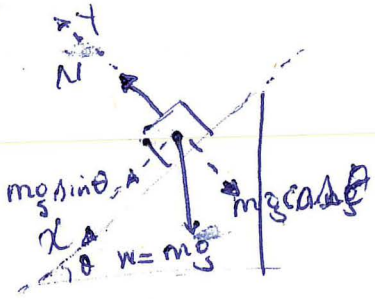
2. A block of mass m is placed on a frictionless inclined plane of angle θ as in the figure below. a, Determine the acc. of the block after it is released from rest at the top of the incline. b, suppose that the

length of the incline is d. How long does it take the block to reach the bottom of the incline and what is its speed just as it gets there?

Solution:

a,





The 2nd fig. is the free body diagram of the block which shows all the forces that act on the block.

Now let us introduce the x-y coordinate system s.t. the x-axis is ∥ to the incline and the y-axis is ⊥ to it. Further, we assume that the +ve x-direction is along the direction of motion of the block/incline. Since there is no motion in the ⊥ direction, $a_y = 0$. So now applying Newtons 2nd law we write

$$\Sigma F_x = mg\sin\theta = ma_x$$

$$\Sigma F_y = N - mg\cos\theta = ma_y = 0.$$

(i.e.) $a_x = g\sin\theta$ and $N = mg\cos\theta$.

If $\theta = 0$, $a_x = 0$. This means the weight does not cause motion in a horizontal direction. If, however, $\theta = 90°$, then the x-axis becomes a vertical line and we have $a_x = g$, which means free fall.

b, Since now a_x is const. we apply the kinematic eq.

$$x - x_0 \equiv d = v_{x0}t + \tfrac{1}{2}a_x t^2 = \tfrac{1}{2}a_x t^2 \text{ since } v_{x0} = 0.$$

This gives

$$t = \sqrt{2d/a_x} = \sqrt{2d/g\sin\theta}.$$

Also,

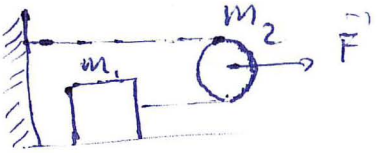
$$v_x^2 = v_{x0}^2 + 2a_x(x - x_0)$$

or

$$v_x^2 = 2g\sin\theta\, d$$

$$\therefore \quad v_x = \sqrt{2gd\sin\theta}$$

3- Ex: A horizontal force $\vec{F}$ is applied to a frictionless pulley of mass m_2 as in the figure below. The horizontal surface is smooth. a, Show that the acc. of the block of mass m_1 is twice the acc. of the pulley. Find b, the acc. of the pulley and the block and c, the tension in the strings.



Solution:

a,

Applying Newton's 2nd law and assuming that the positive direction is to the right we have

One can see from these diagrams that m_1 is accelerated by a force T (the tension in the string) whereas the acc. of m_2 is reduced by a force $2T$ which is twice as great as the accelerating force of m_1. Thus the acc. of m_1 must be equal to twice the acc. of m_2 ior, $a_1 = 2a_2$

b, Applying Newton's 2nd law we have

$$T = m_1 a_1 \quad , \quad F - 2T = m_2 a_2$$

$$\Rightarrow$$

$$F = 2T + m_2 a_2 = 2m_1 a_1 + m_2 \frac{a_1}{2} = \left(\frac{4m_1 + m_2}{2}\right) a_1$$

or

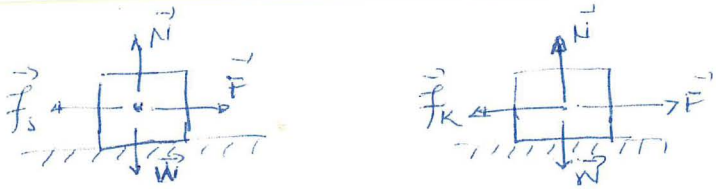
$$a_1 = 2\left(\frac{F}{4m_1 + m_2}\right) \quad \text{and} \quad a_2 = \frac{F}{4m_1 + m_2}$$

c, $$T = m_1 a_1 = \frac{2m_1}{4m_1 + m_2} F .$$

3-7. Forces of Friction

When ever the surface of one body moves over that of the other, each body exerts a frictional force on the other, parallel to the surfaces. The force on each body is opposite to the direction of its motion relative to the other.

Consider a block on a horizontal table, as in the fig (a) below. If we apply an external horizontal force $\vec{F}$ to



the block, acting to the right, the block will remain stationary if $|\vec{F}|$ is not too large. The force that keeps the block from moving acts to the left and is the frictional force $\vec{f}_s$. As long as the block is in equil.

$$|\vec{f}_s| = |\vec{F}| \text{ or } f_s = F$$

Since the block is stationary, we call this frictional force the force of static friction, $\vec{f}_s$. Experiments show that this force arises from the roughness of the two surfaces, so that contact is made only at a few points.

If we increase the magnitude of the applied force $\vec{F}$, as in fig (b) above, the block will eventually slip. When the block is on the verge of slipping, f_s is max. When F exceeds $f_{s,max}$ the block moves and accelerates to the right. When the block is in motion, the retarding frictional force is called the force of kinetic friction, $\vec{f}_K$. The unbalanced force $F - f_K$ produces an acc. to the right. If $F = f_K$, the block moves to the right with const. velocity/speed. If the applied force is removed, the frictional force acting to the left would decelerate the block and eventually bring it to rest.

In a simplified model, we can imagine that the force of kinetic friction is less than $f_{s,max}$ because of the reduction in roughness of the two surfaces when the objects is in motion.

Experimentally, one finds that both f_s and f_K are proportional to the normal force acting on the block and depend on the roughness of the two surfaces in contact. The experimental observations are summ-

arized as follows:-

1. The force of static friction between any two surfaces in contact is opposite the applied force and can have values given by

$$0 \leq f_s \leq \mu_s N$$

$f_s = 0$ when the applied force is zero and $f_s = \mu_s N$ when the block is on the verge of slipping. μ_s is a const. called the coefficient of static friction.

2. The force of kinetic friction is opposite the direction of motion and is given by

$$f_K = \mu_K N$$

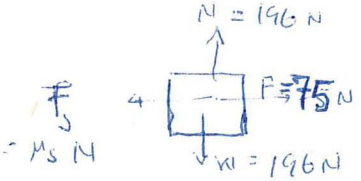
μ_K is called the coeff. of kinetic friction.

3. The values of μ_s and μ_K depend on the nature of the surfaces, but μ_K is generally less than μ_s for a given surface. Further more, μ_s and μ_K are nearly independent of the area of contact between the surfaces. The coeff. of kinetic friction varies with speed, and this is neglected in most problems we will discuss in the future.

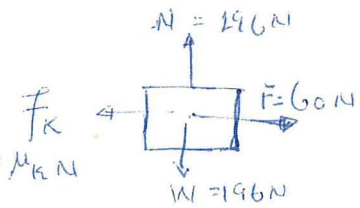
Examples:

1. A 20 kg block is initially at rest on a rough, horizontal surface. A horizontal force of 75 N is required to set the block in motion. After it is in motion, a horizontal force of 60 N is required to keep the block moving with const. speed. Find the coeffn. of static and kinetic friction from this information

Solution:



(a) Block at rest



b. Block moving at const. speed

- In fig. (a) Since the block is at rest

$$-\mu_s N + F = 0 \Rightarrow \mu_s = F/N$$

or

$$\mu_s = \frac{75\ N}{196\ N} = 0.38$$

- In fig. (b) the body moves at a const. velocity

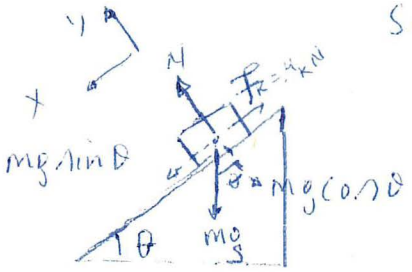
$$-\mu_K N + F = 0 \Rightarrow \mu_K = F/N$$

or

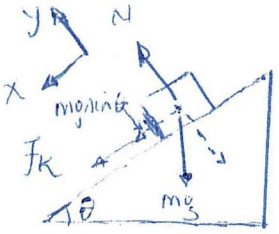
$$\mu_K = 60\,N / 196\,N = 0.31$$

2. A block slides on a rough incline. The coeff. of kinetic friction between the block and the plane is μ_K. a, If the block accelerates down the incline, show that the acc. of the block is given by $a = g(\sin\theta - \mu_K \cos\theta)$. b, If the block is projected up the incline, show that its deceleration is $a = -g(\sin\theta + \mu_K \cos\theta)$.

Solution:



(a)



(b)

a, In fig. (a) the block accelerates down the incline which is taken to be the +ve x-direction. From Newtons 2nd law

$$\Sigma F_x = mg\sin\theta - F_K = ma$$

$$\Sigma F_y = N - mg\cos\theta = 0$$

The 2nd eq. gives $N = mg\cos\theta$ Now, since $F_K = \mu_K N$ and using the expression of N in the first eq. we find

$$mg\sin\theta - \mu_K mg\cos\theta = ma$$

or

$$a = g(\sin\theta - \mu_K \cos\theta).$$

b. For the upward movement in fig. (b)

$$\Sigma F_x = mg\sin\theta + f_K = -ma$$

Again from the eq. for the y-direction, $N = mg\cos\theta$.

$$\Rightarrow mg\sin\theta + \mu_K mg\cos\theta = -ma$$

01

$$a = -g(\sin\theta + \mu_K \cos\theta)$$

3-8 Uniform Circular ~~Motion~~ Motion; Dynamical Consideration

In Sec. 2.3 which deals with the kinematics of uniform circular motion, we pointed out that if a ~~body~~ particle is moving at uniform speed v in a circle of radius, it experiences a centripetal acc. $\vec{a}_r$ whose magnitude is v^2/r. The direction of $\vec{a}_r$ is always radially inward toward the center of rotation. Thus if $\hat{e}_r$ is a unit vector which points outward along a radius, we may write

$$\vec{a}_r = -v^2/r\, \hat{e}_r$$

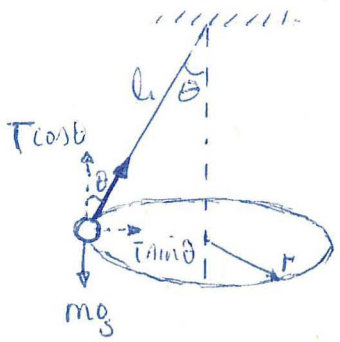
Thus $\vec{a}_r$ is variable because even though its magnitude remains const., its direction changes continuously. Now Newton's second law applied to the circular motion of the particle gives

$$\sum_{\text{(along } \hat{e}_r\text{)}} \vec{F} \equiv \vec{F}_c = -\frac{mv^2}{r}\hat{e}_r$$

Forces responsible for uniform circular motion are called centripetal forces because they are directed "toward the center" of the circular motion.

Example - The Conical pendulum:

A small body of mass m is suspended from a string of length l. The body revolves in a horizontal circle of radius r with const. speed v, as in the figure below. Since the string sweeps out the surface of a cone the system is known as a conical pendulum. Find the speed of the body and the period of revolution T_p.



Solution:

The free-body diagram of the mass m is shown in the above figure, where the tension, $\vec{T}$, has been resolved into a vertical comp., $T\cos\theta$, and a comp. $T\sin\theta$ acting toward the center of rotation. Applying Newton's second law for the problem we find

$$T\cos\theta - mg = 0 \quad \text{, no motion in the vertical direction}$$
$$T\cos\theta = mg$$
$$T\sin\theta = mv^2/r \quad \text{, in the radial direction}$$

Dividing the 2nd eq. by the first we get

$$\tan\theta = v^2/rg$$

But $r = l\sin\theta$, and

$$\tan\theta = v^2/lg\sin\theta$$

or

$$v = \sqrt{lg\sin\theta\tan\theta}$$

The period of revolution T_p is given by

$$T_p = 2\pi r/v = 2\pi l\sin\theta / \sqrt{lg\sin\theta\tan\theta}$$

or

$$T_p = 2\pi\sqrt{l\cos\theta/g}$$

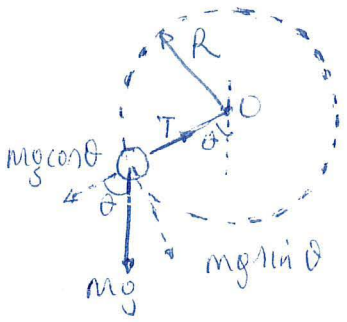
3.9 Nonuniform Circular Motion

If a particle moves with varying speed in a circular path, there arises a tangential acc. $\vec{a}_t$ in addition to the centripetal acc. $\vec{a}_r$ and it is given by

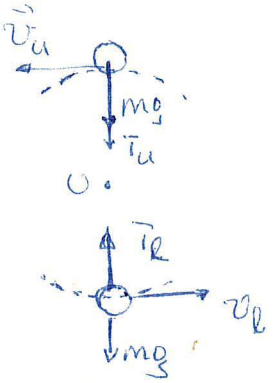
$$\vec{a}_t = d\vec{v}/dt \quad , \vec{v} \text{ tangential velocity.}$$

Therefore, the force acting on the particle must also have a tangential and a radial comp. That is, since the total acc. is given by $\vec{a} = \vec{a}_r + \vec{a}_t$, the total force is given by $\vec{F} = \vec{F}_r + \vec{F}_t$.

Example - A small sphere of mass m is attached to the end of a cord of length R, which rotates in a vertical circle about a fixed point O, as shown below. Determine the position tension in the cord at any instant that the speed of the sphere is v when the cord makes an angle θ with the vertical.



(a)



(b)

Solution:

The speed is not constant (or uniform) since there is a tangential comp. of acc. arising from the weight of the sphere. The free body diagram shows that the only forces acting on the sphere are the weight $m\vec{g}$ and the constraint force in $\vec{T}$. The weight $m\vec{g}$ is resolved into two comps., a radial and a tangential comp. Now, applying the 2nd law we write

$$F_t = mg\sin\theta = m a_t \quad \text{tangential direction}$$

or

$$a_t = g\sin\theta$$

and

$$F_r = T - mg\cos\theta = \frac{mv^2}{R}$$

or

$$T = m\left(\frac{v^2}{R} + g\cos\theta\right)$$

Limiting cases: At the top of the path $\theta = \pi$ where

$$T_u = m\left(\frac{v_u^2}{R} - g\right), \text{ min. value of } T.$$

Note that here $a_t = 0$, and so the acc. is radial directed downward as in fig (b).

At the bottom of the path, where $\theta = 0$ we have

$$T_l = m\left(\frac{v_l^2}{R} + g\right), \text{ max. value of } T.$$

Again $a_t = 0$, and the acc. is radial and is directed upward.

Chapter 4 Work and Energy

4.1 Introduction: A fundamental prob. of dynamics is to find how a particle moves under the action of the forces that act on it. This means given the forces that act on the particle how do we find its position as a function of time. In the previous chapter we discussed the answer to this question for the special case of a constant force. there we indicated that if the const. force $\vec{F}$ that acts on a particle of mass m is known, the acc of the particle is given by

$$\vec{a} = \vec{F}/m$$

If we assume that in addition to the force $\vec{F}$ the mass is also const., the acc. $\vec{a}$ will be const. both in magnitude ~~as and~~ well as in direction and to describe the motion the kinematic equations of chapter 2 (example 2) can be applied. Suppose now that a particle moves at a const acc. a along the x-axis. then the speed of the particle is given from the eq.

$$v = v_0 + at$$

and the position from (with $x_0 = 0$)

$$x = v_0 t + \tfrac{1}{2} a t^2$$

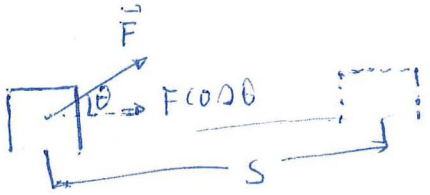
which are special cases of the 3-D equations of example 2 of chapter 2. The last eq gives us directly what we usually want to know, namely, $x(t)$, the position of the particle as a function of time

However the problem is more difficult when the force acting on the particle is not const. Of course, the acc of the particle is obtained from Newtons second law. But to obtain the speed and position of the particle, we can no longer apply the formulas previously developed for const acc. because the acc now is not const. To solve such problems, we use the mathematical method of integration, which we consider in this chapter.

4.2 Work Done by a Constant Force

Let us consider an object (particle) that undergoes a displacement $\vec{s}$ along a straight line under the action of a constant

force $\vec{F}$, which makes an angle θ with $\vec{s}$, as in the figure below. The work done by the const. force is de-



fined as the product of the comp. of the force along the displacement and the magnitude of the displacement. So if W represents this work, we write

$$W = (F\cos\theta)s = Fs\cos\theta \equiv \vec{F}\cdot\vec{s}$$

Note that this eq. refers only to the work done on the ~~particle~~ body by the particular force $\vec{F}$. The work done on the ~~particle~~ body by the other forces must be calculated separately. The total work done on the ~~particle~~ body is the sum of the works done by the separate forces.

Work is a scalar quantity since it is the dot product of two vectors $\vec{F}$ and $\vec{s}$.

unit:

$$[W] = [F\cdot s] = \text{N·m (called Joule - J) in SI; dyne·cm}$$

in the cgs system and lb-ft in the British system.

Example: A box is dragged across a rough floor by a const. force of magnitude 50 N. The force makes an angle of 37° with the horizontal. A frictional force of 10N retards the motion, and the box is displaced a dist. of 3m to the right.

a, calculate the work done by the 50 N force.

From the def.

$$W_F = F\cdot s\cos\theta$$

and for the given values

$$W = 50\,N\cdot 3m\cdot\cos 37° = 120\,N\,m = 120\,J.$$

b, calculate the work done by the frictional force

$$W_f = 10\,N\cdot 3m\cos 180° = -10\,N\cdot 3m = -30\,Nm = -30\,J$$

c) Determine the net work done on the box by all the forces acting on it.

Since the normal force N and the force of gravity mg are both ⊥ to the displacement ($\theta = 90°$), they do no work on the box. Therefore, the net work done on the body is the sum of the works in (a) and (b):

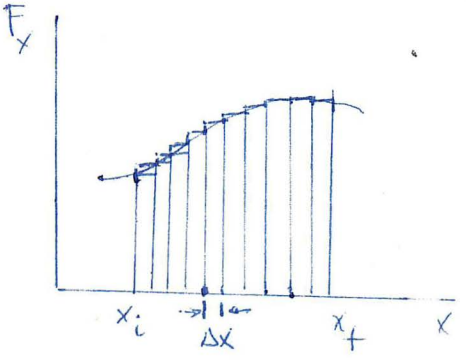
$$W_{net} = W_F + W_T = 120\,J - 30\,J = 90\,J$$

4.3 Work Done by a Variable Force, 1-D case

Consider an object moving along the x-axis under the action of a varying force as in the figures (a) and (b) below. What is the work done by this force in moving the object from an initial position x_i to a final position x_f?

Here we cannot use the formula in Sec. 4.2 to calculate the work since that applies only when $\vec{F}$ is const. both in magnitude as well as in direction. However, if we imagine that the object undergoes a very small displacement Δx, described in fig. (a), then the x-comp. of the force, F_x, is approximately const. over this

(a)



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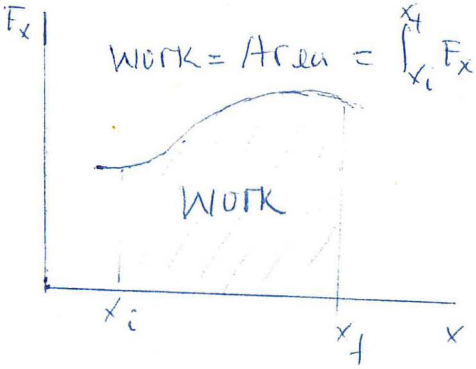


interval and we can express the work done by the force for this small displacement as

$$\Delta W = F_x\,\Delta x$$

Note that this is just the area of the shaded triangle rectangle in fig. (a) above. Now, if we imagine that the region under the curve of F_x vs x that is in between the vertical lines $x = x_i$ and $x = x_f$ is divided into a large no. of such rectangles, then the total work done for the displacement from x_i to x_f is approximately equal to the sum of the areas of these rectangles:

$$W \approx \sum_{x_i}^{x_f} F_x \Delta x$$

If the displacements Δx are allowed to approach zero, then the no. of terms in the sum increases without limit, but the value of the sum approaches a definite limit value, equal to the true area of the region described above. In the notation of calculus this sum is represented as

$$\lim_{\Delta x \to 0} \sum_{x_i}^{x_f} F_x \Delta x = \int_{x_i}^{x_f} F_x \, dx$$

This is called a definite integral since the limits of the integral are clearly indicated as x_i and x_f. So now the total work, being the area under the curve of F_x vs x curve between x_i and x_f, is given as

$$W = \int_{x_i}^{x_f} F_x \, dx$$

If more than one force acts on the object, the total work done is just the work done by the resultant force. If the resultant force in the x direction is expressed as ΣF_x, the net work done on the object as it moves from x_i to x_f is

$$W_{net} = \int_{x_i}^{x_f} (\Sigma F_x) \, dx$$

This expression can be generalized for 3-D as follows. Suppose the force $\vec{F}$ acting on the object has components along the x-, y-, and z-directions. Then, if this force produces 3-D motion so as to displace the object from an initial position $\vec{r}_i$ to a final position $\vec{r}_f$, the work done by $\vec{F}$ is given as

$$W = \int_{\vec{r}_i}^{\vec{r}_f} \vec{F} \cdot d\vec{r}$$

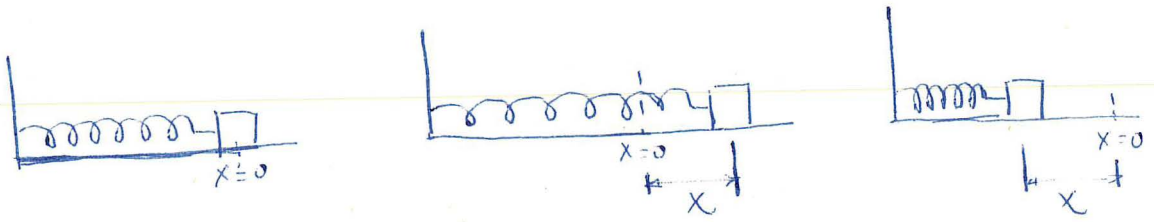
where $d\vec{r} = dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k}$, $\vec{F} = F_x(x,y,z)\,\vec{i} + F_y(x,y,z)\,\vec{j} + F_z(x,y,z)\,\vec{k}$

so that

$$W = \int_{(x_i,y_i,z_i)}^{(x_f,y_f,z_f)} (F_x\,dx + F_y\,dy + F_z\,dz)$$

Example: work done by a spring

The spring is a physical system for which the force varies with position. The spring with a mass attached at its one end with the other end fixed to a wall is shown in the figure below. The mass lies on a horizontal and ~~and~~ frictionless surface. If the spring is either stretched or compressed from



(a) (b) (c)

its unstretched (or equil.) position by a small dist. x, and released, the spring will exert a force on the body given by

$$F_s = -kx$$

k is a const. called the stiffness or force const. of the spring. This force law for springs is known as Hookes law. The negative sign in the above eq. shows that the force exerted by the spring is always directed opposite the displacement. For example, when $x > 0$ as in fig. (b) above the spring force is to the left, or negative. When $x < 0$ as in fig. (c) the spring force is to the right, or positive. of course, when $x = 0$ as in fig. (a), the spring is unstretched and $F_s = 0$. Since the spring force always acts toward the equil. position, it is sometimes called a restoring force. once the mass is displaced some dist. x_m from equil. and then released, it will move from $-x_m$ through zero to $+x_m$.

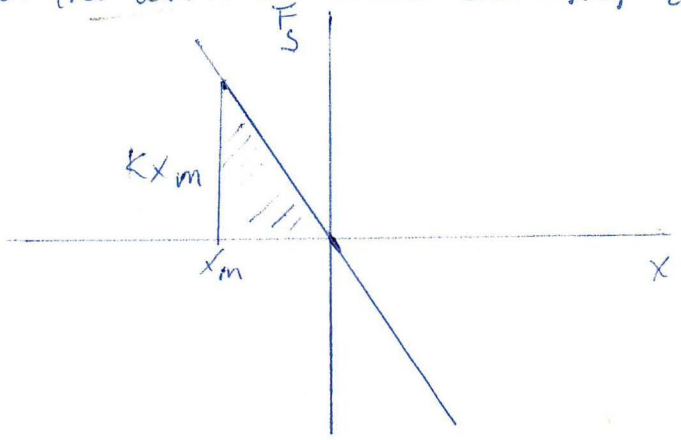
Suppose that the block is pushed to the left a dist. x_m from equil. as in fig. (c) above, and then released. Let us calculate the work done by the spring as the body moves from $x_i = -x_m$ to $x_f = 0$. So we get

$$W = \int_{x_i}^{x_f} F_s\,dx = \int_{-x_m}^{0} (-kx)\,dx = \tfrac{1}{2} k x_m^2 \quad \ldots (*)$$

That is, the work done by the spring is positive since the spring force is in the same direction as the displacement (both are to the right). However, if we consider the work done by the spring as the body moves from $x_i = 0$ to $x_f = x_m$, we find that $W = -\frac{1}{2} k x_m^2$, since for this part of the motion, the displacement is to the right and the spring force is to the left. Therefore, the net work done by the spring as the

body moves from $x_i = -x_m$ to $x_f = x_m$ is zero.

Plotting F_s vs x as in the fig. below we arrive at the same result for the work done by the spring, that is the work calculated above, is equivalent to the area



of the shaded triangle in this fig., with base x_m and height kx_m. The area of the triangle is $\frac{1}{2} k x_m^2$, which equals the workdone by the spring.

If the mass undergoes an arbitrary displacement from $x = x_i$ to $x = x_f$, the work done by the spring is given by

$$W = \int_{x_i}^{x_f} (-kx)\, dx = \tfrac{1}{2} k x_i^2 - \tfrac{1}{2} k x_f^2$$

If an external agent does work by stretching the spring very slowly from $x_i = 0$ to $x_f = x_m$, this work can be calculated by noting that the applied force $\vec{F}_a$ is equal and opposite to the spring force, $\vec{F}_s$, at any value of the displacement, so that $F_a = -(-kx) = kx$.

Thus the work done by the external agent is given by

$$W_{F_a} = \int_0^{x_m} F_a\, dx = \int_0^{x_m} kx\, dx = \tfrac{1}{2} k x_m^2$$

Which equals the negative of the work done by the spring for this displacement.

4.4 Work and Kinetic Energy

Consider a particle of mass m moving along the x-axis which is acted upon by a const force $\vec{F}_x$ until its velocity changes from an initial value v_i to a final value v_f. In this process, suppose that the total dist. moved by the particle is s. Now the work W done on the particle is

$$W = F_x \cdot s$$

But from Newton's second law

$$F_x = m a_x$$

which gives

$$W = (m a_x) x.$$

From the eqs. for rectilinear motion

$$x = \frac{1}{2}(v_i + v_f)t, \quad a_x = \frac{v_f - v_i}{t}$$

Then we find

$$W = m\left(\frac{v_f - v_i}{t}\right)\left(\frac{v_i + v_f}{2}\right)t = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2$$

One half of the product of the mass and the square of the velocity of a particle is defined as the kinetic energy of the particle. If K denotes the kinetic energy, we write

$$K = \frac{1}{2}mv^2$$

Also,

$$K = \frac{1}{2}\frac{(mv)^2}{m} = \frac{p^2}{2m}$$

where p is the mom. of the particle.

Finally, using the def. above we have

$$W = K_f - K_i = \Delta K$$

That is, the work done by the const. force $\vec{F}$ in displacing the particle equals the change in kinetic energy of the particle. This is known as the work-energy theorem. One can see that kinetic energy is a scalar quantity having the units of work.

The last eq. above i.e., the work-energy theorem is derived for the case of a const. force, but we can show that it is valid even when the force is varying and has more than one comp. From the def. of work if the force $\vec{F}$ exerted on a particle acts over a small displacement $d\vec{r}$, the corresponding work done dW is given as

$$dW = \vec{F} \cdot d\vec{r}$$

or

$$dW = m\vec{a} \cdot d\vec{r} = m\frac{d\vec{v}}{dt} \cdot d\vec{r}$$

According to the def. of velocity

$$\vec{v} = \frac{d\vec{r}}{dt} \quad \text{or} \quad d\vec{r} = \vec{v}\,dt$$

and this gives

$$dW = m\frac{d\vec{v}}{dt}\cdot\vec{v}\,dt = m\vec{v}\cdot d\vec{v}.$$

Assuming the velocity changes from $\vec{v}_i$ to $\vec{v}_f$, the total work done is

$$W = \int dW = m\int_{\vec{v}_i}^{\vec{v}_f}\vec{v}\cdot d\vec{v} = \frac{1}{2}m\int_{\vec{v}_i}^{\vec{v}_f} d\vec{v}^2$$

or

$$W = \tfrac{1}{2}m\vec{v}_f^{\,2} - \tfrac{1}{2}m\vec{v}_i^{\,2} \ldots$$

or

$$W = K_f - K_i = \Delta K.$$

Examples:

1. A 30 g bullet initially traveling 500 m/s penetrates 12 cm into a wooden block. What average force does it exert?

Solution:

If F_b represents the magnitude of the force exerted by the bullet, the work-energy theorem gives (assuming motion along the x-axis)

$$F_b x = K_i - K_f \equiv \text{loss in K.E. of the bullet}$$

or

$$F_b = \frac{K_i - K_f}{x} = \frac{\frac{1}{2}\cdot 3\times10^{-2}\,\text{Kg}\times 2.5\times10^{5}\,\text{m}^2/\text{s}^2}{1.2\times10^{-1}\,\text{m}} = 31{,}250\,\text{N}.$$

2. * A force acts on a 3 kg particle in such a way that the position of the particle as a function of time is given by $x = 3t - 4t^2 + t^3$, where x is in meters and t is in seconds. Find the work done by the force during the first 4 s.

solution:

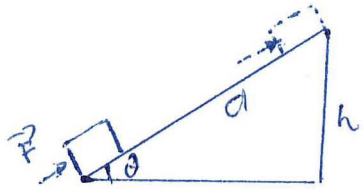
$$v(t) = \frac{dx(t)}{dt} = 3 - 8t + 3t^2, \text{ in m/s}.$$

The initial velocity is $v_i = v(0) = 3$ m/s and the final velo-

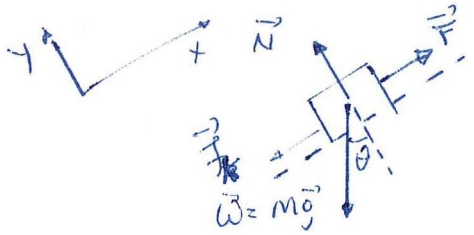
city is $v_f = v(4s) =$ 19 m/s. Therefore, the work done in the time interval of 4s is

$$W = \tfrac{1}{2}\cdot 3\,kg\,(19^2 - 3^2)\,m^2/s^2 = 528\,J.$$

3. A block of mass m is pushed up a rough incline by a const. force $\vec{F}$ acting parallel to the incline, as in the figure below. The block is pushed a dist. d up the incline. a, calculate the work done by the force of gravity for this displacement. b, calculate the work done by the applied force $\vec{F}$. c, Find the work done by the force of kinetic friction if the coeff. of friction is μ.



Solution:



a, The force of gravity is the weight $\vec{w}$ of the block which has a comp. parallel to the plane and a comp. perpendicular to the plane. With respect to the x-y coordinate system shown above these are $-mg\sin\theta$ and $mg\cos\theta$, resp. Therefore, it is the parallel comp. of $\vec{w}$ which does work on the block. or the work done by gravity is

$$W_g = (-mg\sin\theta)\,d = -mgh$$

b, Since $\vec{F}$ is along the displacement, its work is

$$W_F = \vec{F}\cdot\vec{S} = Fd$$

c,

$$f_K = -\mu_K N = -\mu_K mg\cos\theta$$

and

$$W_f = (-\mu mg\cos\theta)\,d = -\mu mgd\cos\theta$$

The net work done on the block is

$$W_{net} = W_g + W_F + W_f = [F - mg(\sin\theta + \mu\cos\theta)]\,d.$$

4.5 Power

Power is defined as the rate at which work is done.

If ΔW represents the work done by an external force $\vec{F}$ in a time interval of Δt, then the average power $\bar{P}$ during this interval is given according to the def. as

$$\bar{P} = \Delta W / \Delta t$$

The instantaneous power, P is given by the limit of $\bar{P}$ as Δt approaches zero. or

$$P = \lim_{\Delta t \to 0} \frac{\Delta W}{\Delta t} = \frac{dW}{dt}$$

But dW is the work of the force $\vec{F}$ over a small displacement $d\vec{r}$, i.e.,

$$dW = \vec{F} \cdot d\vec{r}$$

which gives

$$P = \vec{F} \cdot \frac{d\vec{r}}{dt} = \vec{F} \cdot \vec{v}$$

unit:

$$[P] = J/s \text{ called watt (W) in SI.}$$

In the British system the unit is the horsepower (hp), where

$$1 \text{ hp} = 550 \text{ ft} \cdot \text{lb/s} = 746 \text{ W}.$$

Also,

$$1 \text{ KWh} = 10^3 \text{ W} \times 3600 \text{ S} = 3.6 \times 10^6 \text{ J}$$

which means the kilowatt-hour can be defined as a new unit of energy (or work). It represents the energy converted or consumed in 1h at the constant rate of 1kW.

Examples:

1. A 55 kg woman ($W = mg = 120$ lb) runs up a flight of stairs having a rise of 4.5 m (= 15 ft) in 3.5 s. What average power must she supply?

Solution

Work $W = 55 \text{ kg} \times 9.8 \text{ m/s}^2 \times 4.5 \text{ m} = 120 \text{ lb} \times 15 \text{ ft}$

$= 2425.5 \text{ J} = 1800 \text{ lb-ft}$

and

$$P = \frac{2425.5 \text{ J}}{3.5 \text{ s}} = 693 \text{ W}, \quad \frac{1800 \text{ lb-ft}}{3.5 \text{ s}} = 514.3 \frac{\text{lb-ft}}{\text{s}}.$$

2. A body of mass m accelerates uniformly from rest to a speed v_f in time t_f. a) Show that the work done on the body as a function of time t, in terms of v_f and t_f, is

$$\frac{1}{2} m \frac{v_f^2}{t_f^2} t^2$$

b) As a function of time t, what is the instantaneous power delivered to the body? c. What is the instantaneous power at the end of 10s delivered to a 1500 kg body which accelerates to 90 km/h in 10s?

Solution:

a) Since the force (and hence the acc.) is const. we have

$$W = F \cdot s = mas = m \frac{v_f}{t_f} \cdot \frac{1}{2} \frac{v_f}{t_f} t^2 = \frac{1}{2} m \frac{v_f^2}{t_f^2} t^2$$

b) $$P = \frac{dW}{dt} = m \frac{v_f^2}{t_f^2} t$$

c) $$P(10s) = \left[1500\,kg \cdot (25\,m/s)^2 / (10s)^2\right] 10s = 9375\,W$$

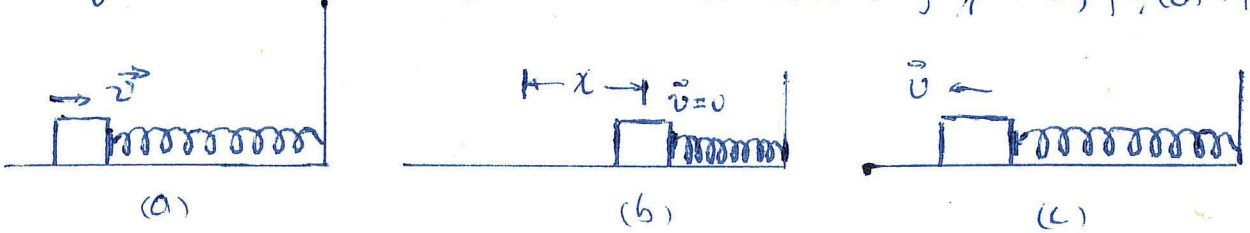
Chapter 5 potential Energy and Conservation of Energy

5.1 Conservative Forces

Let us consider a spring fastened at one end to a rigid wall as in the figure below. In fig (a) we slide a block of mass m with velocity $\vec{v}$ directly toward the spring; if we assume that the horizontal surface is frictionless and the spring obeys Hooke's law, we may write for the restoring force F,

$$F = -kx.$$

After the block touches the spring, the speed and hence the kinetic energy of the block decreases until eventually the block comes to rest and then is forced back to its initial position by the spring. In the reverse motion the block gains speed and kinetic energy. And, when it reaches its position of initial contact with the spring, we find that it has the same speed and kinetic energy as it had originally; only the direction of motion has changed. The block loses kinetic energy during one part of its motion but gains it all back during the other part of its motion as it returns to its starting point, fig(c). Thus



(a) (b) (c)

upon a completion of a round trip the kinetic energy remains the same; it has been conserved. The elastic force exerted by a spring that obeys Hookes law and other forces that act in this same way are called Conservative. The force of gravity is also Conservative; if we throw a ball vertically upward, it will (if the air resistance is negligible) return to our hand with the same kinetic energy that it had before when it left our hand. Then, we may generally state the following:

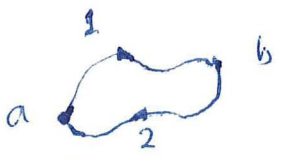
A force is conservative if the kinetic energy of a particle on which it acts remains the same when the particle returns to its initial position after any round trip.

We can define conservative force from another point of view, that of the work done by the force on the particle. In the first example above, the work done by the spring force on the block while the spring was being compressed was negative, because the force exerted on the block by the spring (to the left in fig. (a)) was directed opposite to the displacement of the block (to the right in fig. (a)). While the spring was being extended the work that the spring force did on the block was positive (force and displacement in the same direction). In that example, then the net work done on the block by the spring force during a complete cycle or round trip, is zero. So, the second statement of a conservative force may be presented as follows:

A force is conservative if the work done by the force on the particle that moves through any round trip is zero.

The work-energy theorem shows that this second way of defining conservative forces is fully equivalent to our first definition. From this theorem if there is no change in the kinetic energy of a particle that has moved through any round trip, then $\Delta K = 0$ and it follows that $W = 0$, and the resultant force acting must be conservative.

A still another way of defining conservative forces is the following. Suppose that a particle goes from a to b along path 1 and back from b to a along path 2 as in fig. (a) below. Several forces may act on the particle during this round trip; we consider each force separately. If the



(a)

(b)

force considered is conservative, the work done on the particle by that particular force for the round trip is zero, or

$$W_{ab,1} + W_{ba,2} = 0 \quad \text{or} \quad W_{ab,1} = -W_{ba,2}$$

However, if the particle is forced to go from a to b along path 2, as shown in fig. (b) above, we merely reverse the direction of the previous motion along 2, so that

$$W_{ab,2} = -W_{ba,2}.$$

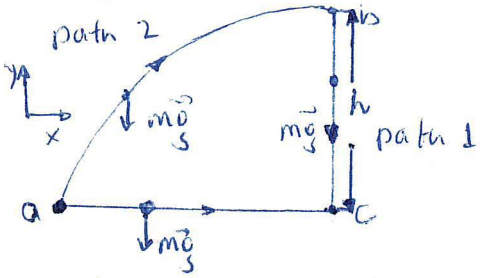
Therefore,

$$W_{ab,1} = W_{ab,2}.$$

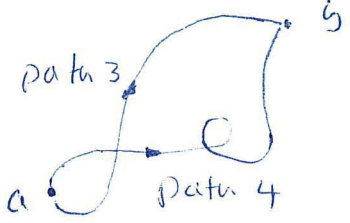
paths 1 and 2 can be any paths as long as they go from a to b; and a and b can be chosen to be any two points. We always find the same result as long as the force is conservative. Hence, we now present a third equivalent def. of conservative forces:

A force is conservative if the work done by it on a particle that moves between two points depends only on these points and not on the path followed.

To illustrate this third def. of a conservative force, consider the gravitational force. Work done by the gravitational force as a stone of mass m is taken from point a to point b along the two paths shown in fig. (a) below.



(a)



(b)

$$W_{ab,1} = W_{ac,1} + W_{cb,1}$$

But $W_{ac,1} = 0$ since the displacement is horizontal and the force of gravity is vertical; and,

$$W_{cb,1} = \int_c^b m\vec{g} \cdot d\vec{r} = -\int_c^b mg\,dy = -mgh.$$

(i.e.)

$$W_{ab,1} = -mgh$$

- For the second arbitrary path, the work is found as follows.
Replace this path by a series of alternate horizontal and vertical segments. By increasing the no. of segments we can make this broken path arbitrarily close to path 2. Now, on the horizontal segments of this path no work will be done. The total length of the vertical segments is, so that the work done on the broken path, and thus on path 2, is indeed -mgh. By a similar argument we can, once again, conclude that the works done for path 3 and path 4 in fig. (b) are the same, -mgh.

5.2 Nonconservative Forces

For the description of a nonconservative force we can consider the examples given in sec. 5.1.

Our first example is thus the spring-mass system that was discussed in that section. But now we assume that the horizontal surface on which the mass moves is rough and there acts the force of friction while the mass is in motion. The frictional force opposes the motion of the block no matter which way the block is moving, and we find that the block returns to its starting point with less kinetic energy than it had initially. Since we showed that the spring force was conservative, we must attribute this new result to the action of the friction force. We say that this force and other forces that act in this same way, are nonconservative. Then, we state that

A force is nonconservative if the kinetic energy of the particle changes after a round trip.

We can define nonconservative force from another point of view, that of the work done by the force on the particle. In the example above the work done on the block by the force of friction is negative for each portion of the cycle (whether the block moves to the right or (the left) because this force always opposes the motion of the block. Hence the work done by friction in a round trip is not zero. In general, we present the second def. of a nonconservative force as follows:

A force is nonconservative if the work done by the force on a particle that moves through any round trip is not zero.

Once again, the work-energy theorem shows that this second way of defining nonconservative forces is fully equivalent to our first definition. From that theorem we see

that if $\Delta K \neq 0$, then so is the work done, i.e., $W \neq 0$ and at least one of the forces acting on the particle must be non-conservative.

Finally, as opposed to the third def. of a conservative force we present the third def of a nonconservative force as

A force is nonconservative if the work done by that force on a particle that moves between two points depends on the path taken between those points.

This fact can be easily verified ~~of course~~ always in the following way. If we push a block over a (rough) table between any two fixed points a and b by various paths, the dist. traversed varies and so does the work done by the frictional force. That is, the work of friction depends on the path.

5.3 Potential Energy

In section 5.1 we found that the work done by a conservative force does not depend on the path taken by the particle. The work done is a function only of the particle's initial and final coordinates; it does not depend on the velocity of the particle or on time.

Under these circumstances (conservative forces acting) we introduce the concept of energy of configuration or potential energy U. This is the energy possessed by a system when it is in a given configuration (or position) and hence it does not depend on the speed of the system. But when the configuration of the system changes both its kinetic energy and potential energy change in such a way that the change in U is equal and opposite to the change in K. Or,

$$\Delta K + \Delta U = 0$$

Alternatively, we say that any change in the kinetic energy K of the system is compensated for by an equal but opposite change in the potential energy U of the system so that their sum remains const. throughout the motion. or

$$K + U = \text{const.}$$

The potential energy of a system represents a form of stored energy that can be fully recovered and converted into kinetic energy. We cannot associate a potential energy with a nonconservative force such as the force of friction because the kinetic energy of a system in which such forces act does not return to its initial value when the system returns to its initial configuration.

The two eqs. above apply to a closed system of interacting

objects, such as the (block + spring) system of sec 6.5. In this example, because we have taken the spring to be effectively massless, the kinetic energy may be associated with the moving block alone. The block slows down (or speeds up) because a force is exerted on it by the spring; it is appropriate, then, to associate the potential energy of the system with this force, that is to say, with the spring. Thus, in this simple case we say that kinetic energy, localized in the block, decreases during the first part of the motion while the elastic potential energy, localized in the spring, increases during this same time interval.

If W represents the work done by the conservative resultant force on a particle as it moves from point a to b, the work-energy theorem gives

$$W = \Delta K = -\Delta U$$

For motion in one dimension, this eq becomes

$$\Delta U = -W = -\int_{x_0}^{x} F_x\, dx = -\int_{x_0}^{x} F(x)\, dx$$

the particle moving from x_0 to x.

Now we know that the potential energy U depends on the position of the particle only and the second eq of this section is written as

$$K + U = \tfrac{1}{2}mv^2 + U(x) = E$$

in which E, which remains const. as the particle is moving, is called the mechanical energy. Suppose that the particle moves from point a (where its position is x_0 and its speed is v_0) to point b (where its position is x and its speed is v); the total mechanical energy E must be the same for each system configuration when the force is conservative, or from the last eq above

$$\tfrac{1}{2}mv^2 + U(x) = \tfrac{1}{2}mv_0^2 + U(x_0)$$

Since the quantity on the right depends on fixed values, initial position x_0 and speed v_0, it is const. so that the quantity on the left which depends on arbitrary position coordinate x and speed v, is also constant. This is the const. total mechanical energy E. Thus, this eq is often called the law of conservation of mechanical energy for conservative forces. This eq is useful in that instead of starting with Newton's laws, we can simplify problem

solving when conservative forces alone are involved just by starting with it. This relation is derived from Newton's laws, of course, but it is one step closer to the solution.

For 1-D motion we can also write the relation between force and potential energy (4th eq. of this section) as

$$F(x) = -\frac{dU(x)}{dx}$$

So the potential energy is a function of position whose negative derivative gives the force.

For motion of a particle from position a to b along the x-axis under the action of a single conservative force we have

$$\Delta U = U_b - U_a$$

or

$$U_b = \Delta U + U_a = -\int_{x_a}^{x_b} F(x)\,dx + U_a.$$

If b represents an arbitrary position x, so that $U_b = U(x)$, we give meaning to $U(x)$ by choosing point a to be some reference position, described by $x_a = x_0$, and by arbitrarily assigning a value to the potential energy $U_a = U(x_0)$ when the body is at that point. Then, our last eq. above becomes

$$U(x) = -\int_{x_0}^{x} F(x)\,dx + U(x_0)$$

The potential energy when the body is at the reference position, i.e., $U(x_0)$, is usually given the value zero.

§8.4 1-D Conservative Systems

Now we calculate the potential energy in our 1-D motion for two kinds of conservative forces, the force of gravity for motions near the earth's surface and the elastic restoring force of a stretched spring that obeys Hooke's law.

For the force of gravity we take the 1-D motion to be vertical, along the y-axis. Suppose that the positive direction of the y-axis is upward. Then $\vec{F} = F_y\hat{j} = -mg\hat{j}$, a constant. So for a particle of mass m taken to a point a dist. y above the surface of the earth the potential energy is given by using the last eq in Sec 8.3 as

$$U(y) = \int_0^y \vec{F}\cdot d\vec{y} + U(0) = -\int_0^y F(y)\,dy + U(0)$$

$$= -\int_0^y (-mg)\,dy + U(0)$$

$$= mgy + U(0)$$

If we take $U(0)=0$, then we find

$$U(y) = mg_0 y.$$

The relation $F(y) = -dU/dy$ is satisfied, for $-(d(mg_0y)/dy) = -mg_0$. For the two positions y and $y=0$ of the particle, the eq. of the conservation of mechanical energy gives

$$\tfrac{1}{2}mv^2 + mg_0 y = \tfrac{1}{2}mv_0^2$$

which is equivalent mathematically to the well known result

$$v^2 = v_0^2 - 2g_0 y$$

If, however, the particle moves from a height h_1 to a height h_2, we can use the eq. of the conservation of mechanical to obtain

$$\tfrac{1}{2}mv_1^2 + mgh_1 = \tfrac{1}{2}mv_2^2 + mgh_2$$

Thus, the mechanical energy E is const. and is conserved during the motion, even though the kinetic energy and potential energy vary as the configuration of the system (particle + earth) changes.

For the spring force we consider a block of mass m attached to an elastic spring which moves on a horizontal frictionless surface. If we take $x_0 = 0$ as the position of the end of the spring when unextended, the force exerted on the block when the spring is stretched a dist. x from its unstretched unextended length is $F = -kx$. The potential energy is given by

$$U(x) = -\int_0^x F(x)\,dx + U(0) = -\int_0^x (-kx)\,dx + U(0)$$

Choosing $U(0) = 0$, the potential energy as well as the force is zero when the spring is unextended, and

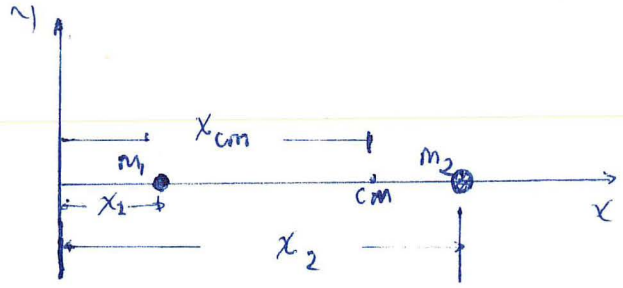
$$U(x) = -\int_0^x (-kx)\,dx = \tfrac{1}{2}kx^2$$

The result is the same whether we stretch or compress the spring, that is, whether x is plus or minus.

The relation $F(x) = -dU/dx$ is again satisfied, for $-d(\tfrac{1}{2}kx^2)/dx = -kx$. For the two positions x and $x=0$ of the block, the eq. of conservation of mechanical energy gives

$$\tfrac{1}{2}mv^2 + \tfrac{1}{2}kx^2 = \tfrac{1}{2}mv_0^2$$

To define the center of mass consider first the simple case of a system of two particles m_1 and m_2 at distances x_1 and x_2, resp., from the origin of the x-y coordinate system as in the figure below.



Now the position of the center of mass (cm) is defined to be given by x_{cm}.

$$x_{cm} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}.$$

The center of mass lies between the two particles and on the line joining them.

If the two particles are in space similar expressions will hold for the y- and z- coordinates of the cm. In general, if we have n particles of masses $m_1, m_2, \ldots m_n$ lying in space, the x-, y- and z- coordinates of the cm are

$$x_{cm} = \frac{m_1 x_1 + m_2 x_2 + \cdots + m_n x_n}{m_1 + m_2 + \cdots + m_n} = \frac{\sum m_i x_i}{\sum m_i}$$

$$y_{cm} = \frac{\sum m_i y_i}{\sum m_i}$$

$$z_{cm} = \frac{\sum m_i z_i}{\sum m_i}.$$

But since $\sum m_i$ is the total mass of the system we can also write

$$x_{cm} = \sum m_i x_i / M, \quad y_{cm} = \sum m_i y_i / M, \quad z_{cm} = \sum m_i x_i / M.$$

The cm can also be located by its position vector $\vec{r}_{cm}$, the rectangular comps. of which are x_{cm}, y_{cm} and z_{cm}. Therefore,

$$\vec{r}_{cm} = x_{cm}\vec{i} + y_{cm}\vec{j} + z_{cm}\vec{k} = \frac{\sum m_i x_i \vec{i} + \sum m_i y_i \vec{j} + \sum m_i z_i \vec{k}}{M}$$

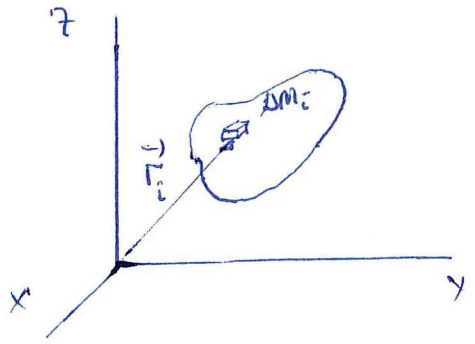
or,

$$\vec{r}_{cm} = \frac{\sum_i m_i \vec{r}_i}{M}$$

where $\vec{r}_i$ is the position vector of the i^{th} particle, defined by

$$\vec{r}_i = x_i \vec{i} + y_i \vec{j} + z_i \vec{k}$$

To give the def. of the cm of the so called rigid body we can think of such a body to be a system of a large no. of particles (the figure below). The particle separation is very small, and so the body can be considered



to have a continuous mass distribution. By dividing the body into elements of mass Δm_i, with coordinates x_i, y_i, z_i, we see that the X coordinate of the cm is approximately

$$x_{cm} \approx \sum_i x_i \Delta m_i / M$$

with similar expressions for y_{cm} and z_{cm}. If we let the no. of elements, n, approach infinity, then x_{cm} will be given precisely. In this limit, we replace the sum by an integral and replace Δm_i by the differential element dm, so that

$$x_{cm} = \lim_{\Delta m_i \to 0} \frac{\sum_i x_i \Delta m_i}{M} = \frac{1}{M} \int_{vol.} x \, dm, \quad \text{definite integral.}$$

Likewise for y_{cm} and z_{cm} we get

$$y_{cm} = \frac{1}{M} \int y \, dm \quad \text{and} \quad z_{cm} = \frac{1}{M} \int z \, dm.$$

In vector form

$$\vec{r}_{cm} = \frac{1}{M} \int \vec{r} \, dm.$$

Since a rigid body is a continuous distribution of mass,

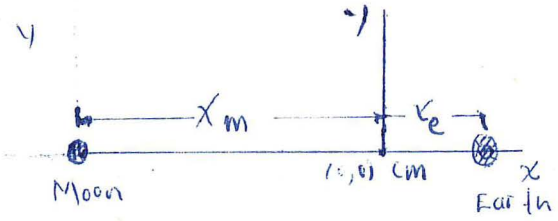
each portion is acted upon by the force of gravity. The net effect of all of these forces is equivalent to the effect of a single force, Mg, acting through a special point, called the center of gravity (cg). If $\vec{g}$ is const. over the mass distribution, then the cg coincides with the cm.

Examples:

1. The mass of the moon is about 0.0123 times the mass of the earth. The earth-moon separation measured from their centers is about 3.84×10^8 m. Determine the location of the center of mass of the earth-moon system as measured from the center of the earth.

Solution:

Suppose that the cm lies at the origin of the x-y coordinate system with both the moon and the earth lying on the x-axis.



$$x_{em} = x_e + x_m$$

From the def. of the cm we write

$$x_{cm} = \frac{-x_m m_m + m_e x_e}{m_m + m_e}$$

But $x_{cm} = 0$ and hence

$$-x_m m_m + m_e x_e = 0$$

which gives

$$x_e = \frac{m_m x_m}{m_e} = \frac{m_m (x_{em} - x_e)}{m_e}$$

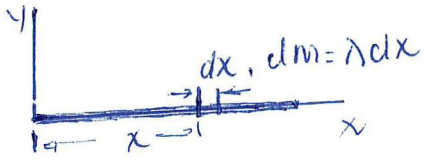
or

$$x_e = \frac{m_m x_{em}}{m_e + m_m} = \frac{0.0123 m_e \times 3.84 \times 10^8 m}{m_e + 0.012 m_e}$$

$$= 4.67 \times 10^6 m.$$

2. a, Show that the cm of a uniform rod of mass M and

length ℓ lies midway between its ends figure below.



Solution:

By symmetry $y_{cm}, z_{cm} = 0$ if the rod is placed along the x-axis. In the figure λ denotes the mass per unit length (or $\lambda = M/\ell$) which is a const. for a uniform rod. Then since an arbitrary mass element is a dist. x from the origin

$$x_{cm} = \frac{1}{M}\int_0^\ell x\,dm = \frac{1}{M}\int_0^\ell x\lambda\,dx = \frac{\lambda}{2M}x^2\Big|_0^\ell$$

or

$$x_{cm} = \frac{\lambda \ell^2}{2M}$$

Since $\lambda = M/\ell$, this gives

$$x_{cm} = \ell/2.$$

6.2 Motion of a System of Particles

We now discuss the motion of a group of particles whose masses are $m_1, m_2, \ldots m_n$, and whose total mass is M. However, as pointed earlier the motion of such a system can be described in terms of the motion of the cm of the system. From the def. of the cm we have

$$M\vec{r}_{cm} = m_1\vec{r}_1 + m_2\vec{r}_2 + \ldots + m_n\vec{r}_n$$

or

$$\vec{r}_{cm} = \sum_i m_i\vec{r}_i / M.$$

Differentiating this eq. with respect to time, we obtain

$$\vec{v}_{cm} = \frac{d\vec{r}_{cm}}{dt} = \frac{1}{M}\sum_i m_i\frac{d\vec{r}_i}{dt} = \frac{1}{M}\sum_i m_i\vec{v}_i$$

where $\vec{v}_i$ is the velocity of the first particle, and so on, and $\vec{v}_{cm}$ is the velocity of the cm. Rearranging gives

$$M\vec{v}_{cm} = \sum_i m_i\vec{v}_i$$

In chapter 3, the product of the mass m and the velocity $\vec{v}$ of a particle is defined to be the linear mom. $\vec{p}$ of the particle. Then, the right side of the last eq. equals the total mom. of the system. Therefore, we conclude that the total mom. of the system equals the total mass multiplied by the velocity of the cm.

If we now differentiate the velocity of the cm with respect to time, we get the acc. of the cm:

$$\vec{a}_{cm} = \frac{d\vec{v}_{cm}}{dt} = \frac{1}{M}\sum_i m_i \frac{d\vec{v}_i}{dt} = \frac{1}{M}\sum_i m_i \vec{a}_i$$

Rearranging this expression and using Newton's second law, we get,

$$M\vec{a}_{cm} = \sum_i m_i \vec{a}_i = \sum_i \vec{F}_i$$

where $\vec{F}_i$ is the force on particle i.

Hence the total mass of the group of particles times the acc. of its cm is equal to the vector sum of all the forces acting on the group of particles.

The forces on any particle in the system may include both external forces (from outside the system) and internal forces (from within the system). However, by Newton's third law, the force of particle 1 on particle 2, for example, is equal to and opposite the force of particle 2 on particle 1. Thus, when we sum over all internal forces, they cancel in pairs and the net force on the system is due only to external forces. Thus we write the above eq. in the form

$$\sum \vec{F}_{ext} = M\vec{a}_{cm} = d\vec{P}/dt \,, \quad \vec{P} \text{ total mom.}$$

That is, the resultant external force on the system of particles equals the total mass of the system multiplied by the acc. of the cm. If we compare this to Newton's 2nd law for a single particle, we see that the cm moves like an imaginary particle of mass M under the influence of the resultant external force on the system.

Finally, we see that if the resultant external force is zero, then it follows from the last eq. above that

$$\frac{d\vec{P}}{dt} = M\vec{a}_{cm} = 0$$

So that

$$\vec{P} = M\vec{v}_{cm} = \text{const.} \quad (\text{when } \Sigma \vec{F}_{ext} = 0).$$

That is, the total linear mom. of a system of particles is conserved if there are no external forces acting on the system. Therefore, it follows that for an isolated system of particles, both the total mom. and velocity of the cm are const. in time.

Chapter 8 Linear Momentum and Collisions

8.1 Linear Momentum and Impulse

Consider a particle of mass m that is acted upon by a force $\vec{F}$ resultant force $\vec{F}$. If $\vec{p}$ is the linear mom. of the particle, Newton's second law gives

$$\vec{F} = \frac{d\vec{p}}{dt}$$

As for a system of particles, the linear mom. of this particle is conserved if the resultant force acting on it is zero. Of course, if the particle is isolated (that is, if it does not interact with its environment), then by necessity, $\vec{F} = 0$ and $\vec{p}$ remains unchanged.

This eq. can be written as

$$d\vec{p} = \vec{F}\,dt$$

To obtain the change in the mom. of a particle we may integrate this expression. If the mom. of the particle changes from $\vec{p}_i$ at time t_i to $\vec{p}_f$ at a later time t_f, integration gives

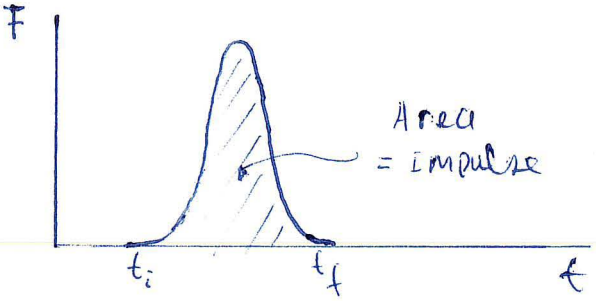
$$\int_{\vec{p}_i}^{\vec{p}_f} d\vec{p} = \vec{p}_f - \vec{p}_i = \Delta\vec{p} = \int_{t_i}^{t_f} \vec{F}\,dt$$

The quantity on the right is called the impulse of the force $\vec{F}$ for the time interval $\Delta t = t_f - t_i$. Then impulse is a vector quantity defined by

$$\vec{I} = \int_{t_i}^{t_f} \vec{F}\,dt = \Delta\vec{p}.$$

Therefore, the impulse of the force $\vec{F}$ equals the change in the mom. of the particle. This statement, known as the impulse mom. theorem, is equivalent to Newton's second law. From the def. we see that impulse is a vector quan-

tity having a magnitude equal to the area under the force-time curve, as shown below. In this figure, it is assumed



that the force varies in time in general manner shown and is nonzero in the time interval $\Delta t = t_f - t_i$. The direction of the impulse vector is the same as the direction of the change in mom. Note that impulse is not a property of the particle itself, but is a quantity that measures the degree to which an external force changes the mom. of the particle. Therefore, when we say an impulse is given to a particle, it is implied that mom. is transfered from an external agent to that particle.

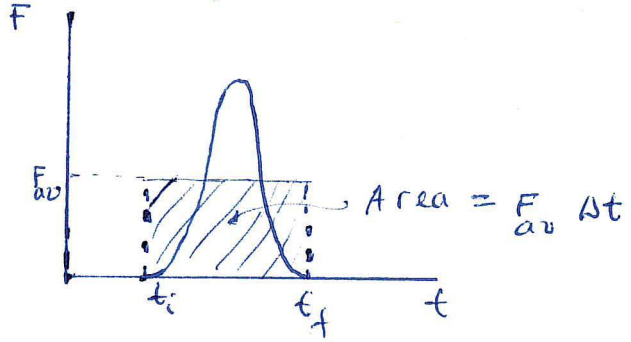
Since the force can generally vary in time as in the above figure, it is convenient to define a time-averaged force $\vec{F}_{av}$, given by

$$\vec{F}_{av} = \frac{1}{\Delta t} \int_{t_i}^{t_f} \vec{F}\, dt$$

where $\Delta t = t_f - t_i$. Therefore, we write

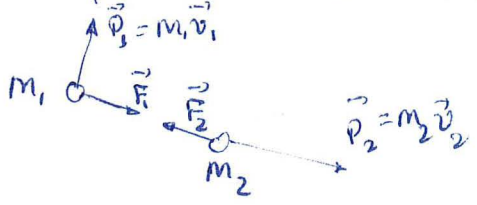
$$\vec{I} = \Delta \vec{p} = \vec{F}_{av} \Delta t$$

This average force, described in the figure below can be thought of as the const. force that would give the same impulse to the particle in the time interval Δt as the actual time-varying force gives over this same time interval.

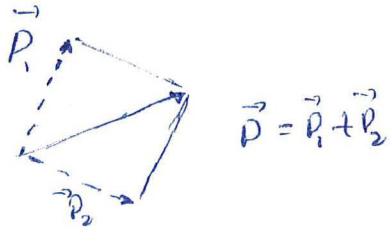


#2 Conservation of Linear Momentum for a Two Particle System

Consider two particles that can interact with each other but are isolated from their surroundings (fig. (a)). That is,



(a)



(b)

the particles exert forces on each other, but no external forces are present. If $\vec{p}_1$ & $\vec{p}_2$ are the momenta of m_1 & m_2, resp., Newton's second law tells us that

$$\vec{F}_{21} = \frac{d\vec{p}_1}{dt} \quad \text{and} \quad \vec{F}_{12} = \frac{d\vec{p}_2}{dt}$$

where $\vec{F}_{21}$ is the force on particle 1 due to particle 2 and $\vec{F}_{12}$ is the force on particle #2 due to particle 1. These forces could be gravitational, electrostatic or of some origin. Then, Newton's third law indicates shows that $\vec{F}_{21}$ and $\vec{F}_{12}$ are equal in magnitude and opposite in direction. qfj That is,

$$\vec{F}_{21} + \vec{F}_{12} = 0$$

or,

$$\frac{d\vec{p}_1}{dt} + \frac{d\vec{p}_2}{dt} = \frac{d}{dt}(\vec{p}_1 + \vec{p}_2) = 0$$

Since the time derivative of the total mom., $\vec{P} = \vec{p}_1 + \vec{p}_2$, is zero, we conclude that the total mom., $\vec{P}$, must remain const. Therefore,

$$\vec{P} = \vec{p}_1 + \vec{p}_2 = \text{const.}$$

In comp. form this eq. says that the total momenta in the x, y, and z-directions are all independently conserved. or,

$$P_{ix} = P_{fx}, \quad P_{iy} = P_{fy}, \quad P_{iz} = P_{fz}$$

So the conservation of linear mom. for the two particle system is often stated as follows:

If two particles of masses m_1 & m_2 form an isolated system, then the total momentum of the system is conserved, regardless of the force between them provided the forces obey Newton's third law. More simply, whenever two

particles collide their total mom. remains const. provided they are isolated.

Suppose $\vec{v}_{1i}$ & $\vec{v}_{2i}$ are the initial velocities of particles 1 & 2, & $\vec{v}_{1f}$ & $\vec{v}_{2f}$ are their velocities at some later time. So the eq. of the conservation of linear momentum given of this isolated system reads

$$m_1\vec{v}_{1i} + m_2\vec{v}_{2i} = m_1\vec{v}_{1f} + m_2\vec{v}_{2f}$$

or

$$\vec{p}_{1i} + \vec{p}_{2i} = \vec{p}_{1f} + \vec{p}_{2f}.$$

That is the total mom. of the isolated system at all times equals its initial total mom. This law may also be described in a diff. way. Since we require that the system be isolated, the only forces acting must be internal to the system (the action-reaction pairs). In other words, if there are no external forces present, the total mom. of the system remains const. Therefore, mom. conservation for an isolated system is an alternative and equivalent statement of Newtons third law.

The law of conservation of mom. is more fundamental than the law of conservation of energy. That is so because M.E. is only conserved for an isolated system when conservative forces alone act on a system and mom. is conserved for an isolated ~~system~~ two-particle system regardless of the nature of the internal forces.

§3 Collisions

When two particles (of masses m_1 & m_2) collide they produce impulsive forces on each other. The impulsive force due to the collision is assumed to be much larger than any external forces present. If $\vec{F}_{21}$ is the force on m_1 due to m_2, then the change in mom. of m_2 due to the collision is given by

$$\Delta\vec{p}_1 = \int_{t_i}^{t_f} \vec{F}_{21}\,dt$$

Likewise, if $\vec{F}_{12}$ is the force on m_2 due to m_1, the change in mom. of m_2 is given by

$$\Delta\vec{p}_2 = \int_{t_i}^{t_f} \vec{F}_{12}\,dt$$

However, Newtons third law states that the force on m_2 due to m_2 is equal to and opposite to the force on m_2 due to m_1, or $\vec{F}_{21} = -\vec{F}_{12}$. Hence, we conclude that

$$\Delta\vec{p}_1 = -\Delta\vec{p}_2$$

or

$$\Delta \vec{P}_1 + \Delta \vec{P}_2 = 0$$

Since the total mom. of the system is $\vec{P} = \vec{P}_1 + \vec{P}_2$, we conclude that the change in the mom. of the system due to collision is zero, i.e.,

$$\vec{P} = \vec{P}_i + \vec{P}_2 = \text{const.}$$

This is precisely what happens if there are no external forces acting on the system. However, the result is also valid if we consider the motion just before and just after the collision. Since the impulsive forces due to the collision are internal, they do not affect the total mom. of the system. Therefore, we conclude that for any type of collision, the total mom. of the system just before the collision equals the total mom. of the system just after the collision.

However, even though the total mom. is always conserved during a collision, the total K.E. is generally not conserved. This is so because some of the K.E. is converted into heat and internal elastic P.E. when the bodies are deformed during the collision.

We define an inelastic collision as a collision for which K.E. is not conserved. Only mom. is conserved in an inelastic collision. When two objects collide and stick together after the collision, the collision is called perfectly inelastic.

An elastic collision is defined as a collision in which both mom. and K.E. are conserved.

4 collisions in one Dimension

Perfectly inelastic collisions:

Consider two particles of masses m_1 and m_2 moving with initial velocities $\vec{v}_{1i}$ and $\vec{v}_{2i}$ along a straight line, as shown in the figure below. We assume the particles collide head-on,

Before collision

After collision

so that they will be moving along the same line of motion after the collision. After collision, the two particles move off as a unit with a velocity $\vec{v}_f$. Only mom. is conserved and we write

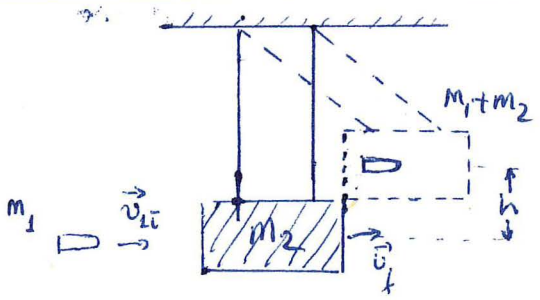
$$m_1 \vec{v}_{1i} + m_2 \vec{v}_{2i} = (m_1 + m_2) \vec{v}_f$$

or,

$$\vec{v}_f = \frac{m_1 \vec{v}_{1i} + m_2 \vec{v}_{2i}}{m_1 + m_2}$$

Example: The ballistic pendulum

The ballistic pendulum (fig. below) is a system used to measure the velocity of a fast-moving projectile, such as a rifle bullet. The bullet is fired into a large block of wood suspended from some light wires. The bullet is stopped by the block, and the entire system swings through a height h. Since the collision is perfectly inelastic and mom. is conserved, i.e., since the



block is originall at rest we have

$$m_1 v_{1i} = (m_1+m_2) v_f \quad , \quad \vec{v}_{1i} \text{ \& } \vec{v}_f \text{ same direction}$$

(or)

$$v_f = \frac{m_1 v_{1i}}{m_1+m_2}$$

The K.E. just after the collision is

$$K_f = \frac{1}{2}(m_1+m_2) v_f^2 = \frac{1}{2}(m_1+m_2) \frac{m_1^2 v_{1i}^2}{(m_1+m_2)^2}$$

or,

$$K_f = \frac{m_1^2 v_{1i}^2}{2(m_1+m_2)}$$

Note that this K.E. is less than the initial K.E. of the bullet. However, after the collision, energy is conserved and K.E. at the bottom is transformed into P.E. at the height h of the bullet and the block; i.e.,

$$\frac{m_1^2 v_{1i}^2}{2(m_1+m_2)} = (m_1+m_2) g h$$

or,

$$v_{1i} = \left(\frac{m_1+m_2}{m_1}\right)\sqrt{2gh}$$

Elastic collisions:

Suppose that two particles of masses m_1 & m_2 undergo an elastic head-on collision as in the figure below. In this case, both mom. and K.E. are conserved; therefore,

Before collision After collision

We can write

$$m_1\vec{v}_{1i} + m_2\vec{v}_{2i} = m_1\vec{v}_{1f} + m_2\vec{v}_{2f} \quad \text{, vector eq.}$$

$$\tfrac{1}{2}m_1\vec{v}_{1i}^2 + \tfrac{1}{2}m_2\vec{v}_{2i}^2 = \tfrac{1}{2}m_1\vec{v}_{1f}^2 + \tfrac{1}{2}m_2\vec{v}_{2f}^2 \quad \text{, scalar eq.}$$

Suppose that the masses and initial velocities of both particles are known. These two eqs. can be solved for the final velocities in terms of the initial velocities, since there are two eqs. and two unknowns. Solving for $\vec{v}_{1f}$ and $\vec{v}_{2f}$ we find

$$\vec{v}_{1f} = \frac{m_1 - m_2}{m_1 + m_2}\vec{v}_{1i} + \frac{2m_2}{m_1 + m_2}\vec{v}_{2i}$$

$$\vec{v}_{2f} = \frac{2m_1}{m_1 + m_2}\vec{v}_{1i} + \frac{m_2 - m_1}{m_1 + m_2}\vec{v}_{2i}$$

Note that the appropriate signs for $\vec{v}_{1i}$ and $\vec{v}_{2i}$ must be included in these two eqs. since they are vectors. For example, if m_2 is moving to the left initially, as in the above figure, then $\vec{v}_{2i}$ is negative.

Cases:

- $m_1 = m_2$ gives $\vec{v}_{1f} = \vec{v}_{2i}$ and $\vec{v}_{2f} = \vec{v}_{1i}$ i.e., the particles exchange velocities if they have equal masses.
- m_2 is at rest i.e. $\vec{v}_{2i} = 0$ gives

$$\vec{v}_{1f} = \frac{m_1 - m_2}{m_1 + m_2}\vec{v}_{1i}, \quad \vec{v}_{2f} = \frac{2m_1}{m_1 + m_2}\vec{v}_{1i}$$

In addition to the cond. $\vec{v}_{2i} = 0$, if $m_1 \gg m_2$, we find from these eqs. that

$$\vec{v}_{1f} \approx \vec{v}_{1i}, \quad \vec{v}_{2f} \approx 2\vec{v}_{1i}$$

Also, if $m_2 \gg m_1$, we get (again from these same eqs.)

$$\vec{v}_{1f} \approx -\vec{v}_{1i}, \quad \vec{v}_{2f} \approx 0.$$

Example:
A body of 2 kg mass makes an elastic collision with another body at rest and continues to move in the original direction but with one-fourth of its original speed. What is the mass of the struck body?

$$m_1 v_{1i} = m_1 \cdot \frac{1}{4} v_{1i} + m_2 v_{2f} \quad \text{— mom. conservation}$$

$$\frac{1}{2} m_1 v_{1i}^2 = \frac{1}{2} m_1 \frac{v_{1i}^2}{16} + \frac{1}{2} m_2 v_{2f}^2 \quad \text{— K.E. conservation}$$

The first eq. gives

$$v_{2f} = \frac{3}{4} \frac{m_1}{m_2} v_{1i}$$

(i.e.)

$$\tfrac{1}{2} m_1 v_{1i}^2 = \frac{1}{2} m_1 \frac{v_{1i}^2}{16} + \tfrac{1}{2} m_2 \frac{9}{16} \frac{m_1^2}{m_2^2} v_{1i}^2$$

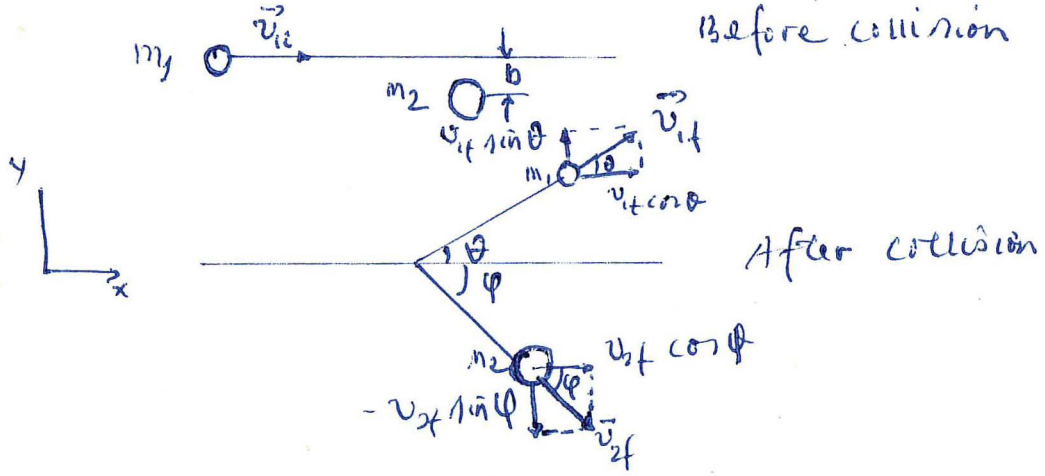
$$\frac{15}{16} m_1 v_{1i}^2 = \frac{9}{16} \frac{m_1^2}{m_2} v_{1i}^2$$

or

$$m_2 = \frac{9}{15} m_1 = \frac{9}{15} (2kg) = \frac{6}{5} kg = 1.2 kg.$$

§.5 Two-Dimensional Collisions

Consider a 2-D prob. in which a particle of mass m_1 collides with a particle of mass m_2 initially at rest, fig. below. The collision is not head-on, but glancing. The dist. b in the figure is called the impact parameter. As



it is seen in the figure, if b is zero, the collision is head-on. After the collision, m_1 moves at an angle θ wrt the horizontal and m_2 moves at an angle φ wrt the horizontal. Initially m_1 moves at an velocity $\vec{v}_{1i}$ along the x-axis and m_2 is at rest. Applying the conservation of mom. for the x- and y-comps. we find

$$m_1 v_{1i} = m_1 v_{1f}\cos\theta + m_2 v_{2f}\cos\varphi$$
$$0 = m_1 v_{1f}\sin\theta - m_2 v_{2f}\sin\varphi$$

Further, if we assume that the collision is elastic K.E. is conserved. or,

$$\tfrac{1}{2}m_1 v_{1i}^2 = \tfrac{1}{2}m_1 v_{1f}^2 + \tfrac{1}{2}m_2 v_{2f}^2$$

If we know the initial velocity, v_{1i}, and the masses, we are left with four unknowns. However, since there are three independent eqs., one of the four remaining quantities (v_{1f}, v_{2f}, θ, or φ) must be given to determine the motion after the collision from conservation principles alone.

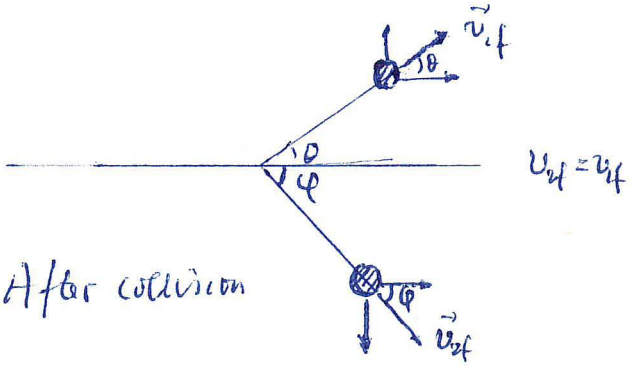
Examples:

1. A proton moving with a velocity $v_0\hat{i}$ collides elastically with another proton initially at rest. If both protons have the same speed after the collision, find a, the speed of each proton after the collision in terms of v_0 and b, the direction of the velocity vectors after the collision.

Solution:

a,

Before collision



After collision

$$m_1 v_{1i} = m_1 v_0 = m_1 v_{1f}\cos\theta + m_2 v_{1f}\cos\varphi$$
$$0 = m_1 v_{1f}\sin\theta - m_2 v_{1f}\sin\varphi \quad \Big\} \text{ Conserv. of mom.}$$
$$\tfrac{1}{2}m_1 v_0^2 = \tfrac{1}{2}m_1 v_{1f}^2 + \tfrac{1}{2}m_2 v_{1f}^2 \quad \text{- conserv. of K.E.}$$

Since the particles are of the same mass identical (two protons), $m_1 = m_2 = m_p$ and from the above eqs. we get

$$v_0 = v_{1f}\cos\theta + v_{1f}\cos\varphi$$
$$0 = v_{1f}\sin\theta - v_{1f}\sin\varphi \quad \Big\} \text{ Mom. conserv.}$$
$$\therefore v_0^2 = v_{1f}^2 + v_{1f}^2 \quad \text{- K.E. conserv.}$$

$\Rightarrow$ $\sin\theta = 1/\sqrt{2}\sin\varphi$ Squaring and adding the first two

The 3rd eq. above gives

$$v_0^2 = 2v_{1f}^2$$

or,

$$v_{1f} = \frac{v_0}{\sqrt{2}} = v_{2f}$$

b, squaring and adding the first two eqs. we find

$$\begin{aligned} v_0^2 &= v_{1f}^2\cos^2\theta + v_{1f}^2\cos^2\varphi + 2v_{1f}^2\cos\theta\cos\varphi \\ &\quad + v_{1f}^2\sin^2\theta + v_{1f}^2\sin^2\varphi - 2v_{1f}^2\sin\theta\sin\varphi \\ &= v_{1f}^2 + v_{1f}^2 + 2v_{1f}^2(\cos\theta\cos\varphi - \sin\theta\sin\varphi) \\ &= \frac{v_0^2}{2} + \frac{v_0^2}{2} + v_0^2\cos(\theta+\varphi) \\ &= v_0^2 + v_0^2\cos(\theta+\varphi) \end{aligned}$$

or,

$$\cos(\theta+\varphi) = 0$$

(i.e.)

$$\theta + \varphi = \pi/2$$

But again from the x-comp. conserv. of the y-comp. of the mom. we have

$$\sin\theta - \sin\varphi = 0 \quad \text{or} \quad \sin\theta = \sin\varphi$$

$\Rightarrow$ $\sin\theta = \sin(\pi/2 - \varphi) = \cos\varphi = \sin\varphi$

(i.e.)

$$\tan\varphi = 1. \text{ and } \varphi = \pi/4$$

Finally,

$$\theta = \pi/2 - \pi/4 = \pi/4 = \varphi.$$

7.6 Motion with a Variable Mass

In physics there are systems for which the mass is variable. The simplest example is that of a raindrop. While it falls, moisture may condense on its surface or water may evaporate, resulting in a change of mass. Suppose that the mass of the

drop is m when it is moving with velocity $\vec{v}$ and that moisture, whose velocity is $\vec{v}_0$, condenses on the drop at the rate dm/dt. The total rate of change of mom. is the sum of $m\,d\vec{v}/dt$, corresponding to the acc. of the drop, and $(dm/dt)(\vec{v}-\vec{v}_0)$, corresponding to the rate of gain of mom. of the moisture. Thus, the eq. of motion of the drop, from Newton's 2nd law, is

$$\vec{F} = m\frac{d\vec{v}}{dt} + (\vec{v}-\vec{v}_0)\frac{dm}{dt}$$

However, if we consider a system of variable mass m which moves at a const. velocity $\vec{v}$, the required eq. of motion is

$$\vec{F} = \vec{v}\,\frac{dm}{dt}.$$

A typical system that obeys this eq. is a rocket, whose mass decreases because it consumes the fuel it carries. In the following example we shall analyze the dynamics of a rocket.

Example: Discuss the motion of a rocket

Solution:

A rocket is a missile which moves under the action of a continuous force derived from the exhaust produced in the combustion chamber within itself. The rocket at take off has a certain amount of fuel which it uses gradually, and therefore its mass decreases.

Let $\vec{v}$ be the velocity of the rocket relative to an observer on earth and $\vec{v}'$ be the exhaust velocity of the gases, also relative to the earth. Then the exhaust velocity of the gases relative to the rocket is

$$\vec{v}_e = \vec{v}' - \vec{v}, \text{ which is usually const.}$$

Let m_0 be the mass of the (rocket + fuel) during take off, time $t = 0$, and m be the mass of the rocket + fuel at any later time. During a very small time interval dt, the mass of the system experiences a small change dm, which is negative because the mass decreases. In the same time interval the velocity of the rocket changes by $d\vec{v}$. The mom. of the system at time t is $\vec{p} = m\vec{v}$. The mom. at time $t+dt$, since $-dm$ is the positive value of the mass of the expelled gases, is

$$\vec{p}' = \underbrace{(m+dm)(\vec{v}+d\vec{v})}_{\text{Rocket}} + \underbrace{(-dm)\,\vec{v}'}_{\text{Gases}} = m\vec{v} + m\,d\vec{v} - (\vec{v}'-\vec{v})\,dm$$

or,

$$\vec{p}' = m\vec{v} + m d\vec{v} - \vec{v}_e dm$$
(neglected)

where we have the second-order term $dm\, d\vec{v}$. The change in mom. in the time dt is

$$d\vec{p} = \vec{p}' - \vec{p} = m d\vec{v} - \vec{v}_e dm$$

and

$$\frac{d\vec{p}}{dt} = m\frac{d\vec{v}}{dt} - \vec{v}_e \frac{dm}{dt}.$$

If $\vec{F}$ represents the external force acting on the rocket, the eq. of motion, according to Newton's 2nd law is

$$m\frac{d\vec{v}}{dt} - \vec{v}_e \frac{dm}{dt} = \vec{F}$$

The 2nd term on the left is called the thrust of the rocket, since it is equal to the "force" due to the escaping gases. In solving this eq., it is generally assumed that $\vec{v}_e$ is a const. Also, neglecting air resistance and the variation of gravity with altitude, we may write $\vec{F} = m\vec{g}$, so that the last eq. above becomes

$$\frac{d\vec{v}}{dt} - \frac{\vec{v}_e}{m}\frac{dm}{dt} = \vec{g}$$

To simplify, suppose that the motion is vertical. Then $\vec{v}$ is directed upward and $\vec{v}_e$ and $\vec{g}$, downward, and we write

$$\frac{dv}{dt} + \frac{v_e}{m}\frac{dm}{dt} = -g, \text{ scalar eq.}$$

Multiplying by dt and integrating from the beginning of the motion ($t=0$), when the velocity is v_0 and the mass is m_0, up to an arbitrary time t, we have

$$\int_{v_0}^{v} dv + v_e \int_{m_0}^{m} \frac{dm}{m} = -g\int_0^t dt$$

Then

$$v - v_0 + v_e \ln\frac{m}{m_0} = -gt$$

or,

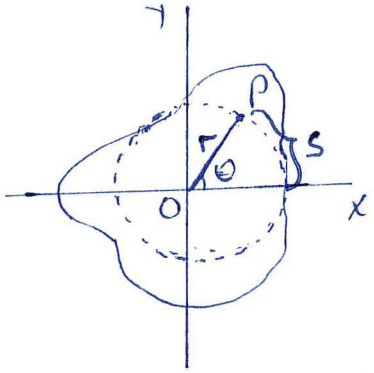
$$v(t) = v_0 + \ln\left(\frac{m_0}{m}\right) - gt$$

If t is the time required for burning all the fuel, then, in this eq., m is the final mass and v is the final velocity attained by the rocket. In general $v_0 = 0$, and the last term (in many cases) is negligible.

Chapter 8 Rotational Kinematics

8.1 Angular velocity and Angular Acceleration

The figure below shows a planar rigid body of arbitrary shape confined to the x-y plane and rotating about a fixed axis through O ⊥ to the plane of the figure. A particle on the



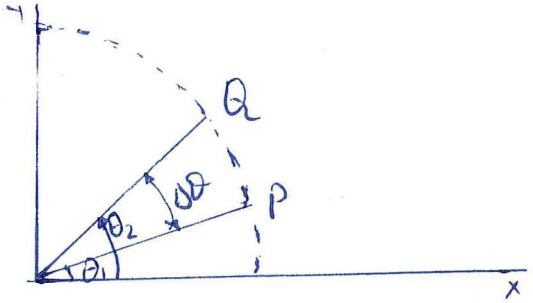
body at P is at a fixed dist. r from the origin and rotates in a circle of radius r centered at O. In fact, every particle on the body undergoes circular motion about O. As the particle moves along the circle from the +ve x-axis (θ=0) to the point P, it moves through an arc length S, which is related to the angular position through the relation

$$S = r\theta \quad \text{or} \quad \theta = S/r.$$

The angle θ in here is thus the ratio of an arc length to the radius of the circle, and it is represented by a pure no. Now, θ is said to be expressed in a unit called a radian (rad). One revolution corresponds to a rotation through a full circle which has a degree measure of 360°, i.e.,

$$\theta = 1\,\text{rev.} = 360^\circ = \frac{2\pi r}{r}\ 2\pi \left(= \frac{2\pi r}{r}\right) \text{rad.}$$

The fig. below shows the movement of the particle from a certain initial position P to another position Q in a time interval Δt.



Suppose that P is the position of the particle at time t_1 and Q is its position at a later time t_2. P and Q are

oriented at angles θ_1 and θ_2 from the x-axis, resp. Now, the angular displacement of the particle in a time interval $t_2 - t_1 = \Delta t$ is $\theta_2 - \theta_1 = \Delta\theta$. Then the average angular velocity $\bar{\omega}$ (omega) of the particle is defined as the ratio of the angular displacement $\Delta\theta$ to the time interval Δt:

$$\bar{\omega} = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{\Delta\theta}{\Delta t}$$

The instantaneous angular velocity is defined as the limit approached by this ratio as Δt approaches zero:

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$

For a rigid body all radial lines fixed in it $\perp$ to the axis of rotation rotate through the same angle in the same time, so that the angular speed ω about this axis is the same for each particle in the body. Thus ω is characteristic of the body as a whole.

Unit:

$$[\omega] = [\theta/t] = \text{rev/sec.}, \text{ degree/min.}, \text{ rad/sec.}, \text{ etc.}$$

If the instantaneous angular velocity of the body changes from ω_1 to ω_2 in the time interval Δt, the body has an angular acc. The average angular acc. $\bar{\alpha}$ (alpha bar) of a rotating body is defined as the ratio of the change in angular velocity $\Delta\omega$ $(= \omega_2 - \omega_1)$ to the time interval Δt $(= t_2 - t_1)$:

$$\bar{\alpha} = \frac{\omega_2 - \omega_1}{t_2 - t_1} = \frac{\Delta\omega}{\Delta t}$$

The instantaneous angular acc. is defined as the limit of this ratio as Δt approaches zero:

$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt}$$

Because ω is the same for all particles in the rigid body, it follows from this eq. that α must be the same for each particle and thus α, like ω, is a characteristic of the body as a whole.

Unit:

$$[\alpha] = [\omega/t] = \text{rev/min/min}, \text{ rad/s}^2 \text{ etc.}$$

The motion of a particle (or a rigid body) about a fixed axis has a formal correspondence to the translational motion of a particle (or a rigid body) along a fixed direction.

The kinematical variables are θ, ω, and α in the first case and x, v, and a in the second. These quantities correspond in pairs: θ to x, ω to v, and α to a.

8.2 Rotation with Constant Angular Acceleration

For translational motion, the simplest type of accelerated motion is rectilinear motion with const. acc. Likewise, for rotational motion about a fixed axis, the simplest type of motion is motion with const. angular acc. Let us now develop the kinematic relations for rotational motion with const. acc.:

1. $$\alpha = \frac{d\omega}{dt} \quad \text{or} \quad d\omega = \alpha\, dt$$

If $\omega_1 = \omega_0$ at the initial time, say $t_1 = 0$ and $\omega_2 = \omega$ at the later time $t_2 = t$, we integrate as follows

$$\int_{\omega_0}^{\omega} d\omega = \int_0^t \alpha\, dt$$

i.e.,
$$\omega - \omega_0 = \alpha t$$

or
$$\omega = \omega_0 + \alpha t$$

2. $$\omega = d\theta/dt$$

Then,
$$\omega_0 + \alpha t = d\theta/dt$$

and
$$d\theta = (\omega_0 + \alpha t)\, dt.$$

If at $t_1 = 0$, $\theta_1 = \theta_0$, and at $t_2 = t$, $\theta_2 = \theta$, we have

$$\int_{\theta_0}^{\theta} d\theta = \int_0^t (\omega_0 + \alpha t)\, dt$$

i.e.,
$$\theta - \theta_0 = \omega_0 t + \tfrac{1}{2}\alpha t^2$$

or,
$$\theta = \theta_0 + \omega_0 t + \tfrac{1}{2}\alpha t^2$$

3. From the result in (1) $t = (\omega - \omega_0)/\alpha$ and using this expression in the expression of θ in (2) we find

$$\theta = \theta_0 + \omega_0\left(\frac{\omega-\omega_0}{\alpha}\right) + \frac{1}{2}\alpha\left(\frac{\omega-\omega_0}{\alpha}\right)^2$$

$$= \theta_0 + \frac{\omega\omega_0 - \omega_0^2}{\alpha} + \frac{\omega^2 - 2\omega\omega_0 + \omega_0^2}{2\alpha}$$

or,

$$\theta - \theta_0 = \frac{2\omega\omega_0 - 2\omega_0^2 + \omega^2 - 2\omega\omega_0 + \omega_0^2}{2\alpha} = \frac{\omega^2 - \omega_0^2}{2\alpha}$$

or,

$$\omega^2 - \omega_0^2 = 2\alpha(\theta - \theta_0)$$

Notice that these kinematic expressions for rotational motion under const. angular acc. are of the same form as those for linear motion under const. linear acc. with the substitutions $x \to \theta$, $v \to \omega$, and $a \to \alpha$.

XP/3 ~~Relation/xxx/p/xx Relation Angular Angular~~

Examples:

1. The turn table of a record player rotates at $33\frac{1}{3}$ rev/min. and takes 90s to come to rest when switched off. Calculate a, its angular acc. and b, the no. of revolutions it makes before coming to rest.

solution:

a, $\omega_0 = 3.49$ rad/sec, $\omega = 0$ and $t = 90s$ Then,

$$\alpha = \frac{0 - 3.49\ \text{rad/sec}}{90\ \text{sec}} = -0.04\ \text{rad/s}^2$$

b,

$$\theta = \omega_0 t + \tfrac{1}{2}\alpha t^2 \quad ; \quad \omega_0 = 0$$

or,

$$\theta = (3.49\ \text{rad(sec.)})(90\text{sec.}) - \tfrac{1}{2} \times 4\times10^{-2}\ \frac{\text{rad}}{s^2} \times (90s)^2$$

$$= 152.1\ 5\ \text{rad}$$

$$= 24.2\ \text{rev.}$$

2. Rotating Wheel

A wheel rotates with a const. anoular acc of $3.5\ \text{rad/s}^2$. If the anoular velocity of the wheel is 2 rad/s at $t_1 = 0$, a, what angle does the wheel rotate through in 2S? b, what is the angular velocity at $t = 2S$?

0,

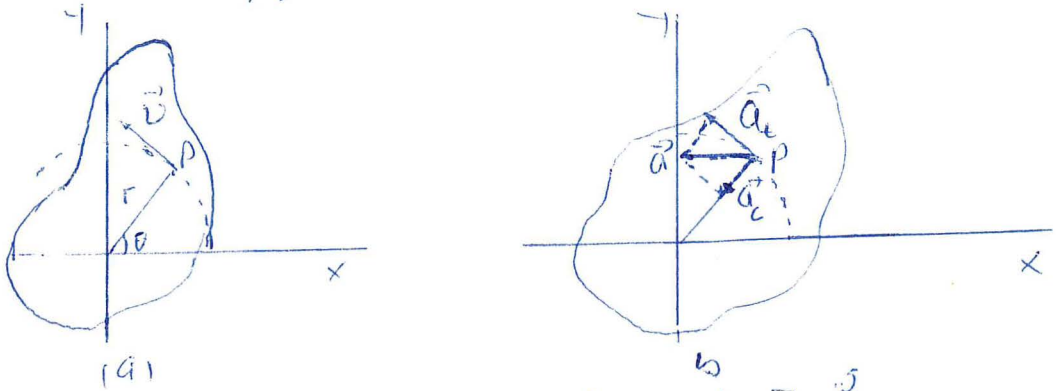
$$\theta - \theta_0 = \omega_0 t + \tfrac{1}{2}\alpha t^2 = 2\frac{rad}{s}(2s) + \tfrac{1}{2}\left(3.5\frac{rad}{s^2}\right)(2s)^2$$

$$= 11\ rad = 630^\circ = 1.76\ rev.$$

b, $\omega = \omega_0 + \alpha t = 2\ rad/s + (3.5\ rad/s^2)(2s) = 9.0\ rad/s.$

8.3 Relationship Between Linear and Angular Quantities

Consider the figure shown below.



We can first relate the angular velocity of the rotating body to the tangential speed, v, of a point on the body. Since P moves in a circle, the linear velocity vector is always tangent to the circular path, and hence the same is tangential velocity. If s is the dist. traveled by this point along the circular path, by def., the tangential velocity of the point P is given by

$$v = ds/dt$$

But

$$s = r\theta, \text{ where } r \text{ is const.}$$

and

$$v = \frac{ds}{dt} = r\frac{d\theta}{dt}$$

or,

$$v = r\omega$$

This result indicates that, although all particles of a rigid body have the same angular velocity, their linear velocities are diff. corresponding to their diff. distances from the axis of rotation.

The relation between the angular acc. α and the linear (or tangential) acc. a_t is found from the above eq., i.e.,

$$a_t = \frac{dv}{dt} = r\frac{d\omega}{dt}$$

or,

$$a_t = r\alpha$$

Again, the tangential acc. a_t has diff. values for diff. particles of the rigid body that are located at diff. distances from the axis of rotation regardless of the fact that these particles all have the same angular acc. α.

In chapter 4 we found that a point rotating in a circular path has a centripetal, or radial, acc. of magnitude v^2/r and directed toward the center of rotation. Since $v = r\omega$ for the point P on the rotating body, we can express this comp. of the total acc. as

$$a_r = \frac{v^2}{r} = r\omega^2$$

Therefore, the magnitude of the total acc. of the point P on the rotating rigid body is given by

$$a = \sqrt{a_t^2 + a_r^2} = \sqrt{r^2\alpha^2 + r^2\omega^4} = r\sqrt{\alpha^2 + \omega^4}.$$

Example:

A cylinder of radius 12 cm starts from rest and rotates about its axis with a const. angular acc. of 5 rad/s². At $t = 3$ s, what is a, its angular velocity? b, the linear velocity of a point on its rim? c, the radial and tangential components of acc. of a point on its rim?

Solution:

a) $\omega = \omega_0 + \alpha t = \alpha t = (5\ \text{rad/s}^2)(3\ \text{s}) = 15\ \text{rad/s}$.

b) $v = r\omega = 0.12\ \text{m} \times 15\ \text{rad/s} = 1.8\ \text{m/s}$

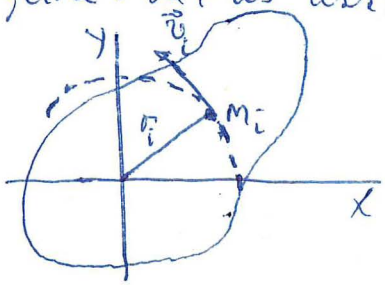
c) $a_r = \frac{v^2}{r} = \frac{(1.8\ \text{m/s})^2}{0.12\ \text{m}} = 27\ \text{m/s}^2$

$a_t = r\alpha = 0.12\ \text{m} \times 5\ \text{rad/s}^2 = 0.6\ \text{m/s}^2$

Chapter 9 Rotational Dynamics and the Conservation of Angular Momentum

9.1 Kinetic Energy of Rotation and Moment of Inertia

The figure below shows a rigid body rotating about the z-axis that is taken to be directed out of the plane of the figure. Let us assume that this rigid body rotates with



an angular velocity ω. Such an extended body may be considered as a collection of small particles of masses $m_1, m_2, \ldots$ that are located at diff. distances $r_1, r_2, \ldots$ from the axis of rotation. Each particle has some K.E., determined by its mass and velocity. If the mass of the i^{th} particle is m_i and its speed is v_i, the K.E. of this particle is

$$K_i = \tfrac{1}{2} m_i v_i^2$$

But from the relations between linear and angular quantities we have $v_i = r_i \omega$ which is diff. for diff. particles that are located at diff. distances from the axis of rotation. The total K.E. of the rotating body is the sum of the K.E.s of the individual particles:

$$K = \sum K_i = \sum \tfrac{1}{2} m_i v_i^2 = \frac{1}{2} \sum_i m_i r_i^2 \omega^2$$

Since ω is common to every particle we may take it out of the summation:

$$K = \frac{1}{2} \left(\sum_i m_i r_i^2 \right) \omega^2$$

The quantity in the parentheses is a property of the rigid body called the moment of inertia, I:

$$I = \sum_i m_i r_i^2$$

Using this notation, the expression of the K.E. becomes

$$K = \tfrac{1}{2} I \omega^2$$

unit:

$$[I] = [m r^2] = kg \cdot m^2 \text{ in SI and } g \cdot cm^2 \text{ in cgs units}$$

The expression of the K.E. of rotation, $\frac{1}{2} I \omega^2$, is analogous to that of the expression of the K.E. of linear motion, $\tfrac{1}{2} m v^2$, with I corresponding to m and ω corresponding to v.

9.1 Calculation of Moments of Inertia For Rigid Bodies

For a body that is not composed of discrete point masses but is instead a continuous distribution of matter, the summation $I = \sum_i m_i r_i^2$ becomes an integration. We imagine the body to be subdivided into infinitesimal elements, each of mass Δm. We let, as before, r be the dist. from such an element to the axis of rotation. Then the rotational inertia (or moment of inertia) is obtained from (in the limit $\Delta m \to 0$)

$$I = \int_{V.B.} r^2 \, dm$$

where the integral is taken over the whole body.

To evaluate the moment of inertia using the above eq., it is necessary to express the element of mass dm in terms of coordinates. It is common to define a mass density in various forms. For a 3-D body, we shall commonly use the local volume density, that is, mass per unit volume. In this case we can write

$$\rho = \lim_{\Delta V \to 0} \frac{\Delta m}{\Delta V} = \frac{dm}{dV}$$

or,

$$dm = \rho\, dV$$

Therefore, the moment of inertia can be expressed in the form

$$I = \int \rho r^2 dV$$

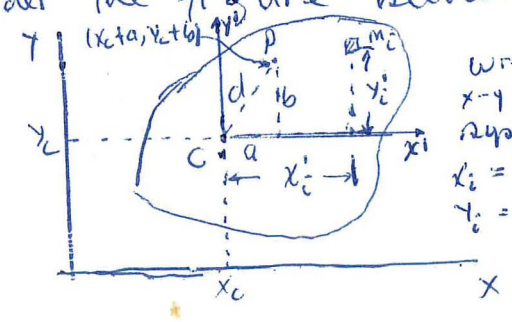
If the body is homogeneous, then ρ is const. and the integral can be evaluated for a known geometry. If ρ is not const., then its variation with position must be specified. When dealing with an object in the form of a sheet of uniform thickness t, it is convenient to define a surface density $\sigma = \rho t$, which signifies mass per unit area. Finally, when mass is distributed along a uniform rod of cross-sectional area A, we use linear density, $\lambda = \rho A$, where λ is defined as mass per unit length.

The calculation of moments of inertia about an arbitrary axis can be somewhat cumbersome, even for a highly symmetric body, such as a sphere. In this regard, there is an imp. theorem, called the parallel-axis theorem, that often simplifies the calculation of moments of inertia. Suppose that the moment of inertia about any axis through the cm is I_c. The parallel axis theorem states that the moment of inertia about any axis that is parallel to and a dist. d away from the axis that passes through the cm is given by

$$I = I_c + Md^2, \quad M \text{ is the total mass of the body.}$$

Proof of the Parallel Axis Theorem:

Consider the figure below.



wrt the x-y coordinate system

$x_i = x_c + x_i'$

$y_i = y_c + y_i'$

C - center of mass, coinciding with x'-y' syst.

P - An axis that is a dist. d from the axis through the cm.

m_i - A typical mass element whose coordinate wrt the axis through the cm is (x_i', y_i').

So, the moment of inertia about the parallel axis through P is

$$I_p = \sum_i m_i[(x_i' - a)^2 + (y_i' - b)^2$$

$$= \sum_i m_i(x_i'^2 + y_i'^2) - 2a\sum_i m_i x_i' - 2b\sum_i m_i y_i' + (a^2+b^2)\sum_i m_i$$

From the def. of the cm.

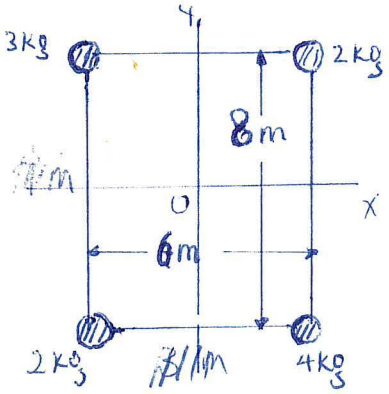
$$\sum_i m_i x_i' = 0, \quad \sum_i m_i y_i' = 0.$$

The first term is simply the moment of inertia about an axis through the cm I_c and the last term is Md^2. Hence it follows that

$$I_p = I_c + Md^2.$$

Examples:

1. The four particles in the figure below are connected by light, rigid rods. If the system rotates in the x-y plane about the z-axis with an angular velocity of 8 rad/s, calculate a, the moment of inertia of the system about the z-axis and b, the K.E. of the system.



a, Each particle is a dist. say, d, away from the z-axis:

$$d = \sqrt{(3m)^2 + (4m)^2} = \sqrt{25}\ m = 5m$$

so,

$$I = (3kg + 2\times 2kg + 4kg)(5m)^2$$

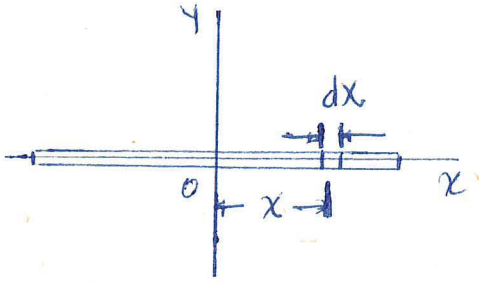
$$= 275\ kg\cdot m^2$$

b, $K = \frac{1}{2} I\omega^2 = \frac{1}{2} \times 275\ kg\cdot m^2 \times (8\ rad/s)^2$

$$\simeq 8,800\ J.$$

2. Uniform Rigid Rod

A uniform rod (shown below) of length L and mass M rotates about an axis perpendicular to the rod (y-axis) passing through its center with an angular velocity ω. a, calculate its moment of inertia about this axis. b, its K.E. of rotation.



a, The shaded element of width dx has a mass dm equal to the mass per unit length multiplied by the element of length, dx, ie., $dm = (M/L)\,dx$. So now with $r = x$, we have

$$I_y = \int r^2\,dm = \int_{-L/2}^{L/2} x^2 \frac{M}{L}\,dx = \frac{M}{L}\int_{-L/2}^{L/2} x^2\,dx = \frac{M}{L}\left[\frac{x^3}{3}\right]_{-L/2}^{L/2}$$

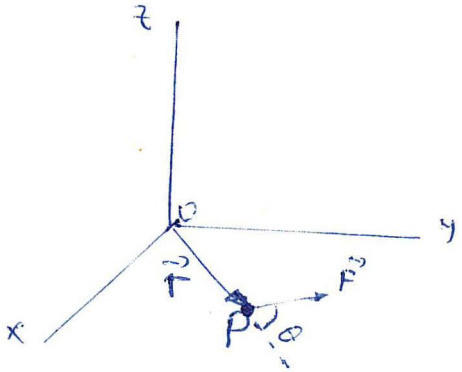
or,

$$I_y = \tfrac{1}{12} ML^2.$$

5, $K = \frac{1}{2} I_y \omega^2 = \frac{1}{24} ML^2\omega^2$.

9.2 Torque

In translational motion, the acc. of a body is associated with a resultant force acting on the body that acts on the body. In rotational motion, however, the angular acc. of a body is associated with a different dynamical quantity called the torque. To define torque property consider the figure below where a particle at the point P in the body with position vector $\vec{r}$ is acted upon by a force $\vec{F}$. This force acting at some dist. from the origin O



of the x-, y- and z- coordinate system, has a tendency of rotating the particle at P wrt the origin O; that is, $\vec{F}$ has the ability to rotate the particle and it is this effect of a force which is called torque. Now, the torque $\vec{\tau}$ acting on the particle about wrt the origin O is defined as

$$\vec{\tau} = \vec{r} \times \vec{F}$$

The magnitude of the torque vector is

$$\tau = rF\sin\theta$$

where θ is the angle between $\vec{r}$ and $\vec{F}$; its direction is normal to the plane formed by $\vec{r}$ and $\vec{F}$ - the right hand rule.

Notice from the above eq that the torque produced by a force depends not only on the magnitude and on the direction of the force but also on the point of application of the force relative to the origin, that is, on the vector $\vec{r}$. In particular, when the particle is at the origin so that the line of action of $\vec{F}$ passes through the origin, $\vec{r}$ is zero and the torque $\vec{\tau}$ about the origin is zero. Now, there is no rotation and hence no angular acc. regardless of the magnitude of the force exerted on the particle. This fact tells us that rotational motion is the result of a nonzero torque acting on the body.

We can also write the magnitude of $\vec{\tau}$ as

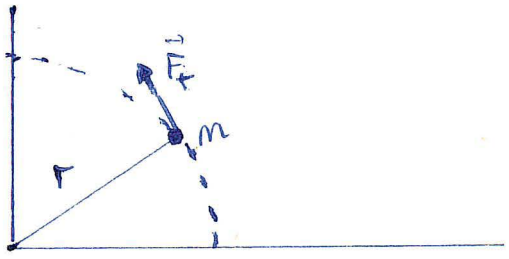
$$\tau = (r \sin\theta) F = r_\perp F \quad \text{or} \quad \tau = r(F\sin\theta) = r F_\perp$$

The quantity $r\sin\theta$, the $\perp$ dist. between O and the line of action of $\vec{F}$, is called the moment arm (or lever arm) of $\vec{F}$, on the other hand, the quantity $F\sin\theta$ ($= F_\perp$) is the comp. of $\vec{F}$ at right angles to $\vec{r}$ and it is only this comp. that contributes to the torque.

9.3 Relationship between Torque and Angular Acceleration

Here we shall first discuss the case of a particle rotating about some fixed point under the influence of an external force and extend the ideas developed in this situation to the case of a rigid body rotating about a fixed axis.

Then, as shown in the figure below, consider a particle of mass m rotating in a circle of radius r under the influence of a tangential force $\vec{F}_t$. This force, as discussed



in chapter 3, brings about a change of the magnitude of the tangential velocity v_t resulting in a tangential acc. a_t. From Newton's 2nd law we thus write

$$F_t = m\frac{dv_t}{dt} = ma_t$$

The torque about the origin due to the force $\vec{F}_t$ is the product of the magnitude of the force, F_t, and the moment arm of the force:

$$\tau = F_t r = m a_t r$$

Since $a_t = r\alpha$ (α is the angular acc.), the torque can be expressed as

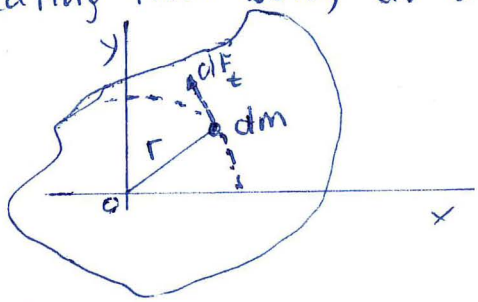
$$\tau = m r \alpha r = (mr^2)\alpha$$

But we recall that the quantity mr^2 is the moment of inertia I of the rotating particle about the origin so that

$$\tau = I\alpha.$$

That is, the torque acting on the particle is proportional to its angular acc. The constant of proportionality is I. It follows now that $\tau = I\alpha$ is the rotational analogue of Newton's 2nd law for translational motion, $F = ma$.

To extend this discussion to a rigid body rotating about a fixed axis, the figure below is considered. Treating this body as a collection of an infinite no. of



(particles) mass elements, each having an infinitesimal mass dm, its rotational motion is described as follows. If a force $d\vec{F}_t$ acts on a typical mass element of the rigid body, this force produces rotational motion since it is tangent to the circular path of the different mass element (see above). On each mass element tangent to its circle of rotation, there will result a tangential acc. $\vec{a}_t$ of this each and from Newton's 2nd law we have

$$dF_t = (dm)\, a_t$$

The torque $d\tau$ associated with the force $d\vec{F}_t$ acting about the origin is

$$d\tau = r\, dF_t = r(dm) a_t$$

once again, $a_t = r\alpha$, and we find

$$d\tau = (r\,dm)\,r\alpha = (r^2 dm)\,\alpha .$$

We know, however, that although each point of a rigid body has diff. tangential acc. a_t, the angular acc. α is the same for all of them. With this fact at hand, the total torque acting on the rigid body about O is found by integrating (adding) such contributions as

$$\tau_{net} = \int r^2 dm\,\alpha = \alpha \int r^2 dm$$

We recall that the quantity $\int r^2 dm$ is the moment of inertia I of the rigid body about O. Therefore, we write

$$\tau_{net} = I\alpha$$

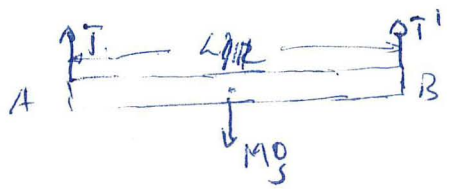
and we see that the net torque about the rotation axis through O is proportional to the angular acc. of the body.

Finally, one should note that the result $\tau = I\alpha$ would also apply if the forces acting on the mass elements had radial comps. as well as tangential comps. This is because the line of action of all radial comps. must pass through the axis of rotation, and hence would produce zero torque about that axis.

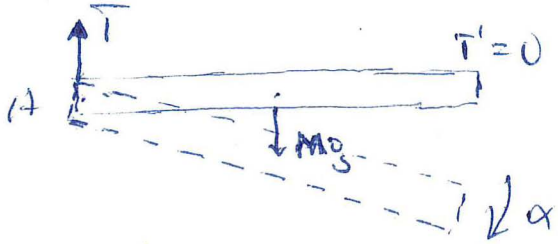
Examples:

1. A uniform horizontal plank of mass M and length L is supported at each end by vertical ropes. At the instant one of the ropes breaks, show that a, the angular acc. of the plank is $3g/2L$ b, the acc. of the cm is $3g/4$ and c, the tension in the supporting rope is $Mg/4$.

Solution



(a)



(b)

a, Fig. (a) shows the situation before one of the ropes say, the right rope breaks. So this is a situation of equil. In fig (b) the right rope is broken and the rod immediately starts to rotate clockwise about its left end. Then the torque about this end is

$$\tau_A = Mg \cdot \frac{L}{2}$$

The moment of inertia of the rod about this same end from the parallel-axis theorem is given by

$$I_A = I_{cm} + M(\tfrac{L}{2})^2 = \tfrac{1}{12}ML^2 + M\tfrac{L^2}{4} = \tfrac{1}{3}ML^2$$

From Newtons 2nd law for rotation

$$\tau_A = I_A \alpha$$

or,

$$Mg \cdot \frac{L}{2} = \frac{1}{3}ML^2\alpha$$

i.e.,

$$\alpha = 3g/2L$$

b, From the relation between tangential acc. and angular acc. we have

$$a_t = r\alpha$$

But the cm of the rod, coinciding with its geometrical center, is a dist. $\frac{L}{2}$ from end A. Thus,

$$a_t = \frac{L}{2} \cdot \frac{3g}{2L} = (3/4)g$$

c, Applying Newtons 2nd law to the gross motion of the rod (motion of the cm) we have

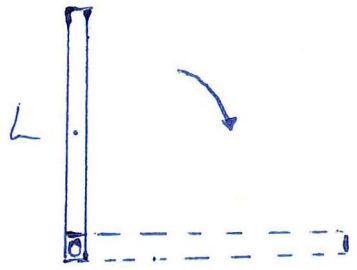
$$Mg - T = Ma_t$$

or,

$$T = Mg - Ma_t = Mg - \tfrac{3}{4}Mg = Mg/4.$$

2. A long uniform rod of length L and mass M is pivoted about a horizontal, frictionless pivot pin through one end. The rod is released from rest in a vertical position as in the figure below. At the instant the rod is horizontal, find a, the angular velocity of the rod, b, its angu-

lar acc., c, the x and y comps. of the reaction acc. of its cm, and d, the comps. of the reaction force at the pivot.



solution:

a, When the rod changes its vertical position to the horizontal position shown above, its p.E. will be converted to its rotational k.E. about O.

$$Mg \cdot \frac{L}{2} = \frac{1}{2} I_O \omega^2 = \frac{1}{2} \cdot \frac{1}{3} ML^2 \omega^2$$

or,

$$\omega = \sqrt{3g/L}$$

b,

$$\tau_O = I_O \alpha$$

$$Mg \cdot \frac{L}{2} = \frac{1}{3} ML^2 \alpha$$

or,

$$\alpha = 3g/2L$$

c, The x-comp. of the acc. of the cm is equal to the centripetal acc. of the cm; i.e.,

$$a_x = a_c = \omega^2 r = (3g/L) \cdot L/2 = (3/2)g$$

The y-comp. of this acc. is just the tangential acc.
or,

$$Mg - F = Ma_t \qquad a_t = r\alpha$$

or,

$$a_t = \frac{L}{2} \cdot \frac{3g}{2L} = 3g/4$$

d, The x-comp. of the reaction force produces the centripetal acc. so that

$$R_x = M a_x = M \cdot \frac{3}{2} g_s = \frac{3}{2} M g_s$$

And for the y-comp. we have

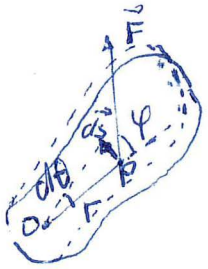
$$Mg - R_y = Ma_t = M \cdot \frac{3}{4} g$$

or,

$$R_y = Mg - \frac{3}{4} Mg = Mg_s/4 .$$

10·3 Work and Energy in Rotational Motion

Consider a rigid body pivoted at the point O in the figure below.



Suppose a single external force $\vec{F}$ is applied at the point P. The work done by $\vec{F}$ as the body rotates through an infinitesimal distance $ds = r d\theta$ in a time dt is

$$dW = \vec{F} \cdot \vec{ds} = (F \sin\varphi) r d\theta$$

where $F \sin\varphi$ is the tangential comp. of $\vec{F}$, or the comp. of the force F along the displacement. Note that the radial comp. of $\vec{F}$ in the above figure does no work since it is $\perp$ to the displacement.

But $rF\sin\varphi$ is the magnitude of the torque due to $\vec{F}$ about the origin O. Therefore,

$$dW = \tau d\theta$$

The rate at which work is done by $\vec{F}$ for the rotation about the fixed axis is

$$\frac{dW}{dt} = \tau \frac{d\theta}{dt}$$

But dW/dt is, by def., the instantaneous power, P, delivered by the force. Furthermore, since $d\theta/dt = \omega$, we write

$$P = \tau\omega .$$

This expression is analogous to $P = Fv$ in the case of linear motion, and the expression $dW = \tau\, d\theta$ is analogous to $dW = F dx$.

The Work-Energy Theorem in Rotational Motion:

From Newton's second law for rotational motion we have

$$\tau = I\alpha$$

Using the chain rule from calculus we further write

$$\tau = I\alpha = I\frac{d\omega}{dt} = I\frac{d\omega}{d\theta}\frac{d\theta}{dt} = I\omega\frac{d\omega}{d\theta}$$

which gives

$$\tau\, d\theta = dW = I\omega\, d\omega$$

If θ changes from an initial value θ_0 to a final value θ, and ω changes from ω_0 to ω, integration gives

$$W = \int_{\theta_0}^{\theta} \tau\, d\theta = \int_{\omega_0}^{\omega} I\omega\, d\omega = \frac{1}{2}I\omega^2 - \frac{1}{2}I\omega_0^2$$

which is analogous to the expression for the work-energy theorem in translational motion with m replaced by I and v replaced by ω. That is, the net work done by external forces in rotating a rigid body about a fixed axis equals the change in the body's rotational K.E.

Example:

A wheel 1m in diameter rotates on a fixed, frictionless, horizontal axis. Its moment of inertia about this axis is 5kg·m². A const. tension of 20 N is maintained on a rope wrapped around the rim of the wheel, so as to cause the wheel to accelerate. If the wheel starts from rest at $t=0$, find a, the angular acc. of the wheel, b, the wheel's angular speed at $t=3$s, c, the K.E. of the wheel at $t=3$s, and d, the length of the rope unwound in the first 3s.

Solution:

a, From Newton's 2nd law for rotation,

$$\tau = I\alpha \quad \text{or} \quad TR = I\alpha$$

(i.e.)

$$\alpha = \frac{TR}{I} = \frac{20N \times .5m}{5kg \cdot m^2} = 2\ rad/s^2$$

b, From the eqs. for rotation with const. angular acc.

$$\omega = \omega_0 + \alpha t = (2\ rad/s^2)(3s) = 6\ rad/s.$$

c,

$$K = \tfrac{1}{2} I\omega^2 = \frac{1}{2} \cdot 5kg \cdot m^2 \times (6\ rad/s)^2 = 90.0\ J.$$

d, The work done by the torque due to the tension is

$$\tau\theta = \tfrac{1}{2} I\omega^2, \quad \omega_0 = 0.$$

or,

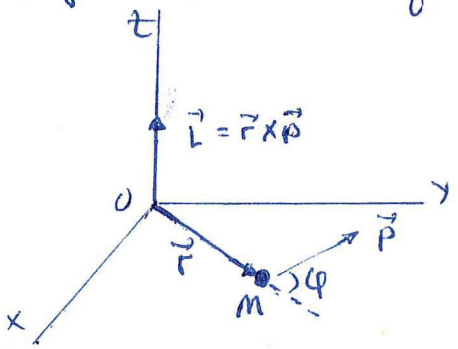
$$\theta = \frac{I\omega^2}{2\tau} = \frac{90\ J}{20N \times .5m} = 4.5\ rad \times 2 = 9\ rad$$

And the length of the rope unwound in the time interval of 3 s is

$$s = r\theta = 0.5m \times 9\ rad = 4.5m.$$

10.4 Angular Mom. of A particle

In the figure below a particle of mass m, located with position vector $\vec{r}$ moves with a velocity $\vec{v}$ and linear mom. $\vec{p}$. The instantaneous angular mom. $\vec{L}$ of the particle relative to the



the origin O is defined by the cross product of its instantaneous position vector $\vec{r}$ and the instantaneous linear mom. $\vec{p}$:

$$\vec{L} = \vec{r} \times \vec{p} \qquad \cdots (*)$$

The magnitude and direction of $\vec{L}$ depend on the choice of the origin. The direction of $\vec{L}$ as determined by the right hand

rule is perpendicular to the plane formed by $\vec{r}$ and $\vec{p}$. For example, in the above figure $\vec{r}$ and $\vec{p}$ are assumed to be in the x-y plane, so that $\vec{L}$ points in the z-direction. Since $\vec{p} = m\vec{v}$, the magnitude of $\vec{L}$ is given by

$$L = rp\sin\varphi = mvr\sin\varphi$$

where φ is the angle between $\vec{r}$ and $\vec{p}$. It follows that $\vec{L}$ is zero when $\vec{r}$ is parallel to $\vec{p}$ ($\varphi = 0$ or 180°). In other words, when the particle moves along a line that passes through the origin, it has zero angular mom. wrt the origin. This is equivalent to stating that it has no tendency to rotate about the origin. On the other hand, if $\vec{r}$ is $\perp$ to $\vec{p}$ ($\varphi = 90^\circ$), then L is a max. and equal to mvr. In this case, the particle has max. tendency to rotate about the origin. In fact, at that instant the particle moves exactly as though it were on the rim of a wheel rotating about the origin in a plane defined by $\vec{r}$ and $\vec{p}$.

We now derive an imp. relation between torque and angular mom. We have from Newtons 2nd law for translational motion that

$$\vec{F} = \frac{d}{dt}(m\vec{v}) = \frac{d\vec{p}}{dt}$$

Taking cross product of $\vec{r}$ with both sides of this eq., we get

$$\vec{r}\times\vec{F} = \vec{r}\times\frac{d\vec{p}}{dt}$$

Since $\vec{r}\times\vec{F}$ is the torque $\vec{\tau}$ of the force $\vec{F}$ acting on the particle about the origin O, we can write

$$\vec{\tau} = \vec{r}\times d\vec{p}/dt$$

Next, let us differentiate (*) wrt time

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r}\times\vec{p})$$

But the derivative of a vector product is taken in the same way as the derivative of an ordinary product, except that we must not change the order of the terms. It then follows

$$\frac{d\vec{L}}{dt} = \frac{d\vec{r}}{dt}\times\vec{p} + \vec{r}\times\frac{d\vec{p}}{dt}$$

But $d\vec{r}/dt$ is the instantaneous velocity $\vec{v}$ of the particle.

Also, $\vec{p} = m\vec{v}$, so that the eq. can be written as

$$\frac{d\vec{L}}{dt} = \vec{v} \times m\vec{v} + \vec{r} \times \frac{d\vec{p}}{dt}$$

Now, $\vec{v} \times m\vec{v} = 0$, since $\vec{v}$ and $m\vec{v}$ are parallel vectors. Therefore,

$$\frac{d\vec{L}}{dt} = \vec{r} \times \frac{d\vec{p}}{dt}$$

And this shows that

$$\vec{\tau} = d\vec{L}/dt \quad \cdots (**)$$

This means the time rate of change of the angular mom. of a particle is equal to the torque acting on it. We see that the above result is the rotational analog of Newtons second law for translational motion, $\vec{F} = d\vec{p}/dt$. Eq.(**), like all vector eqs., is equivalent to three scalar eqs., namely

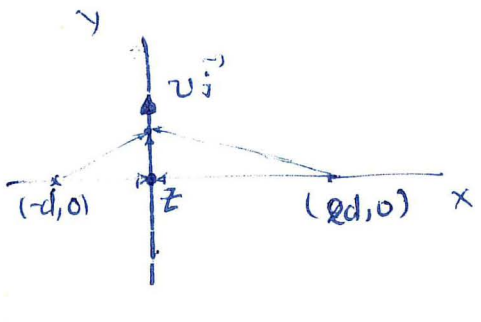
$$\tau_x = \frac{dL_x}{dt}, \quad \tau_y = \frac{dL_y}{dt}, \quad \tau_z = \frac{dL_z}{dt}$$

Hence, the x- comp. of the applied torque is given by the time derivative of the x- comp. of the angular mom., and so on.

Examples:

1. A particle of mass m moves in a straight line with a const. velocity $\vec{v} = v\vec{j}$ along the positive y-axis. Determine the angular mom. of the particle (both magnitude and direction) relative to a, the point having coordinates $(-d, 0)$ b, the point having coordinates $(2d, 0)$, and c, the origin

Solution:



a, the position vector of the particle wrt this point is $\vec{r} = d\vec{i} + y\vec{j}$

$$\frac{d\vec{L}}{} = \vec{L} = \vec{r} \times \vec{p}$$

$$= d\,mv(\vec{i} \times \vec{j}) + mvy(\vec{i} \times \vec{j})$$

$$= d\,mv\,\vec{k} + 0$$

$$\therefore dmv\,\vec{k}$$

(i.e.) $|\vec{L}| = mvd$, $\vec{L}$ points along the +ve z-axis (i.e.) out of the plane of the figure above.

b, $\vec{r} = -2d\vec{i} + y\vec{j}$ and $\vec{L} = \vec{r} \times \vec{p} = -2d\,mv(\vec{i}\times\vec{j})$

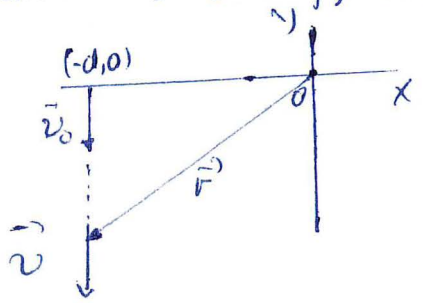
or,

$$\vec{L} = -2d\,mv\,\vec{k}$$

i.e., $|\vec{L}| = 2mvd$ and $\vec{L}$ is directed into the plane of the figure, toward the -ve z-axis.

c, Relative to the origin, $\vec{r} = y\vec{j}$ and the particle will have no angular mom. since $\vec{r}\times\vec{p} = mvy(\vec{i}\times\vec{j}) = 0$

2. A particle of mass m is given a velocity $-v_0\vec{j}$ at the point $(-d, 0)$ and proceeds to accelerate in the presence of earth's gravity (figure below). a, Find an expression for the angular mom. as a function of time wrt the origin. b, Calculate the torque acting on the particle at any time relative to the origin c, using your results to a, and b, verify that $\vec{\tau} = d\vec{L}/dt$.



Solution:

a, Let us find the velocity of the particle as a function of time.

$$m\frac{d\vec{v}}{dt} = -mg\vec{j} \quad \text{or} \quad m\frac{dv}{dt} = -mg, \text{ 1-D motion}$$

$$\Rightarrow \quad v(t) = -gt + c$$

But $v(0) = -v_0 = c$

$$\therefore \quad \vec{v}(t) = -(v_0 + gt)\vec{j}$$

Then we find

$$\vec{p}(t) = -m(v_0 + gt)\vec{j}$$

But $\vec{r}(t) = -d\hat{i} - (v_0 t + \frac{1}{2} g t^2)\hat{j}$.

It follows then

$$\vec{L} = \vec{r} \times \vec{p} = d\,m(v_0 + gt)\hat{k} = m(v_0 + gt)\,d\,\hat{k}$$

b,

$\vec{\tau} = \vec{r} \times \vec{F}$, $\vec{F} = -mg\hat{j}$ so that we get

$$\vec{\tau} = mgd\,\hat{k}$$

c. Using the result from (a)

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}[m(v_0 + gt)d]\hat{k} = mgd\,\hat{k}$$

which equals the result in (b). That is,

$$\vec{\tau} = d\vec{L}/dt$$

10.5 Angular Mom. and Torque for a System of Particles

The total angular mom., $\vec{L}$, of a system of particles about some point is defined as the vector sum of the angular momenta of the individual particles:

$$\vec{L} = \vec{L}_1 + \vec{L}_2 + \cdots + \vec{L}_n = \sum_i \vec{L}_i$$

where n is the no. of particles in the system.

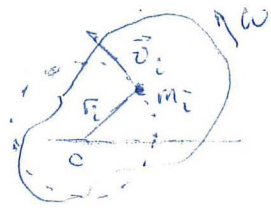
Since the individual momenta of the particles may vary in time, the total angular mom. may also vary in time. From the discussion in the previous section we see that the time rate of change of the total angular mom. equals the vector sum of all torques, including those associated with internal forces between particles and those associated with external forces. However, the net torque associated with internal forces is zero and this is a consequence Newtons third law. Finally, we conclude that the total angular mom. can vary with time only if there is a net external torque on the system, so that we have

$$\sum \vec{\tau}_{ext} = \sum_i \frac{d\vec{L}_i}{dt} = \frac{d}{dt}\sum_i \vec{L}_i = \frac{d\vec{L}}{dt}$$

That is, the time rate of change of the total angular mom. of the system about some origin in an inertial frame equals the net external torque acting on the system about that origin.

This is the rotational analog of $\vec{F}_{ext} = d\vec{p}/dt$ for a system of particles.

Next consider a planar rigid body rotating about a fixed axis O through as in the fig. shown below. Each particle of the rigid



body rotates about the same axis in a circle with the same angular velocity ω. The angular mom. of the i^{th} particle of mass m_i has magnitude $m_i v_i r_i$ about O. Its direction is the same as that of ω. Since $v_i = r_i \omega$, we can also write this angular mom. as $m_i r_i^2 \omega$. The total angular mom. is the sum of all such contributions of the particles of the body so that

$$L = \left(\sum_i m_i r_i^2\right)\omega = I\omega$$

where I is the ~~angular~~ mom. of inertia of the rigid body about the axis of rotation O. Note that this eq. also applies to rotation of a symmetric rigid body about a fixed axis through its CM.

If we differentiate this eq. wrt time assuming I to be const. (true for a rigid body) we find

$$\frac{d\vec{L}}{dt} = I\frac{d\vec{\omega}}{dt}$$

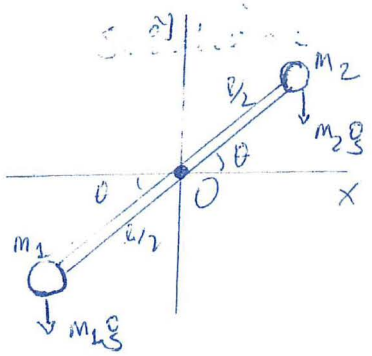
or since $d\vec{\omega}/dt = \vec{\alpha}$, the angular acc., and $\frac{d\vec{L}}{dt} = \sum \vec{\tau}_{ext}$ we write

$$\sum \vec{\tau}_{ext} = I\vec{\alpha}$$

That is, the net external torque acting on a rigid body equals the moment of inertia about the axis of rotation multiplied by its angular acc. For rotation about a fixed axis, $\vec{\alpha}$ is in the same direction as $\vec{\omega}$ if ω is increasing in time and in the opposite direction if ω is decreasing in time.

Example: Rotating Rod

A rigid rod of mass M and length L rotates in a vertical plane about a frictionless pivot through its center (below). Particles of masses m_1 and m_2 are attached at the ends of the rod. a) Determine the angular mom. when the angular velocity is ω. b) Determine the angular acc. of the system when the rod makes an angle θ with the horizontal.



Solution:

a, The moment of inertia of the system equals the sum of the moments of inertia of the two particles and the rod. So

$$I = \frac{1}{12} M\ell^2 + m_1(\ell/2)^2 + m_2(\ell/2)^2 = \frac{\ell^2}{4}\left(\frac{M}{3} + m_1 + m_2\right)$$

Therefore, the angular mom. is given by

$$L = I\omega = \frac{\ell^2}{4}\left(\frac{M}{3} + m_1 + m_2\right)\omega$$

b, due to $m_1 g$ the torque is

$$\tau_1 = m_1 g \frac{\ell}{2}\cos\theta \quad (\text{out of the plane})$$

due to $m_2 g$ the torque is

$$\tau_2 = -m_2 g \frac{\ell}{2}\cos\theta \quad (\text{into the plane})$$

Hence, the net torque τ about O is

$$\tau = \tau_1 + \tau_2 = \frac{\ell}{2}(m_1 - m_2) g \cos\theta$$

and this has to be equal to $I\alpha$. Or,

$$\alpha = \frac{\tau}{I} = \frac{2(m_1 - m_2) g \cos\theta}{\ell\left(\frac{M}{3} + m_1 + m_2\right)}$$

20.6 Conservation of Angular Mom.

We found that the total linear mom. of a system of particles remains const. when the resultant external force acting on the system is zero (Chapter 7). We have an analogous conservation law in rotational motion which states that the total vector angular mom. of a system is const. if the resultant external torque acting on the system is zero. That is,

$$\frac{d\vec{L}}{dt} = \sum \vec{\tau}_{ext} = 0$$

and hence

$$\vec{L} = \text{const.}$$

For a system of particles, we write this conservation law $\sum_i \vec{L}_i = \text{const.}$ If a body undergoes a redistribution of its mass, then its moment of inertia changes and we express this conservation of angular mom. in the form

$$L_i = L_f = \text{constant}$$

For rotations either about a fixed axis or about an axis through the CM of the system such that the axis remains parallel to itself we write

$$I_i \omega_i = I_f \omega_f$$

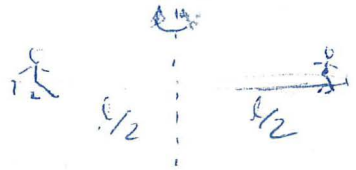
The only requirement is that the net torque acting on the rigid body be zero.

Example:

Two girls each of mass M, sit on opposite ends of a thin rod with length L and mass M (the same as the mass of each girl). The rod is pivoted at its center and is free to rotate in a horizontal circle without friction. a, What is the rotational inertia of the rod plus the girls about a vertical axis through the center of the rod? b, What is the angular mom. of the system if it is rotating with an angular speed ω_0 in a clockwise direction as seen from above? What is the direction of the angular mom? c, While the system is rotating, the girls pull themselves toward the center of the rod until they are half as far from the center as before. What is the resulting angular speed in terms of ω_0? d, What is the change in K.E. of the system due to the girls changing their positions? How is energy conserved during this motion?

Solution

a,



$$I_0 = M(\ell/2)^2 + \frac{M\ell^2}{12} + M(\ell/2)^2 = \frac{7}{12} M\ell^2$$

b,

$$L = I\omega_0 = \frac{7}{12} M\ell^2 \omega_0$$

If one crunch his fingers in the sense of rotation with the thumb being ⊥ to this fingers, the direction of the angular mom. is indicated by the thumb and it is down in our case.

c, When the girls pull in themselves toward the axis, there is no net torque acting and hence angular mom. is conserved. or

$$I_0\omega_0 = I\omega \quad \text{or} \quad \omega = \frac{I_0}{I}\omega_0$$

But,

$$I = M(\tfrac{\ell}{4})^2 + \frac{M\ell^2}{12} + M(\ell/4)^2 = \frac{10}{48} M\ell^2$$

$$\Rightarrow \quad \omega = \frac{7}{12} M\ell^2\omega_0 \Big/ \frac{10}{48} M\ell^2 = \frac{14}{5}\omega_0$$

d,

$$K_0 = \tfrac{1}{2} I_0\omega_0^2 = \frac{7}{24} M\ell^2\omega_0^2 = 0.3\, M\ell^2\omega_0^2$$

$$K = \tfrac{1}{2} I\omega^2 = \frac{5}{48} M\ell^2 \left(\frac{196}{25}\omega_0^2\right) = 0.82\, M\ell^2\omega_0^2$$

(i.e.)

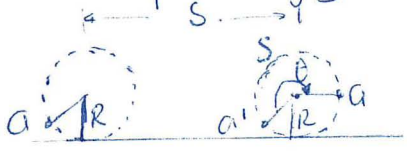
$$K = 2.73\, K_0$$

Work is done by the girls while moving close to the axis and this results in an increase of K.E.

10.7 Rolling Motion of A Rigid Body

Here we analyze the motion of a rigid body that is rotating about a moving axis. Since the general motion of a rigid body is complex, we restrict our discussion to a homogeneous rigid body having a high degree of symmetry, such as a cylinder, sphere or hoop.

Consider a cylinder of radius R rolling on a rough, horizontal surface (fig. below). As the cylinder rotates through an angle θ, its CM moves a dist. $s = R\theta$. Therefore, the velocity and acc. of the cm for pure rolling motion are given by

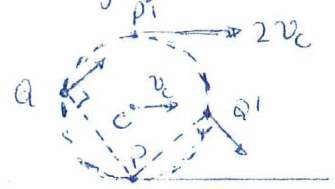


$$v_c = \frac{ds}{dt} = R\frac{d\theta}{dt} = R\omega$$

$$a_c = \frac{dv_c}{dt} = R\frac{d\omega}{dt} = R\alpha$$

The linear velocities of various points on the rolling cylinder

are shown in the fig. below. Note that the linear velocity of



any point is in a direction ⊥ to the line from that point to the contact point. At any instant, the point P is at rest relative to the surface since sliding does not occur. For that reason, the axis through P ⊥ to the diagram is called the instantaneous axis of rotation.

A general point on the cylinder, such as Q, has both horizontal and vertical comps of velocity. However, the points P and P' and the point at the center of mass are unique and of special interest. By def, the cm moves with a velocity $v_c = R\omega$, whereas the contact point P has zero velocity. Thus, it follows that the point P' must have a velocity equal to $2v_c = 2R\omega$, since all points on the cylinder have the same angular velocity.

We can express the total K.E. of the rolling cylinder as

$$K = \tfrac{1}{2} I_P \omega^2$$

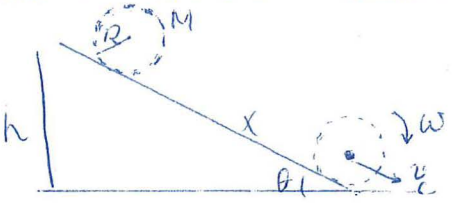
where I_P is the moment of inertia about the axis through P. Applying the parallel axis theorem, we have $I_P = I_c + MR^2$, and we get

$$K = \tfrac{1}{2} I_c \omega^2 + \tfrac{1}{2} MR^2\omega^2$$
$$= \tfrac{1}{2} I_c \omega^2 + \tfrac{1}{2} M v_c^2$$

where $v_c = R\omega$.

We then say that the total K.E. of the cylinder equals the sum of the rotational K.E. about the cm, $\tfrac{1}{2} I_c \omega^2$, and the K.E. the cylinder would have if it were just translating through space without rotating (or the translational K.E. of the cm), $\tfrac{1}{2} M v_c^2$.

Next let us consider the motion of a rigid body down a rough incline as shown below. Energy methods



can be used to treat this prob. First we shall assume that the body rolling down the incline does not slip and is released from rest at the top of the incline. Note that the rolling motion is possible only if a frictional force is present between the object

and the incline to produce a net torque about the cm. Despite the presence of friction, there is no loss of M.E. Since the contact point is at rest relative to the surface at any instant. On the other hand, if the body were to slide, M.E. would be lost as the motion continues.

$$K = \frac{1}{2} I_c \left(\frac{v_c}{R}\right)^2 + \frac{1}{2} M R^2 v_c^2 = \frac{1}{2}(I_c/R^2 + M) v_c^2.$$

If the body rolls down, it loses P.E. Mgh and if it starts from rest at the top, its K.E. at the bottom, given above, must equal its P.E. at the top. Thus,

$$Mgh = \frac{1}{2}(I_c/R^2 + M) v_c^2$$

gives

$$v_c = \sqrt{2gh \, KL + I_c/MR^2)}$$

Example:

If the rigid body shown in the above figure is a solid sphere, calculate the velocity of its cm at the bottom and determine the linear acc. of the cm of the sphere.

Solution:

For a uniform solid sphere, $I_c = \frac{2}{5} MR^2$, and hence

$$v_c = \sqrt{2gh / (1 + \frac{2}{5} MR^2/MR^2)} = \sqrt{\frac{10}{7} gh}$$

or in terms of the displacement, x, s.t. $h = x \sin\theta$, we find

$$v_c^2 = \frac{10}{7} gx \sin\theta$$

Comparing this with the eq. of rectilinear motion with const. acc., $v^2 = 2ax$, we find that the acc. of the cm is

$$a_c = \frac{5}{7} g \sin\theta.$$

If we repeat the calculations for a hollow sphere, a solid cylinder, or a hoop, we will obtain similar results. The const. factors that appear in the expressions for v_c and a_c depend on the moment of inertia about the cm for the specific body.

Chapter 11. Static Equilibrium of a Rigid Body

11.1 The conditions of Equil. of a Rigid Body

For a rigid body to be in equil. two conditions are required: the first and second conditions of equil. The first condition guarantees the translational equil. of a body by stating that for a body in equil. the resultant force acting on the body must be zero. In this case the body will either remain at rest or move uniformly along a straight line as predicted by Newton's first law of motion. This cond. is not sufficient for the description of the equil. of a particle but not for the equil. of a rigid body.

For a rigid body we need an additional cond., the second cond. of equil., which guarantees rotational equil. We may recall from our discussions of rotational motion that if a force acts on a rigid body, this force will bring about rotation of the body about a point wrt which the force could produce a torque. For rotational equil. then such effects should not appear and we state the second cond. of equil. as follows.

A rigid body will be in rotational equil. only if the net torque about any axis must be zero.

We now have two necessary conditions for equil. of a rigid body, which can be presented as follows.

1. The resultant external force must equal zero $\quad \Sigma \vec{F} = 0$

2. The resultant external torque about any axis must be zero $\quad \Sigma \vec{\tau} = 0$

The two vec for eqs are equivalent to six scalar eqs. in which we can have many unknowns. However, to simplify ma Hers we consider here only coplanar forces which lie say, in the x-y plane. In this case, we will have to deal only with three scalar eqs. Two of these are from the first cond. of equil. whereas the 3rd one comes from the second cond. of equil. Thus we write

$$\Sigma F_x = 0 \;, \quad \Sigma F_y = 0 \;; \quad \Sigma \tau_z = 0 \,.$$

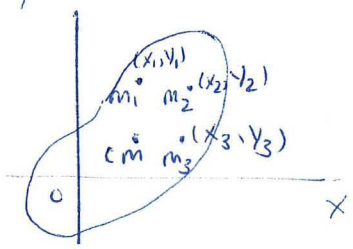
Note that for a body to be in equil. if two forces

act on it they have to be equal in magnitude and opposite in direction and if three forces act on it they have to be concurrent.

11.2 The Center of Gravity

In rigid body problems one of the forces that must be considered is the weight of the body, that is, the force of gravity. In order to calculate the torque of this force, we consider the weight to be concentrated at a single point called the center of gravity, cg. As we shall see, the cg of a body coincides with its cm if the body is in uniform gravitational field.

Consider a planar rigid body of arbitrary shape that lies in the x-y plane as shown below. Suppose that the

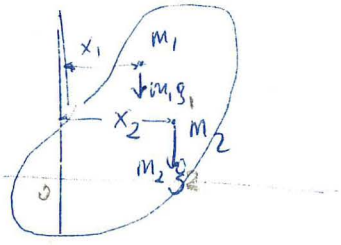


body is divided into a large no. of very small particles of masses m_1, m_2, etc. having coordinates (x_1,y_1), (x_2,y_2) etc. As we may recall from chapter 6, the X-coordinate of the cm is defined as

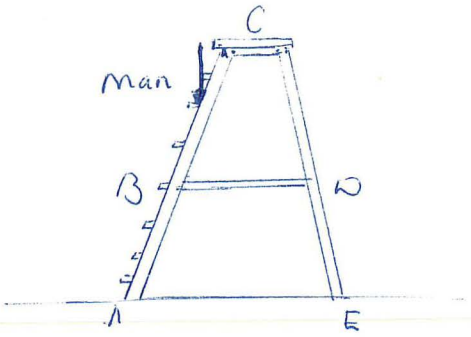
$$x_{cm} = \frac{m_1x_1 + m_2x_2 + \cdots}{m_1 + m_2 + \cdots} = \frac{\sum_i m_i x_i}{\sum_i m_i}$$

The y coordinate is similar to this, with x_i replaced by y_i.

On the other hand, each particle of the body is attracted toward the center of the earth and has a weight. This weight force contributes a torque about the origin O equal to its magnitude multiplied by the corresponding moment arm (see the figure below). For example, the torque due to the weight m_1g_1 is



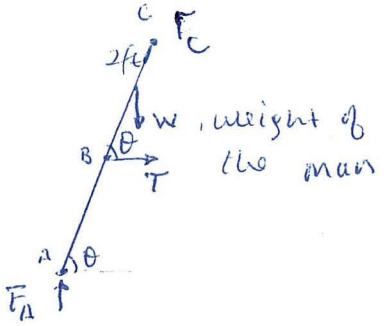
is $m_1g_1x_1$, and so on. We need to locate the position of the single force $\vec{w}$ (the weight of the body). This point is called the cg of the body. Since the effect of $\vec{w}$ is the same as the effect of all the weights of the particles, we write for the torques acting on the body that



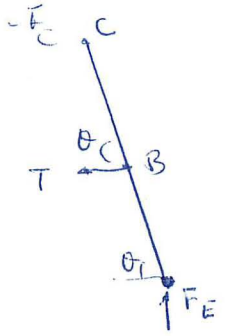
Solution:

a) Since the floor is frictionless, the reaction forces of the floor at points A and E are vertical. So from the first cond. of equil.

$$\Sigma F_y = F_A + F_E - 192\,lb = 0 \qquad \cdots (1)$$



(a)



(b)

$\theta = 71.8^0$

If we calculate torques about C in fig. (a) we have

$$T \times 4 \sin\theta + W \times 2\cos\theta - F_A \times 8\cos\theta = 0 \cdots (2)$$

Again calculating torques about C in fig. (b) we find

$$F_E \times 8\cos\theta - T \times 4\sin\theta = 0 \qquad \cdots (3)$$

$\Rightarrow$ $(F_A = W - F_A)\,8\cos\theta - 4T\sin\theta = 0$

or,

$$F_A = W - \frac{T}{2}\tan\theta$$

Using this in (2), we find

$$4T\sin\theta + 2W\cos\theta - 8W\cos\theta + 4T\sin\theta = 0$$

or,

$$T = \frac{6W}{8\tan\theta} = \frac{6 \times 192\,lb}{8 \times 3.077} \simeq 47\,lb, \quad \theta = 71.8^0$$

From the 2nd eq.

$$R_y = T_y - (m+M)g$$
$$= 2278.5\,N - 200\times 9.8\,N = 318.5\,N$$

chapter 12 Oscillatory Motion

12-1 Oscillations

Any motion that repeats itself in equal intervals of time is called periodic or harmonic motion. If a particle in periodic motion moves back and forth over the same path, we call the motion oscillatory or vibratory.

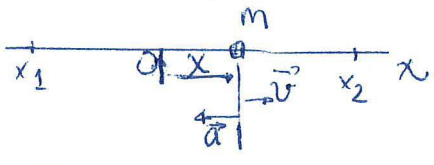
Many oscillating bodies do not move back and forth between two precisely fixed limits because frictional forces dissipate the energy of motion. Thus oscillating bodies such as a pendulum stop eventually stop their vibrations. These motions are called damped harmonic motions. Although we cannot eliminate friction from periodic motions of bodies, we can often cancel out its damping effect by feeding supplying energy to the oscillating system so as to compensate for the energy dissipated by friction.

The period T of a harmonic motion is the time required to complete one round trip of the motion, that is one complete oscillation or cycle. The frequency ν of the motion is the number ~~oscillations~~ (cycles) the oscillating body makes in unit time. Therefore the frequency is the reciprocal of the period, or

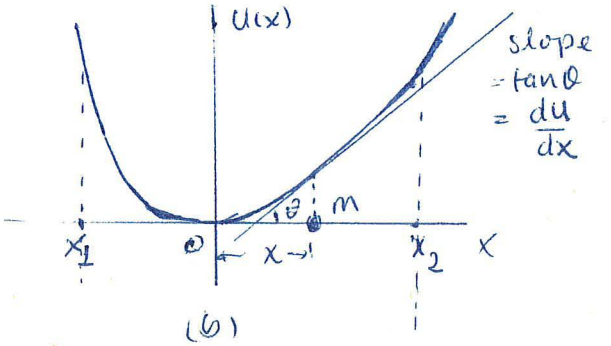
$$\nu = 1/T$$

The SI unit of frequency is the cycle per second or hertz (Hz). The position at which no net force acts on the oscillating particle is called its equilibrium position. The displacement (linear or angular) is the dist. (linear or angular) of the particle from its equilibrium position at any instant.

Let us now consider a particle oscillating back and forth along a straight line between two fixed limits say, x_1 and x_2, as in figure (a) below. Its displacement x changes



(a)



(b)

that the particle cannot move at all outside the limits x_1 and x_2 because, in these regions, U exceeds E. This, as the last eq. above indicates, would require the kinetic energy to be negative, an impossibility.

In general, for any given potential energy U(x), an oscillating particle particle can have various total energies, depending on how we set it into motion initially. Thus, the total energy may be E', rather than E, in which case the limits of oscillation would be x_1' and x_2', as shown in the last figure above, rather than x_1 and x_2.

12-3 Simple Harmonic Motion

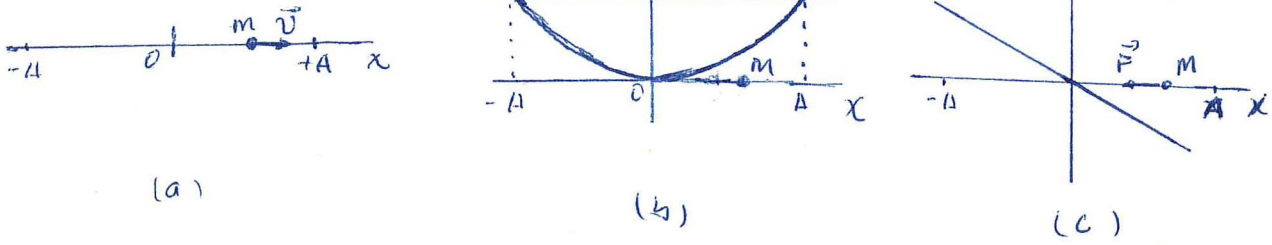
In fig. (a) below we have a particle moving back and forth about an equil. position through a potential that varies as

$$U(x) = \tfrac{1}{2} k x^2$$

where k is a positive const. see fig. (b). The force acting is given by

$$F(x) = -\frac{dU}{dx} = -\frac{d(\frac{1}{2}kx^2)}{dx} = -kx;$$

See fig. (c). Such an oscillating particle is called a



(a) (b) (c)

simple harmonic oscillator and its motion is called simple harmonic motion (SHM). In such a motion, as shown in the first eq. of this section, the P.E. varies quadratically with the displacement, and, as the second eq. of this section shows, the force acting on the particle is proportional to the displacement but is oppositely directed to it. In SHM the limits of oscillation are equally spaced about the equil. position, $\pm A$. This is not true for the more general motion of sec. 12-1 which, although harmonic, is not SHM. The magnitude of the max. displacement, the quantity A, is called the amplitude of SHM.

The expressions of the P.E. and K.E. of the particle above are the same as those of a spring-mass system discussed in chapter 5. In that case k is called the force constant. Hence, a body of mass m attached to a spring of force const.

A', ω, and φ, we have written down the above function as a solution of eq. (*). In order to determine these constants such that eq. (**) is actually the solution of eq. (*), we differentiate eq. (**) twice wrt time. Then

$$\frac{dx}{dt} = -\omega A' \sin(\omega t + \varphi)$$

and

$$\frac{d^2x}{dt^2} = -\omega^2 A' \cos(\omega t + \varphi)$$

putting this in eq. (*) we obtain

$$-\omega^2 A' \cos(\omega t + \varphi) = -\frac{k}{m} A' \cos(\omega t + \varphi)$$

We choose

$$\omega^2 = k/m$$

so that

$$x = A' \cos(\omega t + \varphi)$$

is in fact a solution of the eq. of a simple harmonic oscillator ~~with $\omega^2 = k/m$.~~

The consts A' and φ are still undetermined and, therefore, still completely arbitrary. To find the physical significance of the const. ω we increase t by $2\pi/\omega$ so that $x(t)$ becomes

$$\begin{aligned} x &= A' \cos[\omega(t + 2\pi/\omega) + \varphi] \\ &= A' \cos(\omega t + 2\pi + \varphi) \\ &= A' \cos(\omega t + \varphi). \end{aligned}$$

That is, the function $x(t)$ repeats itself after a time $2\pi/\omega$. Therefore, $2\pi/\omega$ is the period of the motion T. Since $\omega^2 = k/m$, we have

$$T = 2\pi/\omega = 2\pi\sqrt{m/k}.$$

The frequency ν as mentioned in sec. 12-1 is given by

$$\nu = \frac{1}{T} = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}.$$

Hence,

$$\omega = 2\pi\nu = 2\pi/T = \sqrt{k/m}.$$

The quantity ω is called the angular frequency; it differs from the frequency ν by a factor 2π. Its unit is the

Examples

1. A particle executes linear harmonic motion about the point $x = 0$. At $t = 0$ it has displacement $x = 0.37\,cm$ and zero velocity. The frequency of the motion is $0.25\,Hz$. Determine a, the period b, the angular frequency c, the amplitude d, the displacement at time t e, the velocity at time t f, the max. speed g, the max. acc h, the displacement at $t = 3s$ and i, the speed at $t = 3s$.

solution:

a, $T = 1/\nu = \frac{1}{0.25\,Hz} = 4\,s$

b, $\omega = 2\pi\nu = 2\pi \times \frac{1}{4}\,rad/s = \frac{\pi}{2}\,rad/s.$

c, since at time $t = 0$ the displacement is $0.37\,cm$ and the velocity is 0, the amplitude of the oscillation is $A = 0.37\,cm$.

d, The possible solution satisfying the conditions $x = 0.37\,cm$, $v = 0$ at $t = 0$ is

$$x(t) = (0.37\,cm)\cos(\tfrac{\pi}{2}t)$$

e, $v(t) = \frac{dx(t)}{dt} = -0.37 \frac{\pi}{2} \sin(\frac{\pi}{2}t).\,cm/s = (-0.58\,\frac{cm}{s})\sin(\frac{\pi}{2}t)$

f, We can write

$$v(t) = -v_{max}\sin\omega t$$

so that comparison with the expression in (e) gives

$$v_{max} = 0.58\,cm/s.$$

g, $a(t) = \frac{dv(t)}{dt} = (-0.58 \frac{\pi}{2}\,\frac{cm}{s^2})\cos(\frac{\pi}{2}t)$

$$= (-0.91\,\tfrac{cm}{s^2})\cos(\tfrac{\pi}{2}t)$$

which gives

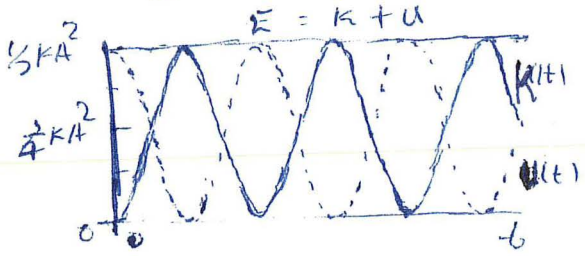
$$a_{max} = 0.91\,cm/s^2$$

h, $x(3s) = (0.37\,cm)\cos(\frac{3}{2}\pi) = 0$

i, $v(3s) = (-0.58\,\frac{cm}{s})\sin(3\frac{\pi}{2}) = 0.58\,cm/s.$

2. A 4 kg block extends a spring 16 cm from its unstretched position. The block is removed and a 0.5 kg body is hung from the same spring. If the spring is then stretched and released, what is its period of

$\frac{1}{2}kA^2$. The max. value $\frac{1}{2}kA^2$ (or $\frac{1}{2}m(\omega A)^2$) is in agreement with the max. speed ωA, mentioned earlier. The plots (curves) of the P.E. and K.E. as functions of time are given as shown below.



Plotted in the condition where $\varphi = 0$ at $t = 0$.

The total M.E., $E = K + U$, is

$$E = \tfrac{1}{2}kA^2\sin^2(\omega t+\varphi) + \tfrac{1}{2}kA^2\cos^2(\omega t+\varphi) = \tfrac{1}{2}kA^2.$$

That is the total M.E. is const., as expected. At the max. displacement the K.E. is zero, but the P.E. has the value $\frac{1}{2}kA^2$. At the equil. position the P.E. is zero, but the K.E. has the value $\frac{1}{2}kA^2$. At other positions the K.E. and P.E. each contribute energy whose sum is always $\frac{1}{2}kA^2$. This const. total energy is shown in the above figure. From this figure it is seen that the average K.E. and P.E. for the motion during one period are exactly equal and each of these average quantities is $\frac{1}{4}kA^2$.

Also,

$$U = \tfrac{1}{2}kA^2\cos^2(\omega t+\varphi) = \tfrac{1}{2}kx^2$$

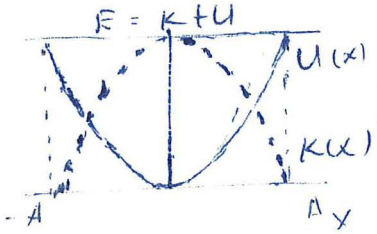
$$K = \tfrac{1}{2}kA^2\sin^2(\omega t+\varphi) = \tfrac{1}{2}k[(A^2 - A^2\cos^2(\omega t+\varphi)]$$

$$= \tfrac{1}{2}k(A^2 - x^2)$$

Then,

$$K + U = \tfrac{1}{2}k(A^2 - x^2) + \tfrac{1}{2}kx^2 = \tfrac{1}{2}kA^2$$

Thus, as a function of displacement the K.E. and P.E. are plotted as in the figure below.

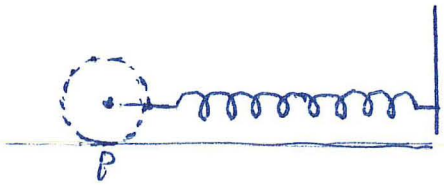


$$v = \frac{dx}{dt} = \sqrt{\frac{2K}{m}} = \sqrt{\frac{k}{m}(A^2 - x^2)}$$

That means the speed is a max. at the equil.

$$T = 2\pi\sqrt{3M/2k},$$

Where M is the mass of the cylinder



Solution:

a, As the cylinder passes through its equilibrium it will have both translational and rotational K.E's. That is, the P.E. of the system when the spring is stretched by 0.25 m will be converted into the rotational K.E. and translational K.E. mentioned above. In short since the frictionless M.S. is conserved in a rolling motion we write

$$\frac{1}{2}kA^2 = \frac{1}{2}I_p\omega^2,$$

P is the instantaneous axis of rotation at the equilibrium position.

But,

$$I_p = I_{cm} + MR^2$$

For a solid cylinder

$$I_{cm} = \tfrac{1}{2}MR^2.$$

This gives

$$I_p = \frac{1}{2}MR^2 + MR^2 = \frac{3}{2}MR^2$$

Also,

$$\omega = \frac{v_{cm}}{R}.$$

Then,

$$\frac{1}{2}kA^2 = \frac{1}{2}\cdot\frac{3}{2}MR^2\frac{v_{cm}^2}{R^2} = \frac{1}{2}\frac{3}{2}Mv_{cm}^2$$

(i.e.)

$$\frac{1}{2}Mv_{cm}^2 = \frac{1}{3}kA^2$$

$\therefore$ Translational K.E. $= \frac{1}{3}kA^2 = \frac{1}{3}\times 3\,(0.25)^2\,J =$

$= 0.0625\,J.$

In the above figure, a particle of mass m hanging from a string of length ℓ is displaced by an angle θ from the vertical position. The forces acting on m are shown here. The gravitational force $m\vec{g}$ is resolved into two components, one along the string and the other perpendicular to it. If released at this position, the particle will move along the arc of a circle of radius ℓ and until it reaches its max. position on the other side of the vertical equil. position. This motion occurs repeatedly, i.e., oscillation takes place. As can be seen from the above figure, the comp. $mg\cos\theta$ of the weight provides the centripetal force for the motion along the circular path and the comp. $mg\sin\theta$ provides the restoring force of the oscillation. We then write

$$F = -mg\sin\theta$$

It is seen that since F is not proportional to the angular displacement θ, the resulting motion is not simple harmonic. However, if the angle θ is small, $\sin\theta$ is very nearly equal to θ in radians. Hence, assuming $\sin\theta \approx \theta$, we find

$$F = -mg\theta = -mg\frac{x}{\ell} = -\frac{mg}{\ell}x$$

where x is the displacement measured along the circle.

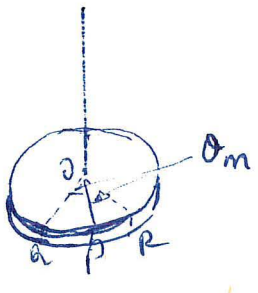
For small displacements, therefore, the restoring force is proportional to the displacement and is oppositely directed. This is exactly the case of SHM. The const. mg/ℓ represents the const. k in $F = -kx$. Thus the period of a simple pendulum when its amplitude is small is

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{m}{mg/\ell}} = 2\pi\sqrt{\ell/g}$$

which is independent of the mass of the suspended particle.

- The Torsional pendulum

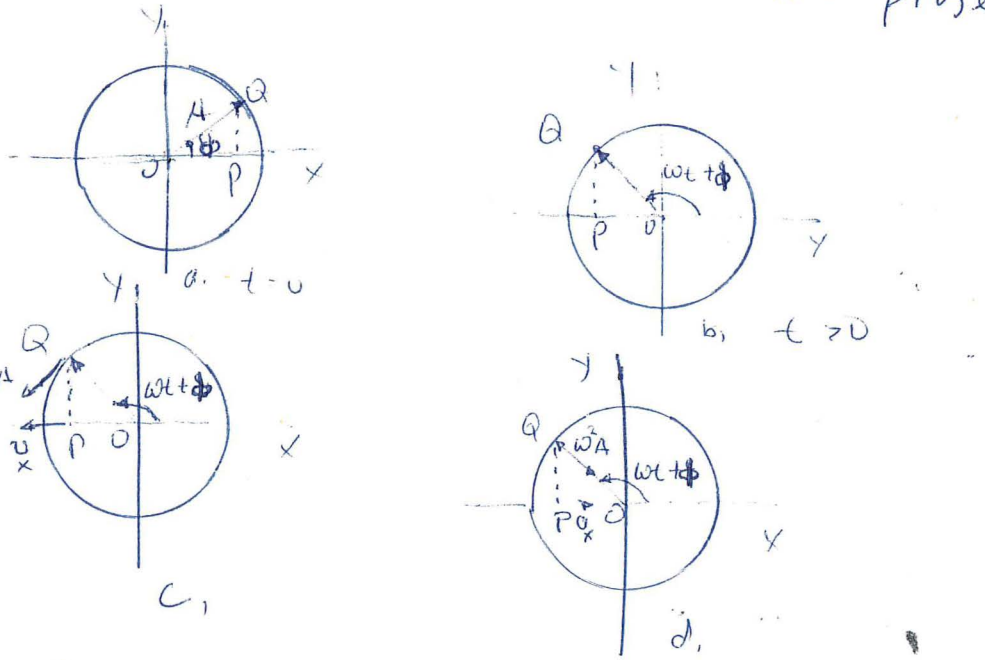
It consists of a disk suspended by a wire attached to the center of mass of the disk, as shown below.



12.5 SHM and Uniform Circular Motion

Let us consider the relation between SHM along a straight line and uniform circular motion. This relation gives a simple geometric meaning to the angular frequency ω and the phase const. ϕ to the angular fre-

Consider the figure below in which a point Q moves around a circle of radius A with a const. angular speed ω. P is the projection of Q on the horizontal diameter, along the x-axis. We call Q the reference point and the circle its motion. The reference circle. As the reference point Q revolves, the projection point



P moves back and forth along the horizontal diameter. The displacement of P is the same as the x-comp. of the displacement of Q; the velocity of P is always the same as the x-comp. of the velocity of Q; and the acc. of P is always the same as the x-comp. of the acc. of Q.

point Q starts its motion at time $t=0$ when its radius vector $\vec{OQ}$ makes an angle ϕ with the x-axis. If Q moves with a const. angular speed ω, the angle that the radius vector $\vec{OQ}$ makes with the x-axis at any later time t will be $\omega t + \phi$. Thus, the x-coordinate of Q at any time will be

$$x = A\cos(\omega t + \phi),$$

This implies that P indeed moves with SHM along the x-axis!

The angular frequency ω of SHM is the same as the

$$F = \sum F' = \frac{GM}{r^2} \sum M' \text{ , all forces have the same direction}$$

or,

$$F = \frac{GmM}{r^2} \quad \text{or} \quad \vec{F} = -\frac{GmM}{r^2}\hat{e}_r \text{ , attractive.}$$

- $b \le r \le a$

A similar calculation gives a result

$$\vec{F} = -GmM'\frac{\hat{e}_r}{r^2}$$

where M' is the mass of the shell of inner radius b and outer radius r. But,

$$\rho = \frac{M}{\frac{4}{3}\pi(a^3-b^3)} = \frac{M'}{\frac{4}{3}\pi(r^3-b^3)}$$

ie,

$$M' = M\,\frac{r^3-b^3}{a^3-b^3}.$$

Therefore,

$$F = \frac{GmM}{(a^3-b^3)}\,\frac{r^3-b^3}{r^2}.$$

In vector form, $\vec{F} = \left(-GmM(r^3-b^3)/(a^3-b^3)r^2\right)\hat{e}_r$.

- $0 \le r \le b$

The example mentioned above has proved that the gravitational force is zero when the particle is inside the shell.

b) The p.E. is given from

$$U(r_f) - U(r_i) = -\int_{r_i}^{r_f} \vec{F}\cdot d\vec{r}$$

- $a \le r \le \infty$

$$U(r) = -\int_{\infty}^{r}\left(-\frac{GmM}{r'^2}\right)dr' = -\frac{GmM}{r'}\Big|_{\infty}^{r}$$

or,

$$U(r) = -\frac{GmM}{r}, \quad U(\infty) = 0.$$

- $b \le r \le a$

$$U(r) - U(a) = \frac{GmM}{(a^3-b^3)} \int_a^r \left(r' - \frac{b^3}{r'^2}\right) dr'$$

$$= \frac{GmM}{(a^3-b^3)} \left(\frac{r'^2}{2} + \frac{b^3}{r'}\right)_a^r$$

$$= \frac{GmM}{(a^3-b^3)} \left(\frac{r^2}{2} + \frac{b^3}{r} - \frac{a^2}{2} - \frac{b^3}{a}\right)$$

and,

$$U(r) = \frac{GmM}{(a^3-b^3)} \left(\frac{r^2-a^2}{2} + b^3 \frac{a-r}{ar}\right) + U(a)$$

From the above result

$$U(a) = -\frac{GmM}{a}$$

(i.e.)

$$U(r) = \frac{GmM}{a(a^3-b^3)} \left[\frac{a}{2}(r^2-a^2) + ab^3 \frac{a-r}{r} - (a^3-b^3)\right]$$

- $0 \le r \le b$

Since the force in this region is zero, the p.E. has to be a const. But at the inner surface of the shell

$$U(b) = \frac{GmM}{a(a^3-b^3)} \left[\frac{a}{2}(b^2-a^2) + b^3 \frac{a-b}{b} - (a^3-b^3)\right]$$

$$= \frac{GmM}{a(a^3-b^3)} \left[\frac{3}{2}ab^2 - \frac{a^3}{2}\right] = \frac{3GmM}{2a(a^3-b^3)}(b^2-a^2)$$

$$= U(r) \text{ for } 0 \le r \le b.$$

c) the gravitation field strength of the sphere will be obtained by dividing each expressions of the gravitational force above by m and the gravitational potential of the sphere will be obtained by dividing the gravitational p.E. of the sphere and the particle by the mass m of the particle.

13.6 p.E. of Systems of Particles

As given in the previous section, if two particles are separated by a dist. r, their potential energy is

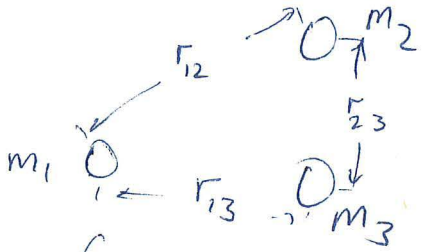
$$U(r) = -W_{gr},$$

in which W_{gr} is the work done by gravitational force as the particles move from an infinite separation to r. Another interpretation of U(r) may be given as follows:

Suppose that the gravitational force acting between the two particles is balanced by a force applied by an external agent. The work done by the external force as the particles move from an infinite separation to separation r is not W_{gr} but $-W_{gr}$; this is because the displacements are the same but the forces are equal and opposite. Thus the above eq. is interpreted as follows. The p.E. of a system of particles is equal to the work that must be done by an external agent to assemble the system, starting from the standard reference configuration.

Thus, if somebody lifts a block of mass m a dist. y above the earth's surface, the work he does in making this configuration is $+mgy$, which is also the p.E. Similarly, the work done by an external agent ~~to~~ bring a body of mass m moves in from infinity to a dist. r from the earth is negative because the agent must exert a restraining force on the body, in agreement with the above eq.

These considerations hold for systems that contain more than two particles. Consider three bodies of masses m_1, m_2, and m_3, figure below. Initially let these particles be infinitely far from one another. To calculate the work done by an external agent in bringing them into the positions shown in the figure, let us first bring m_2 in



toward m_1 from an infinite separation to the separation r_{12}. The work done against the gravitational force exerted by m_1 on m_2 is $-GM_1M_2/r_{12}$. Next, let us bring m_3

in from infinity to the separation r_{13} from m_1 and r_{23} from m_2. The work done against the gravitational force exerted by m_1 on m_3 is $-Gm_1m_3/r_{13}$ and that against the gravitational force exerted by m_2 on m_3 is $-Gm_2m_3/r_{23}$. The total work done in assembling this system is the total P.E. of the system

$$-\left(G\frac{m_1m_2}{r_{12}} + G\frac{m_1m_3}{r_{13}} + G\frac{m_2m_3}{r_{23}}\right).$$

No matter how we assemble the system, i.e., regardless of which bodies are moved or which paths are taken, we always find this same amount of work required to bring the bodies into the configuration shown above from an infinite separation. The P.E. must, therefore, be associated with the system rather than with any one or two bodies. If we want to separate the system into three isolated masses once again, we would have to supply an amount of energy

$$+\left(G\frac{m_1m_2}{r_{12}} + G\frac{m_1m_3}{r_{13}} + G\frac{m_2m_3}{r_{23}}\right).$$

This energy can be regarded as a binding energy holding the mass particles together in the configuration shown. We define binding energy to be that amount of energy that must be added to a system in order to separate the particles of interest to infinite distances from one another.

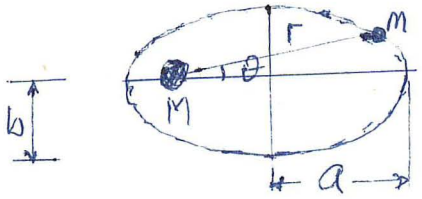
13.7 The Motions of Planets and Satellites

Kepler's three laws of planetary motion:

1. All planets move in elliptical orbits having the sun as one focus (the law of orbits).
2. A line joining any planet to the sun sweeps out equal areas in equal times (the law of areas).
3. The square of the period of any planet is proportional to the cube of the semimajor axis of its orbit (the law of periods).

These laws apply not only to planets moving in the gravitational field of the sun but also to natural and artificial satellites moving in the gravitational field of a planet. The analytical formulations of these laws were given by Isaac Newton from his laws of motion and universal gravitation. Let us now discuss each of them separately.

1. The laws of orbits: The fig. (a) below shows a planet moving in an elliptical orbit around the sun. Since the sun has



a - Semimajor axis
b - Semiminor axis

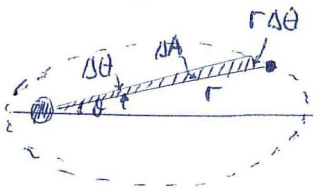
much larger mass M than the mass of the planet we may assume the center of mass of the sun-planet system lies at the center of the sun. So the sun remains at rest at one focus of the elliptical orbit.

The eq. of the orbit may be expressed as

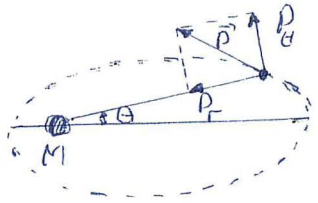
$$\frac{1}{r} = \frac{1}{b^2}\left(a + \sqrt{a^2 - b^2}\cos\theta\right).$$

In the case $b = a$ the ellipse reduces to a circle and r always has the const. value a, no matter what the value of θ.

2. The law of areas: Fig. (a) below, once again, shows a planet moving around the sun. In a small time Δt



(a)



(b)

the line that joins the planet to the sun sweeps through an angle $\Delta\theta = \omega\,\Delta t$. The area ΔA corresponding to this angle of the shaded region in fig. (a) is approximately $\frac{1}{2} r\Delta\theta \cdot r \left(= \frac{1}{2} r^2 \Delta\theta\right)$. This expression becomes more exact in the limit $\Delta t \to 0$. The instantaneous rate dA/dt at which area is being swept out is then found to be

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$$L = p_\theta r = mv_\theta r = m\omega r r = mr^2\omega \quad \cdots (**)$$

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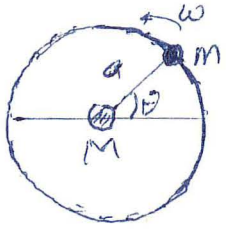
We prove this law for the special case of a circular orbit of radius a, as shown in the figure below.



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Solution:

Let the initial mass of a person, value of the gravitational acc., weight of a person, and speed of a satellite ~~was~~, mass of the earth, radius of the earth be m, g, $w(=mg)$, v, M and R resp. and let the corresponding new quantities be m', g', $w'(=m'g')$, v', M', and $R'(=2R)$.

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where m is the mass of the moon and v is its speed of revolution. Since $v = 2\pi a/T$ we have

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According to Kepler's 3rd law, stated above, $T^2 \propto a^3$. That is

$$F \propto \frac{a}{a^3} = \frac{1}{a^2} \quad \text{or} \quad F \propto \frac{1}{a^2}$$

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3. Three identical bodies of mass M are located at the vertices of an equilateral triangle with side L. At what speed must they move if they all revolve under the influence of one another's gravity in a circular orbit circumscribing the triangle while still preserving the equilateral triangle?

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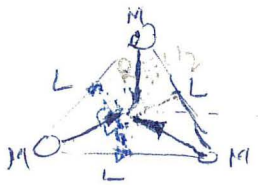

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The radius R of this circle is

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It follows

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i.e.,

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23.8 Energy considerations in the motions of planets and Satellites

Suppose that a planet or Satellite of mass m moves about the Sun or earth of mass M. In the approximation $M \gg m$, the body of mass M will be at rest while the body of mass m moves about it in a circular orbit. The P.E. of the System is

$$U(r) = -\frac{GMm}{r_0}$$

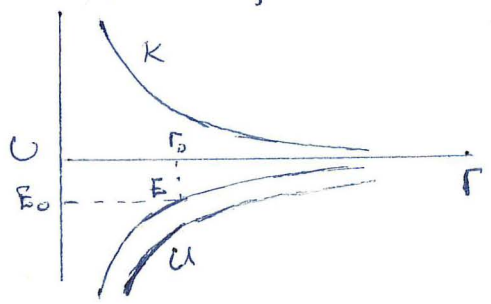
where r_0 is the radius of the circular orbit. The K.E. of the System is

$$K = \frac{1}{2}mv^2 = \frac{r_0}{2}\frac{mv^2}{r_0} = \frac{r_0}{2}\frac{GMm}{r_0^2} = \frac{GMm}{2r_0}.$$

The M.E. is

$$E = K + U = \frac{GMm}{2r_0} - \frac{GMm}{r_0} = -\frac{GMm}{2r_0}.$$

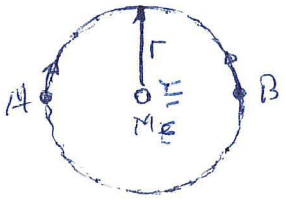
The physical significance of this negative M.E. is that the System is a closed one, the body m always being bound to the attracting body M and never escaping from it (below



Even for elliptical orbits in which r and ω vary, the M.E. is negative. It is also const. corresponding to the fact that gravitational forces are conservative. Hence, both the M.E. and the total angular mom. are const. in planetary motion. We obtain the actual orbit of a planet wrt to the sun by starting with these conservation relations and eliminating the time variable by use of the laws of dynamics and gravitation. The result is that planetary orbits are elliptical.

Examples:

1. Consider two satellites, A and B, of equal mass m, moving in the same circular orbit of radius r around the earth E but in opposite senses of rotation and therefore on a collision course (below). a, In terms of G, M_E, m, and r, find the total M.E. $E_A + E_B$ of the two-satellite-plus-earth system before collision. b, If the collision is completely inelastic so that wreckage remains as one piece of tangled material (mass = 2m), find the total M.E. immediately after collision. c, Describe the subsequent motion of the wreckage.



Solution:

a, $$E_A = K_A + U_A = \frac{GmM_E}{2r} - \frac{GmM_E}{r} = -\frac{GmM_E}{2r}$$

Similarly,

$$E_B = -\frac{GmM_E}{2r}$$

so that

$$E = E_A + E_B = -\frac{GmM_E}{2r} - \frac{GmM_E}{2r} = -\frac{GmM_E}{r}.$$

b, Just after the collision the wreckage remains at rest as a result of the conservation of mom. So the M.E. will be consisting a contribution from the gravitational p.e. only. Thus

$$E = -G\frac{2mM}{r}$$

c, The wreckage falls down since angular mom. has to be conserved. That is since the initial angular mom. is zero (sum of equal and oppositely directed angular mom.) the final has to be zero as well so that the wreckage should simply fall.

$$F = \sum F' = \frac{GM}{r^2} \sum M' \text{ , all forces have the same direction}$$

or,

$$F = \frac{GmM}{r^2} \quad \text{or} \quad \vec{F} = -\frac{GmM}{r^2}\hat{e}_r \text{ , attractive.}$$

- $b \le r \le a$

A similar calculation gives a result

$$\vec{F} = -GmM'\frac{\hat{e}_r}{r^2}$$

where M' is the mass of the shell of inner radius b and outer radius r. But,

$$\rho = \frac{M}{\frac{4}{3}\pi(a^3-b^3)} = \frac{M'}{\frac{4}{3}\pi(r^3-b^3)}$$

ie,

$$M' = M\,\frac{r^3-b^3}{a^3-b^3}.$$

Therefore,

$$F = \frac{GmM}{(a^3-b^3)}\,\frac{r^3-b^3}{r^2}.$$

In vector form, $\vec{F} = \left(-GmM(r^3-b^3)/(a^3-b^3)r^2\right)\hat{e}_r$.

- $0 \le r \le b$

The example mentioned above has proved that the gravitational force is zero when the particle is inside the shell.

b) The p.E. is given from

$$U(r_f) - U(r_i) = -\int_{r_i}^{r_f} \vec{F}\cdot d\vec{r}$$

- $a \le r \le \infty$

$$U(r) = -\int_{\infty}^{r}\left(-\frac{GmM}{r'^2}\right)dr' = -\frac{GmM}{r'}\Big|_{\infty}^{r}$$

or,

$$U(r) = -\frac{GmM}{r}, \quad U(\infty) = 0.$$

- $b \le r \le a$

$$U(r) - U(a) = \frac{GmM}{(a^3-b^3)} \int_a^r \left(r' - \frac{b^3}{r'^2}\right) dr'$$

$$= \frac{GmM}{(a^3-b^3)} \left(\frac{r'^2}{2} + \frac{b^3}{r'}\right)_a^r$$

$$= \frac{GmM}{(a^3-b^3)} \left(\frac{r^2}{2} + \frac{b^3}{r} - \frac{a^2}{2} - \frac{b^3}{a}\right)$$

and,

$$U(r) = \frac{GmM}{(a^3-b^3)} \left(\frac{r^2-a^2}{2} + b^3 \frac{a-r}{ar}\right) + U(a)$$

From the above result

$$U(a) = -\frac{GmM}{a}$$

(i.e.)

$$U(r) = \frac{GmM}{a(a^3-b^3)} \left[\frac{a}{2}(r^2-a^2) + ab^3 \frac{a-r}{r} - (a^3-b^3)\right]$$

- $0 \le r \le b$

Since the force in this region is zero, the p.E. has to be a const. But at the inner surface of the shell

$$U(b) = \frac{GmM}{a(a^3-b^3)} \left[\frac{a}{2}(b^2-a^2) + b^3 \frac{a-b}{b} - (a^3-b^3)\right]$$

$$= \frac{GmM}{a(a^3-b^3)} \left[\frac{3}{2}ab^2 - \frac{a^3}{2}\right] = \frac{3GmM}{2a(a^3-b^3)}(b^2-a^2)$$

$$= U(r) \text{ for } 0 \le r \le b.$$

c) the gravitation field strength of the sphere will be obtained by dividing each expressions of the gravitational force above by m and the gravitational potential of the sphere will be obtained by dividing the gravitational p.E. of the sphere and the particle by the mass m of the particle.

13.6 p.E. of Systems of Particles

As given in the previous section, if two particles are separated by a dist. r, their potential energy is

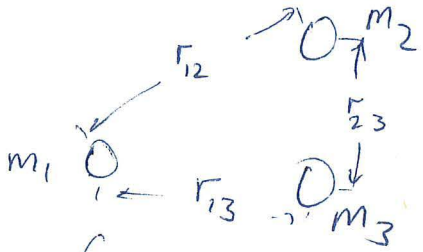
$$U(r) = -W_{gr},$$

in which W_{gr} is the work done by gravitational force as the particles move from an infinite separation to r. Another interpretation of U(r) may be given as follows:

Suppose that the gravitational force acting between the two particles is balanced by a force applied by an external agent. The work done by the external force as the particles move from an infinite separation to separation r is not W_{gr} but $-W_{gr}$; this is because the displacements are the same but the forces are equal and opposite. Thus the above eq. is interpreted as follows. The p.E. of a system of particles is equal to the work that must be done by an external agent to assemble the system, starting from the standard reference configuration.

Thus, if somebody lifts a block of mass m a dist. y above the earth's surface, the work he does in making this configuration is $+mgy$, which is also the p.E. Similarly, the work done by an external agent ~~to~~ bring a body of mass m moves in from infinity to a dist. r from the earth is negative because the agent must exert a restraining force on the body, in agreement with the above eq.

These considerations hold for systems that contain more than two particles. Consider three bodies of masses m_1, m_2, and m_3, figure below. Initially let these particles be infinitely far from one another. To calculate the work done by an external agent in bringing them into the positions shown in the figure, let us first bring m_2 in



toward m_1 from an infinite separation to the separation r_{12}. The work done against the gravitational force exerted by m_1 on m_2 is $-GM_1M_2/r_{12}$. Next, let us bring m_3

in from infinity to the separation r_{13} from m_1 and r_{23} from m_2. The work done against the gravitational force exerted by m_1 on m_3 is $-Gm_1m_3/r_{13}$ and that against the gravitational force exerted by m_2 on m_3 is $-Gm_2m_3/r_{23}$. The total work done in assembling this system is the total P.E. of the system

$$-\left(G\frac{m_1m_2}{r_{12}} + G\frac{m_1m_3}{r_{13}} + G\frac{m_2m_3}{r_{23}}\right).$$

No matter how we assemble the system, i.e., regardless of which bodies are moved or which paths are taken, we always find this same amount of work required to bring the bodies into the configuration shown above from an infinite separation. The P.E. must, therefore, be associated with the system rather than with any one or two bodies. If we want to separate the system into three isolated masses once again, we would have to supply an amount of energy

$$+\left(G\frac{m_1m_2}{r_{12}} + G\frac{m_1m_3}{r_{13}} + G\frac{m_2m_3}{r_{23}}\right).$$

This energy can be regarded as a binding energy holding the mass particles together in the configuration shown. We define binding energy to be that amount of energy that must be added to a system in order to separate the particles of interest to infinite distances from one another.

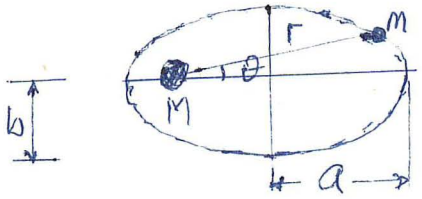
13.7 The Motions of Planets and Satellites

Kepler's three laws of planetary motion:

1. All planets move in elliptical orbits having the sun as one focus (the law of orbits).
2. A line joining any planet to the sun sweeps out equal areas in equal times (the law of areas).
3. The square of the period of any planet is proportional to the cube of the semimajor axis of its orbit (the law of periods).

These laws apply not only to planets moving in the gravitational field of the sun but also to natural and artificial satellites moving in the gravitational field of a planet. The analytical formulations of these laws were given by Isaac Newton from his laws of motion and universal gravitation. Let us now discuss each of them separately.

1. The laws of orbits: The fig. (a) below shows a planet moving in an elliptical orbit around the sun. Since the sun has



a - Semimajor axis
b - Semiminor axis

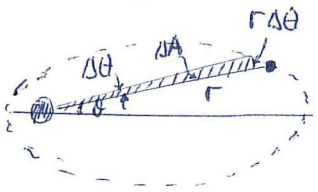
much larger mass M than the mass of the planet we may assume the center of mass of the sun-planet system lies at the center of the sun. So the sun remains at rest at one focus of the elliptical orbit.

The eq. of the orbit may be expressed as

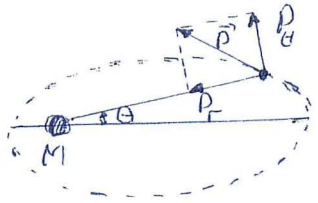
$$\frac{1}{r} = \frac{1}{b^2}\left(a + \sqrt{a^2 - b^2}\cos\theta\right).$$

In the case $b = a$ the ellipse reduces to a circle and r always has the const. value a, no matter what the value of θ.

2. The law of areas: Fig. (a) below, once again, shows a planet moving around the sun. In a small time Δt



(a)



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the line that joins the planet to the sun sweeps through an angle $\Delta\theta = \omega\,\Delta t$. The area ΔA corresponding to this angle of the shaded region in fig. (a) is approximately $\frac{1}{2} r\Delta\theta \cdot r \left(= \frac{1}{2} r^2 \Delta\theta\right)$. This expression becomes more exact in the limit $\Delta t \to 0$. The instantaneous rate dA/dt at which area is being swept out is then found to be

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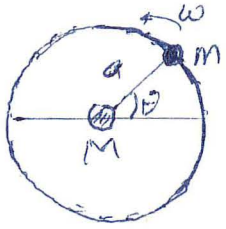

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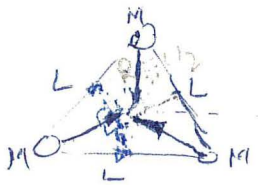

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It follows

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$$U(r) = -\frac{GMm}{r_0}$$

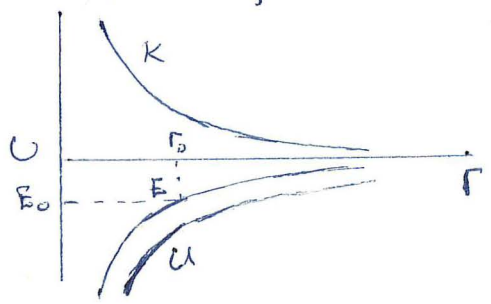
where r_0 is the radius of the circular orbit. The K.E. of the System is

$$K = \frac{1}{2}mv^2 = \frac{r_0}{2}\frac{mv^2}{r_0} = \frac{r_0}{2}\frac{GMm}{r_0^2} = \frac{GMm}{2r_0}.$$

The M.E. is

$$E = K + U = \frac{GMm}{2r_0} - \frac{GMm}{r_0} = -\frac{GMm}{2r_0}.$$

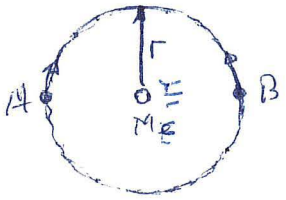
The physical significance of this negative M.E. is that the System is a closed one, the body m always being bound to the attracting body M and never escaping from it (below



Even for elliptical orbits in which r and ω vary, the M.E. is negative. It is also const. corresponding to the fact that gravitational forces are conservative. Hence, both the M.E. and the total angular mom. are const. in planetary motion. We obtain the actual orbit of a planet wrt to the sun by starting with these conservation relations and eliminating the time variable by use of the laws of dynamics and gravitation. The result is that planetary orbits are elliptical.

Examples:

1. Consider two satellites, A and B, of equal mass m, moving in the same circular orbit of radius r around the earth E but in opposite senses of rotation and therefore on a collision course (below). a, In terms of G, M_E, m, and r, find the total M.E. $E_A + E_B$ of the two-satellite-plus-earth system before collision. b, If the collision is completely inelastic so that wreckage remains as one piece of tangled material (mass = 2m), find the total M.E. immediately after collision. c, Describe the subsequent motion of the wreckage.



Solution:

a, $$E_A = K_A + U_A = \frac{GmM_E}{2r} - \frac{GmM_E}{r} = -\frac{GmM_E}{2r}$$

Similarly,

$$E_B = -\frac{GmM_E}{2r}$$

so that

$$E = E_A + E_B = -\frac{GmM_E}{2r} - \frac{GmM_E}{2r} = -\frac{GmM_E}{r}.$$

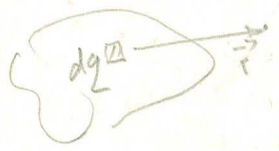
b, Just after the collision the wreckage remains at rest as a result of the conservation of mom. So the M.E. will be consisting a contribution from the gravitational p.e. only. Thus

$$E = -G\frac{2mM}{r}$$

c, The wreckage falls down since angular mom. has to be conserved. That is since the initial angular mom. is zero (sum of equal and oppositely directed angular mom.) the final has to be zero as well so that the wreckage should simply fall.

Calculate the electric field for a single point charge or for a group of point charges. But here we shall consider the method of calculation of the electric field due to a continuous charge distribution. We proceed as follows.

Divide the given charge distribution into infinitesimal elements dq. The field $d\vec{E}$ due to each element at the indicated point is then calculated, treating the elements as point charges. The magnitude of $d\vec{E}$ is given by



$$dE = \frac{1}{4\pi\epsilon_0}\frac{dq}{r^2}$$

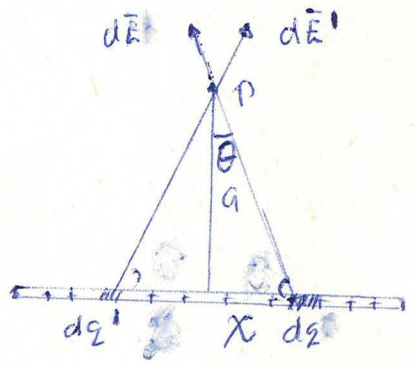
where r is the dist. from the charge element dq to the point P. The resultant field at P is then found by applying the superposition principle i.e., by adding - integrating the contributions of all the charge elements, or,

$$\vec{E} = \int d\vec{E}$$

This integration is a vector operation which can be done component by component. $E_x = \int dE_x$, $E_y = \int dE_y$, $E_z = \int dE_z$.

Example: A thin nonconducting rod of finite length L carries a total charge Q, spread uniformly along it. Find the electric field at a point P on the perpendicular bisector.

Solution



Assume a positive charge distribution. The magnitude of the contribution of dq to the electric fied at P is

$$dE = \frac{1}{4\pi\epsilon_0}\frac{dq}{(x^2+a^2)}$$

But we have a similar charge element dq' to the left of the bisector contributing to the electric field at P the same amount. The x- comps of these contributions add up zero while the y- comps add up to a non zero value.

01.

$$dE_p = 2 \cdot \frac{1}{4\pi\epsilon_0} \frac{dq}{(x^2+a^2)} \cos\theta$$

$$dE_p = \frac{1}{2\pi\epsilon_0} \frac{dq}{(x^2+a^2)} \frac{a}{\sqrt{x^2+a^2}}$$

Since we have a uniform charge distribution, the charge per unit length say, λ is constant. or

$$q/l = \lambda \equiv \text{constant}$$

Then one may write, $dq = \lambda dx$. This gives

$$dE_p = \frac{\lambda a}{2\pi\epsilon_0} \frac{dx}{(x^2+a^2)^{3/2}}$$

$$E_p = \frac{\lambda a}{2\pi\epsilon_0} \int_0^{l/2} \frac{dx}{(x^2+a^2)^{3/2}}$$

From the figure,

$$x/a = \tan\theta \Rightarrow dx/a = \frac{a\, d\theta}{\sin^2\theta}$$

$$x = a\tan\theta \Rightarrow dx = a\sec^2\theta\, d\theta$$

If $x=0$, $\theta=0$ and if $x=l/2$, $\theta=\theta_0$.

$$E_p = \frac{\lambda a}{2\pi\epsilon_0} \int_0^{\theta_0} a\sec^2\theta\, d\theta$$

$$a^3\sec^3\theta\, d\theta$$

$$= \frac{\lambda}{2\pi\epsilon_0 a} \int_0^{\theta_0} \cos\theta\, d\theta$$

$$= \frac{\lambda}{2\pi\epsilon_0 a} \sin\theta \Big|_0^{\theta_0}$$

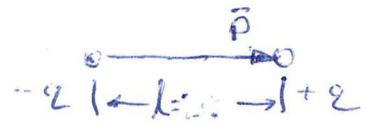
$$= \frac{\lambda}{2\pi\epsilon_0 a} \sin\theta_0$$

But $\sin\theta_0 = \frac{l/2}{(l^2/4 + a^2)^{1/2}} = \frac{l}{(l^2+4a^2)^{1/2}}$

$$\therefore E_p = \frac{q}{2\pi\epsilon_0 a} \frac{l}{\sqrt{l^2+4a^2}}$$

2.5 The Electric Dipole

It is an arrangement of a positive and a negative charge of equal magnitude q that are placed a small distance apart as shown below.

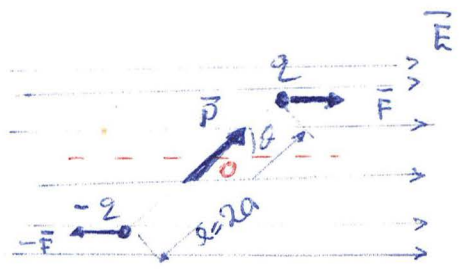


The electric dipole moment $\bar{p}$ is defined by

$$\bar{p} = q\bar{l}$$

where $\bar{l}$ is the displacement from the negative to the positive charge and this vector indicates the direction of the vector $\bar{p}$. So we see that the magnitude of $\bar{p}$ is equal to the product of the magnitude of either charge and their separation, $p = ql$.

In the fig. below we have an electric dipole which is placed in an external uniform electric field $\bar{E}$. The two charges are separated by a dist. $2a$. The dipole moment $\bar{p}$ makes an angle θ wrt the direction of the electric field.



The two forces $\bar{F}$ & $-\bar{F}$ equal and opposite forces are acting on the dipole, where

$$F = qE$$

Even though the forces add up zero, the net torque on the dipole is about an axis through O is nonzero. This torque is given by,

$$\tau = 2(F \frac{l}{2} \sin\theta) = F\, 2a \sin\theta$$

$$\tau = qE\, 2a \sin\theta = q\, 2a\, q\, E \sin\theta$$

or

$$\tau = pE \sin\theta$$

In vector form

$$\bar{\tau} = \bar{P} \times \bar{E}.$$

which is the torque that tries to align the electric dipole with the field.

Work must be done to change the orientation of an electric dipole in an external field and this is stored as the $P \cdot E$ of the system of electric field and dipole. Suppose that the dipole is rotated from its initial orientation θ_0 to a final position θ. Then

$$U = -W_{\text{ext}} = -\int dW = -\int_{\theta_0}^{\theta} \tau_e \, d\theta, \quad \text{ref. of rotational work & } \tau_e \text{ is the torque due to the external agent.}$$

But $\tau_e = -\tau = -PE \sin\theta$ so that

$$U = -\int_{\theta_0}^{\theta} (-PE \sin\theta)\, d\theta = -PE \cos\theta \Big|_{\theta_0}^{\theta}$$

Only the change in $P \cdot E$ is required so that one can take any convenient value of the reference angle θ_0; take $\theta_0 = \pi/2$ where

$$U = -PE \cos\theta$$

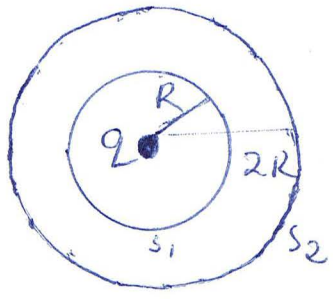
i.e., in vector form

$$U = -\bar{P} \cdot \bar{E}$$

2.6 Gauss Law

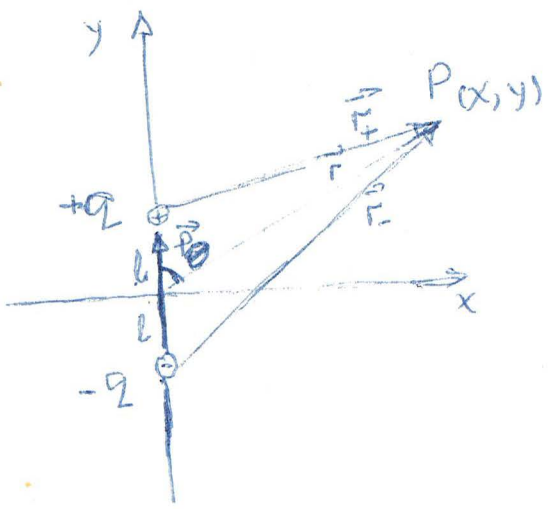
This law is useful in computing the electric field of a charge distribution of exhibiting symmetry as the discussion below may show.

Suppose that a positive point charge Q is surrounded by two concentric spherical surfaces of radii R and $2R$ as shown in the fig. below.



Areas of the spheres S_1 and S_2 are $4\pi R^2$ and $16\pi R^2$, respectively. If the total no. of lines of force emanating from Q is N, the corresponding

Exercise :



Calculate the electric field of the dipole at the point P s.t $r \gg l$. Obtain the values

$$E_x = \frac{1}{4\pi\varepsilon_0} \frac{3pxy}{r^5}$$

$$E_y = \frac{1}{4\pi\varepsilon_0} \frac{p(2y^2 - x^2)}{r^5}$$

Where $r^2 = x^2 + y^2$.

or,

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \left(-\frac{\vec{p}}{r^3} + 3\frac{(\vec{p}\cdot\vec{r})\vec{r}}{r^5} \right), \vec{p} = p\hat{j}$$

lines of force per unit area for each sphere are $n_1 = \frac{N}{4\pi R^2}$ and $n_2 = N/16\pi R^2$.

According to one of the properties of the lines of force from sec. 2·3, these expressions mean that the field is inversely proportional to the square of the radius, i.e. which is in line i coulomb's law. But the area of the sphere is proportional to the square of the radius, so that the product of the two, the field and the area, is independent of the radius. For a sphere of arbitrary radius r, the magnitude of the electric field $\bar{E}$ on the surface is

$$E = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}, \text{ coulomb's law}$$

the surface area is

$$S = 4\pi r^2$$

whence

$$ES = q/\epsilon_0$$

which depends only on the charge q.

On the other hand, dividing the sphere into infinitesimal area elements ΔS and adding all the products $E\Delta S$ of these elements will give the same result, i.e.,

$$\Sigma E\,\Delta S = E\,\Sigma\Delta S \;(E \equiv \text{constant}) = ES = q/\epsilon_0$$

or

$$\Sigma E\,\Delta S = q/\epsilon_0.$$

Instead of a spherical surface, we can have an arbitrary surface enclosing the charge q. But in this case dividing the surface into infinitesimal area elements, we have to add the products of the area elements components of the area elements along the radial directions drawn from q and the corresponding values of the electric field intensity E to obtain the correct value. Or,

$$\Sigma \bar{E}\cdot\Delta\bar{S} = q/\epsilon_0$$

or the limit of infinite no. of divisions,

$$\oint \bar{E}\cdot d\bar{s} = q/\epsilon_0 \quad \cdots (*)$$

By def^n the left hand side of the above eq. is called the electric flux Φ_E over the surface enclosing the charge q.

$$\Phi_E = \oint \bar{E}\cdot d\bar{s}$$

If the surface is not a closed one, the electric flux over this surface is given by

$$\Phi_E = \int \bar{E}\cdot d\bar{s} \quad \cdots (*)$$

If θ be the angle between $\bar{E}$ & $d\bar{s}$ then

i. $\theta < 90^\circ \Rightarrow \Phi_E > 0$; since q is +ve lines of force point only outward here

ii. $\theta > 90^\circ \Rightarrow \Phi_E < 0$; lines of force point inward

iii. $\theta = 90^\circ \Rightarrow \Phi_E = 0$; the incoming and outgoing lines of force are equal.

Re arranging, ~~Gauss~~ eq. (i) is rewritten in the form

$$\epsilon_0 \oint \bar{E} \cdot d\bar{s} = q \qquad (ii)$$

This is called Gauss's law and it relates the electric field with the charge q responsible for the field. Note also that for a surface enclosing a no. of charges, (positive, negative) q must be replaced by the net charge within the surface. Or if

$$Q = \Sigma q, \text{ algebraic sum}$$

then

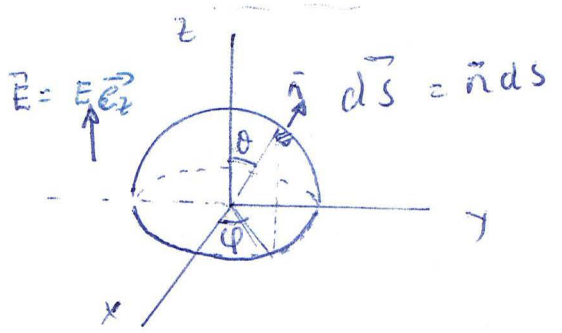
$$\epsilon_0 \oint \bar{E} \cdot d\bar{s} = Q \qquad (iii)$$

The surface enclosing the charge is called the Gaussian surface.

Examples

1. Calculate Φ_E through a hemisphere of radius R. The field $\bar{E}$ is uniform and parallel to the axis of the hemisphere.

Solution



$$\phi_E = \int \bar{E} \cdot d\bar{s} = \int E\, ds \cos\theta$$

In spherical coordinates, $ds = r^2 \sin\theta\, d\theta\, d\varphi$. Or since $r = R$, $ds = R^2 \sin\theta\, d\theta\, d\varphi$. Then

$$\phi_E = E R^2 \int_0^{2\pi} d\varphi \int_0^{\pi/2} \sin\theta \cos\theta\, d\theta \quad , E \text{ is uniform}$$

$$= 2\pi \times \frac{1}{2} E R^2 \int_0^{\pi/2} \sin 2\theta\, d\theta$$

$$= \frac{1}{2} \pi R^2 E \left(-\cos 2\theta\right)_0^{\pi/2}$$

or

$$\phi_E = \pi R^2 E$$

(1.e.)

$$E = \frac{\rho r}{2\varepsilon_0}$$

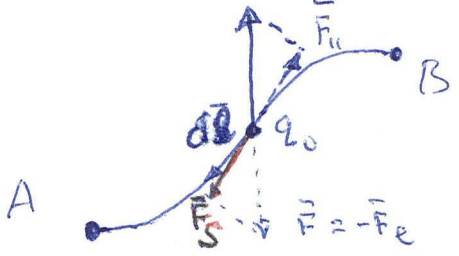
Chapter Three The Electric potential

3.1 Definition of the Electric potential

The region around a charged body can be described not only by a vector field $\bar{E}$ but also by a scalar quantity, the electric potential V. That is by associating the values of this quantity with each and every point around the body, we can as well property describe the region — define its electrical property. These quantities, $\bar{E}$ and V, are intimately related, and often it is a matter of convenience which is used in a given problem.

To find the electric potential difference between two points A and B in an electric field, fig. below, we move a positive test charge q_0 from B to A, always keeping in equilibrium, and measure the work W_{BA} that must be done by the agent moving the charge. The

$$\bar{F}_e = q_0 \bar{E}$$



electric potential difference is defined from

$$V_A - V_B = \frac{W_{BA}}{q_0}$$

where

$$W_{BA} = \int_B^A \bar{F} \cdot d\bar{l}$$

$\bar{F}$ is the force that has to be exerted against the electric force on q_0 by an external agent.

The work W_{BA} may be positive, negative or zero in which the corresponding potential at A will be higher, lower or the same as the potential at B.

If point B is taken to be located at infinity from all the charges, the potential here is arbitrarily assumed to be zero. This allows us to define the electric potential at a point. putting $V_B = 0$ and dropping the subscripts we write

$$V = \frac{W}{q_0}$$

where now W is the work that an external agent must do to move the test charge q_0 from infinity to the point in question.

From the above eq. it is seen that V near an isolated positive charge is positive because positive work must be done by an outside agent to push a (positive) test charge in from infinity. Similarly, the potential near an isolated negative charge is negative because an outside agent must exert a restraining force on (i.e., must do negative work on) a (positive) test charge as it comes in from infinity.

Note Both W_{BA} and $V_A - V_B$ are independent of path followed in moving the test charge from point B to point A. Such path independence of the work holds for all conservative fields. If this were not so, point A would not have a unique electric potential (wrt B as a defined reference position), and the concept of potential would not be useful.

The locus of points, all of which have the same electric potential, is called an equipotential surface. No work is done in moving a test charge between any two points on an equipotential surface. This follows from the eq.

$$V_A - V_B = \frac{W_{BA}}{q_0}$$

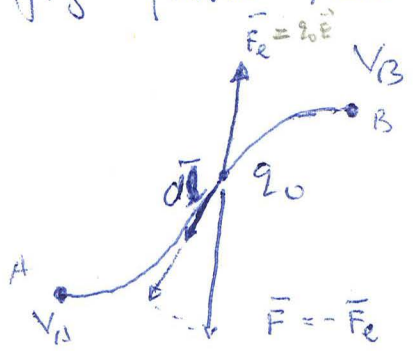
Since W_{BA} must be zero if $V_B = V_A$.

From symmetry, the equipotential surfaces for a spherical charge are a family of concentric spheres. For a uniform field they are a family of planes at right angles to the field. In all cases the equipotential surfaces are perpendicular to the lines

force and thus to $\bar{E}$. If $\bar{E}$ were not at right angles to the equipotential surface, it would have a component lying in that surface. Consequently, work would have to be done in moving a test charge about on the surface. Work cannot be done if the surface is an equipotential, so $\bar{E}$ must be at right angles to the surface.

3.2 Relations between the Electric field and the Electric potential

Consider the fig. from sec. 3.1.



As defined, the p.d. between points A and B is

$$V_A - V_B = \frac{W_{BA}}{q_0}$$

Now,

$$W_{BA} = \int_B^A \bar{F}\cdot d\bar{l} = -q_0\int_B^A \vec{E}\cdot d\bar{l}$$

This integral is to be calculated along a path connecting the points A and B. Then one obtains

$$V_A - V_B = -\int_B^A \vec{E}\cdot d\bar{l} \quad \cdots (*)$$

which gives the specific relation between the electric field and the p.d. Note that if points A and B coincide so that A = B, then the path of integration is a closed curve; and

$$\oint \bar{E}\cdot d\bar{l} = 0$$

indicates that the work done by a static electric field when a charge traverses a closed path is zero. Thus the static electric field corresponds to a conservative force.

Referring to the above fig., eq. (*) may be written in an alternative form as

$$V_A - V_B = -\int_B^A E_\ell \, d\ell$$

where E_ℓ is the comp. of $\bar{E}$ along the path. Also,

$$\int_B^A E_\ell \, d\ell = -\int_B^A dV$$

Suppose now that points A and B are so close that each integral in the eq. above practically reduces to a single term. Then

$$E_\ell \, d\ell = -dV \quad \text{or} \quad E_\ell = -\frac{\partial V}{\partial \ell}$$

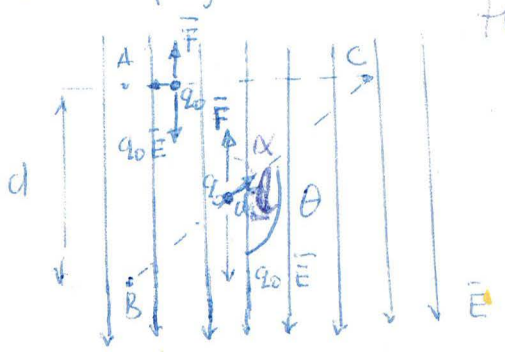
which shows that the comp. of the electric field in a certain direction is equal to the negative of the change of electric potential per unit length in the same direction. For example the rectangular components of the electric field are given by

$$E_x = -\frac{\partial V}{\partial x}, \quad E_y = -\frac{\partial V}{\partial y}, \quad E_z = -\frac{\partial V}{\partial z}$$

Thus if V is known for all points of space, that is, if the function $V(x,y,z)$ is known, the components of $\bar{E}$, and thus $\bar{E}$ itself, can be found by taking derivatives.

Note The partial derivative $\partial V/\partial x$ instructs us to take the derivative of $V(x,y,z)$ considering x as a variable and y and z as constants. Similarly for $\partial V/\partial y$ and $\partial V/\partial z$.

Example: Calculate $V_A - V_B$ for the p.d. of points A and B which are in a uniform electric field $\bar{E}$ as shown in the fig. below. (Use the path indicated in the fig.)



Solution

Consider the arbitrary path shown in the above figure.

$$V_C - V_B = \frac{W_{BC}}{q_0} = -\int_B^C \vec{E}\cdot d\vec{\ell}$$

But $\int_B^C \vec{E}\cdot d\vec{\ell} = \int_B^C E\, d\ell \cos\theta = E\cos\theta \int_B^C d\ell = E\cos\theta\, S_{BC}$

Since E and $\cos\theta$ are constant. But

$$\cos\theta = \cos(180^\circ - \alpha) = -\cos\alpha$$

i.e.)

$$V_C - V_B = E\cos\alpha\, S_{BC} = E\,d$$

Next, we

$$V_A - V_C = -\int_C^A \vec{E}\cdot d\vec{\ell} = \int_C^A E\, d\ell \cos 90^\circ = 0$$

A and C have the same potential - no work is done. which gives

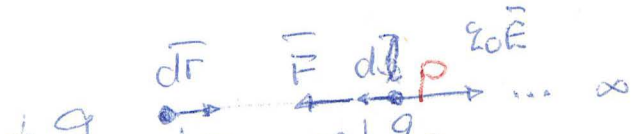
$$V_A - V_B = V_C - V_B + V_A - V_C = E\,d$$

or

$$V_A - V_B = E\,d$$

3.3 Potential Due to a point charge

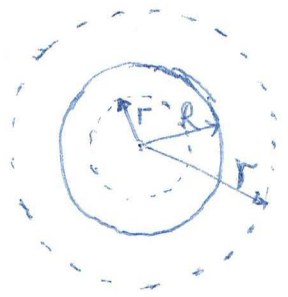
Let us find the potential at a point P which is located a dist. r from a positive point charge Q.



A test charge q_0 is moved in from infinity against the electric field

$$V = -\int_\infty^r \vec{E}\cdot d\vec{\ell} = -\int_\infty^r E\, d\ell \cos 180^\circ = +\int_\infty^r E\, d\ell$$

Example: Electric potential due to a uniform spherical charge distribution



Total charge Q

Volume charge density ρ = const.

Show that

a, $V = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r} = \frac{\rho R^3}{3\varepsilon_0 r}$ | $r > R$

b, $V = \frac{Q}{8\pi\varepsilon_0 R^3}(r^2+3R^2) = \frac{\rho}{6\varepsilon_0}(r^2+3R^2)$

---

a, $r > R$

$E = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}$

$V = -\int_{\infty}^{r} E\,dr = -\frac{Q}{4\pi\varepsilon_0}\int_{\infty}^{r}\frac{dr}{r^2} = \frac{Q}{4\pi\varepsilon_0 r}\Big|_{\infty}^{r} = \frac{Q}{4\pi\varepsilon_0 r}$

or,

$V = \frac{\rho \times \frac{4}{3}\pi R^3}{4\pi\varepsilon_0 r} = \frac{\rho R^3}{3\varepsilon_0 r}$.

b, $r < R$

$E = \frac{Q}{4\pi\varepsilon_0 R^3} r = \frac{\rho r}{3\varepsilon_0}$

$V(r) - V(R) = -\int_{R}^{r} E\,dr = -\frac{Q}{4\pi\varepsilon_0 R^3}\int_{R}^{r} r\,dr = -\frac{Q}{4\pi\varepsilon_0 R^3}\cdot\frac{r^2}{2}\Big|_{R}^{r}$

$V(r) - V(R) = \frac{Q}{8\pi\varepsilon_0 R^3}(R^2 - r^2)$

or,

$V(r) = \frac{Q}{8\pi\varepsilon_0 R^3}(R^2 - r^2) + \frac{Q}{4\pi\varepsilon_0 R} = \frac{Q}{8\pi\varepsilon_0 R^3}(3R^2 - r^2)$.

or,

$V(r) = \frac{\rho}{6\varepsilon_0}(3R^2 - r^2)$

The vector $\bar{r}$ is drawn outward from the charge q so that

$$d\ell = -dr$$

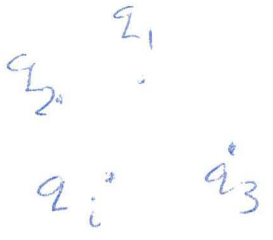
$$\Rightarrow V = -\int_{\infty}^{r} \bar{E}\cdot d\bar{r} = -\int_{\infty}^{r} E\,dr = -\frac{q}{4\pi\varepsilon_0}\int_{\infty}^{r}\frac{dr}{r^2}$$

so assuming the potential to be zero at infinity,

$$\boxed{V = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}}$$

- A group of point charges

If there are several charges $q_1, q_2, \ldots q_N$ then as shown below, the electric potential at a point



P is the algebraic sum of their individual potentials. That is,

$$V = \frac{q_1}{4\pi\varepsilon_0 r_1} + \cdots + \frac{q_N}{4\pi\varepsilon_0 r_N} = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{N}\frac{q_i}{r_i}$$

- A continuous charge distribution

To find the el. potential here divide the given charged body into infinitesimal elements dq and calculate the contribution due to each element at the indicated point (treating it as point charge). So the total potential at this point is given by adding the contributions

$$dV = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r}$$

of these elements. Or

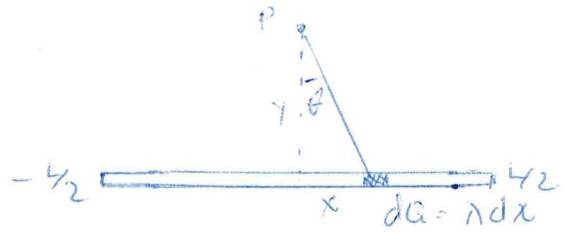
$$V = \frac{1}{4\pi\varepsilon_0}\int\frac{dq}{r}$$

r is the dist. of the point from the charge

Example:

Determine the electric potential along a $\perp$ bisector of a straight line charge with length L and uniform linear charge density λ. Show that when the dist. becomes large compared with L, your result reduces to that for a point charge $Q = \lambda L$ located at the center of the line charge.

solution:



The electric potential at P due to the charge element dQ is given by

$$\delta V = \frac{1}{4\pi\varepsilon_0}\frac{\lambda\,dx}{\sqrt{x^2+y^2}}$$

i.e.,

$$V = \frac{\lambda}{4\pi\varepsilon_0}\int_{-L/2}^{L/2}\frac{dx}{\sqrt{x^2+y^2}}$$

Now,

$$x = y\tan\theta \Rightarrow dx = y\sec^2\theta\,d\theta$$

Let $\theta = \theta_0$ when $x = L/2$. Then,

$$V = \frac{\lambda}{4\pi\varepsilon_0}\int_{-\theta_0}^{\theta_0}\frac{y\sec^2\theta\,d\theta}{y\sec\theta} = \frac{\lambda}{4\pi\varepsilon_0}\int_{-\theta_0}^{\theta_0}\frac{d\theta}{\cos\theta} = \frac{\lambda}{4\pi\varepsilon_0}\int_{-\theta_0}^{\theta_0}\frac{d(\pi/2+\theta)}{\sin(\pi/2+\theta)}$$

Next, if $\theta = \pi/2 + \theta$, we get

$$V = \frac{\lambda}{4\pi\varepsilon_0}\int_{\pi/2-\theta_0}^{\pi/2+\theta_0}\frac{du}{\sin u},$$

The substitution $\tan\frac{u}{2} = v$ gives: $\sin u = \frac{2v}{1+v^2}$, $du = \frac{2\,dv}{1+v^2}$

And,

$$V = \frac{\lambda}{4\pi\varepsilon_0}\ln v = \frac{\lambda}{4\pi\varepsilon_0}\ln\left[\tan\frac{u}{2}\right]_{\pi/2-\theta_0}^{\pi/2+\theta_0} = \frac{\lambda}{4\pi\varepsilon_0}\ln\left\{\tan(\pi/4+\theta/2)\right\}_{-\theta_0}^{\theta_0}$$

i.e.,

$$V = \frac{\lambda}{4\pi\varepsilon_0}\left[\ln\left\{\tan(\pi/4+\theta_0/2)\right\} - \ln\left\{\tan(\pi/4-\theta_0/2)\right\}\right]$$

$$= \frac{\lambda}{4\pi\varepsilon_0}\ln\left\{\frac{\tan(\pi/4+\theta_0/2)}{\tan(\pi/4-\theta_0/2)}\right\} = \frac{\lambda}{2\pi\varepsilon_0}\ln\left\{\frac{1+\tan\theta_0/2}{1-\tan\frac{\theta_0}{2}}\right\}$$

Since $\tan\theta_0/2 = \sqrt{1-\cos\theta_0}\,/\sqrt{1+\cos\theta_0}$ s.t. $\cos\theta_0 = y/\sqrt{y^2+L^2/4}$ we find

$$V(y) = \frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{\sqrt{y^2+L^2/4}}{y} + L/2y\right)$$

If $y \gg L$,

$$V(y) \approx \frac{\lambda}{2\pi\varepsilon_0}\ln(1+L/2y) = \frac{\lambda L}{4\pi\varepsilon_0 y} = Q/4\pi\varepsilon_0 y.$$

element dQ

3.4 Electric potential Energy

The electric p.E. of a system of point charges is defined as the work required to assemble this system of charges by bringing them in from an infinite dist. Here the charges are assumed to be at rest (no K.E.) in both their initial and final positions.

Consider two charges Q_1 and Q_2 separated by a dist r.

Q_1 ○ ---- r ----→ ○ Q_2

Assuming Q_2 is removed to infinity for the moment the electric pitential of Q_1 at the original site of Q_2 is given by

$$V = \frac{1}{4\pi\varepsilon_0}\frac{Q_1}{r}$$

If Q_2 is moved in to the original position a dist. r from Q_1, the work required is, from the def. of electric potential,

$$W = VQ_2$$

This is defined as the p.E. of the system of the two charges, (i.e.)

$$U = W = \frac{1}{4\pi\varepsilon_0}\frac{Q_1 Q_2}{r}$$

or

$$U = \frac{1}{4\pi\varepsilon_0}\frac{Q_1 Q_2}{r_{12}}$$

the subscript on r is indicating that the dist. involved is that between the charges Q_1 and Q_2.

For a system of point charges the procedure is to compute the p.E. for every pair of charges separately and add the results algebraically.

Unit

$$[V] = \left[\frac{W}{Q_0}\right] = \frac{J}{C} \equiv \text{Volts}(V) \text{ in SI}$$

Examples:

1. A particle of mass m, charge $q > 0$, and initial K.E. K is projected (from "infinity") toward a heavy nucleus of charge Q, assumed to have a fixed position in an inertial reference frame. If the aim is "perfect," how close to the center of the nucleus is the particle when it comes instantaneously to rest?

Solution:

$\infty \cdots \overset{q}{\bullet} \longrightarrow \quad \cdots \quad (Q)$

instantaneous position where q is at rest.

The K.E. K of the particle is transformed to the P.E. of the particle - nucleus system. Or,

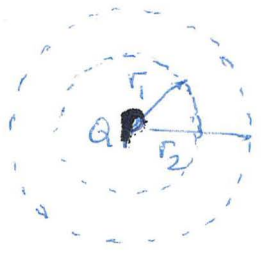
$$K = \frac{1}{4\pi\varepsilon_0}\frac{qQ}{r_0}$$

r_0 being the dist. from the nucleus at which q is at rest. Then,

$$r_0 = \frac{1}{4\pi\varepsilon_0}\frac{qQ}{K}.$$

2. A particle of (+ve) charge Q is assumed to have a fixed position at P. A 2nd particle of mass m and (negative) charge $-q$ moves at a const. speed in a circle of radius r_1, centered at P. Derive an expression for the work W that must be done by an external agent on the 2nd particle in order to increase the radius of the circle of motion, centered at P, to r_2.

Solution:



At $r = r_1$, the M.E. E_1 is

$$E_1 = \frac{1}{4\pi\varepsilon_0}\frac{qQ}{r_1} + \tfrac{1}{2}mv^2$$

Since $-q$ performs circular motion,

$$\frac{1}{4\pi\varepsilon_0}\frac{qQ}{r_1^2} = \frac{mv^2}{r_1} \Rightarrow \tfrac{1}{2}mv^2 = \frac{1}{8\pi\varepsilon_0}\frac{qQ}{r_1}.$$

That is,

$$E_1 = -\frac{1}{4\pi\varepsilon_0}\frac{qQ}{r_1} + \frac{1}{8\pi\varepsilon_0}\frac{qQ}{r_1}$$

$$= -\frac{1}{8\pi\varepsilon_0}\frac{qQ}{r_1}.$$

Similarly,

$$E_2 = -\frac{1}{8\pi\varepsilon_0}\frac{qQ}{r_2}.$$

Thus, the work done by the external agent is

$$W = E_2 - E_1 = -\frac{1}{8\pi\varepsilon_0}\frac{qQ}{r_2} + \frac{1}{8\pi\varepsilon_0}\frac{qQ}{r_1}$$

$$= -\frac{qQ}{8\pi\varepsilon_0}\left(\tfrac{1}{r_2} - \tfrac{1}{r_1}\right).$$

Chapter Four Capacitors and Dielectrics

4.1 Capacitance

The electric capacitance (C) of an isolated (or insulated) conductor is defined as the ratio of its charge (Q) to its potential (V) (i.e., the potential on its surface); or,

$$C = Q/V$$

This eq. indicates how much charge is stored on a conductor for a given potential.

cf. for example, an isolated spherical conductor of radius R contains a total charge Q. the electric potential on its surface is

$$V = \frac{Q}{4\pi\varepsilon_0 R}$$

Whence its capacitance is

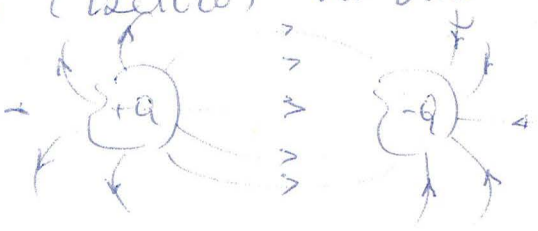
$$C = \frac{Q}{V} = 4\pi\varepsilon_0 R$$

unit

$$[C] = \left[\frac{Q}{V}\right] = \frac{\text{coulomb}}{\text{volt}} \left(\frac{C}{V}\right), \text{ in SI.}$$

The unit C/V is called the farad (F).

The concept of capacitance can also be extended for a system of conductors. Consider the case of two conductors (below) having charges $+Q$ and $-Q$.

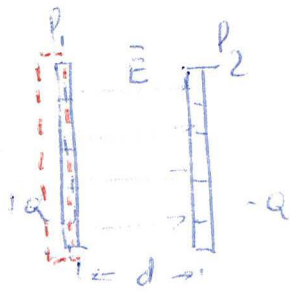


If V_1 and V_2 are their respective potentials so that $V = V_1 - V_2$ is their p.d., the capacitance of the system is defined as

$$C = \frac{Q}{V_1 - V_2} = \frac{Q}{V}$$

This arrangement constitutes what is defined as a capacitor.

Example: Calculate the capacitance of the arrangement of conductors shown below called parallel-plate capacitor shown below. (They have the same surface area A)



Find $\vec{E}$.

$\epsilon_0 \oint \vec{E} \cdot d\vec{S} = Q$. For small d $\vec{E}$ is uniform

$\epsilon_0 \int E\,ds = Q$, $\vec{E}$ is parallel to $d\vec{S}$

$$\epsilon_0 E A = Q$$

or $$E = \frac{Q}{\epsilon_0 A}$$

$$V = -\int_{P_2}^{P_1} \vec{E} \cdot d\vec{\ell}$$, $d\vec{\ell}$ is a line element when going from the -ve plate to the positive plate.

$$V = -\int_{P_2}^{P_1} E\,d\ell \cos 180^\circ = \int_{P_2}^{P} E\,d\ell = E\,d$$

or $$V = \frac{Q\,d}{\epsilon_0 A}$$

We then obtain

$$C = \frac{Q}{V} = \frac{Q}{Qd/\epsilon_0 A} = \frac{\epsilon_0 A}{d}$$

Similarly diff. formulas are derived for other types of capacitors.

7.2 Energy Stored within an Electric Field

A charged capacitor has stored in it an electrical P.E. U equal to the work W required.

to charge it i.e., the work required to bring the charges to the required configuration (or arrangement). This energy can be recovered if the capacitor is allowed to discharge; the charges are allowed to go thereby doing work at the expense of the P.E. stored in the system.

At time t let an amount of charge $q(t)$ be transferred from one plate to the other and let the p.d. between the conductors at this moment be $V(t) = q(t)/C$. If the external agent brings an increment of charge dq, the corresponding work done will be

$$dW = V(t)\, dq(t) = \frac{q(t)}{C}\, dq(t)$$

Suppose then the total charge transferred is Q. Then the total work done is given by

$$W = \int dW = \frac{1}{C}\int_0^Q q\, dq = \frac{Q^2}{2C}$$

This is then the P.E. stored in the capacitor which can be thought of as being residing in the electric field between the conductors. Thus

$$\boxed{U = \frac{Q^2}{2C} = \frac{1}{2}CV^2}$$

For a parallel plate capacitor, the energy per unit volume (energy density) u is

$$u = \frac{U}{Ad} = \frac{\frac{1}{2}CV^2}{Ad} = \frac{1}{2}\frac{\epsilon_0 A}{A}\left(\frac{V}{d}\right)^2 = \frac{1}{2}\epsilon_0 E^2$$

or

$$\boxed{u = \tfrac{1}{2}\epsilon_0 E^2}$$

Exercise: Obtain an expression for the total energy stored in the electric field of a uniformly charged nonconducting sphere.

$U = \frac{3Q^2}{20\pi\epsilon_0 R}$

But not only for the special case of a parallel-plate capacitor this expression holds for other types of capacitors as well.

1.3 The Polarization of Matter (Dielectrics)

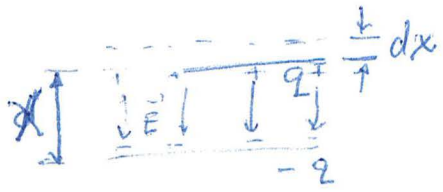
Examples:

1. Show that the plates of a parallel-plate capacitor attract each other with a force given by

$$F = q^2/2\epsilon_0 A$$

Prove this by calculating the work necessary to increase the plate separation from x to $x+dx$ with the charging mechanism being disconnected.

Solution:



The work done by an external agent in separating the plates by dx is equal to the change in the electric p.E. U of the capacitor. Or,

$$dW = dU = d\left(\frac{q^2}{2C}\right) = d\left(\frac{q^2 x}{2\epsilon_0 A}\right) = \frac{q^2}{2\epsilon_0 A}dx.$$

But in terms of the force applied by the external agent we may write

$$dW = F\,dx.$$

So equating the two results we have

$$F\,dx = \frac{q^2}{2\epsilon_0 A}dx$$

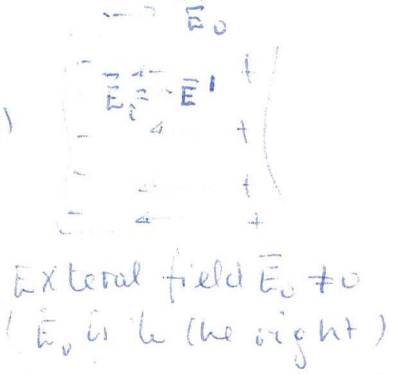
or,

$$F = \frac{q^2}{2\epsilon_0 A}.$$

- A dielectric is a nonconducting material. Let us see what happens when a dielectric is placed in an electric field. In ~~some~~ molecules of some dielectrics (like water, for example) there is some sort of localization of the negative and positive electric charges separately resulting in a permanent electric dipole moment. If such a material is placed within an external electric field, the dipole moments $\bar{P}$ will try to align themselves with the external electric field as in the fig below. Be-

cause the molecules are in constant thermal agitation, the degree of alignment will not be complete but will increase as the applied electric field is increased or as the temp. is decreased.

- Whether or not the molecules have permanent electric dipole moments, they acquire them by induction when placed in an electric field, i.e., the external field tends to separate the negative and positive (called polarization) charge in the atom or the molecule (fig. below). This



induced dipole moment is present only when the electric field is present. It is proportional to the electric field (for moderate field strengths). The resultant electric fied $\bar{E}$ in the above fig is given by

$$\bar{E} = \bar{E}_0 - \bar{E}'$$

- Let us now see the effect of a dielectric on the capacitance of a conductor. Consider a

parallel plate capacitor, for example. It has been found out that if a dielectric is placed in the space between the plates of this capacitor, the capacitance increases by a factor K, called the dielectric constant of the material ~~of which the~~ introduced plates are made. This is true for any capacitor and for the parallel plate capacitor we write

$$C = K\frac{\epsilon_0 A}{d}$$

For capacitors in general we have

$$C = K\epsilon_0 L$$

where L depends on the geometry and has the dimensions of a length. For the parallel plate capacitor $L = A/d$. Also, the introduction of a dielectric increases the charge on a capacitor by a factor of K and decreases the p.d. by a factor of $1/K$. Or

$$Q = KQ_0 \quad \text{and} \quad V = \frac{V_0}{K}$$

where Q_0 and V_0 are the charge and p.d. before the introduction of a dielectric and Q and V are those after the introduction of a dielectric.

If a dielectric slab of dielectric constant K is present within a parallel-plate capacitor, the energy density is given by

$$u = \frac{\frac{1}{2}CV^2}{Ad} = \frac{1}{2}\frac{K\epsilon_0 A}{A}\left(\frac{V}{d}\right)^2 = \frac{1}{2}K\epsilon_0 E^2$$

This relation again is a general one holding for other capacitors as well.

Note also that $V = \frac{V_0}{K}$ implies $E = E_0/K$ for the resultant electric field within the capacitor.

~~If~~ Gauss' law ~~is~~ if a dielectric is inserted in a parallel plate capacitor in the form

$$\epsilon_0 \oint \vec{E} \cdot d\vec{A} = Q$$

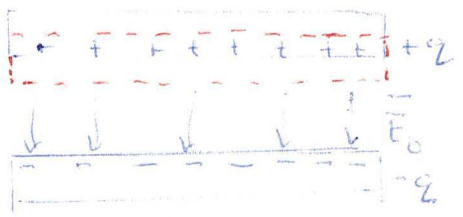
for a parallel plate capacitor where Q is the ~~dielectric~~ charge

4.4 Dielectrics and Gauss's Law

So far our use of Gauss law has been confined to situations in which no dielectric was present. Now let us apply this law to a parallel-plate capacitor filled with a dielectric of dielectric constant K.

Figs. (a) below shows the capacitor with without the dielectric whereas fig. (b) shows the capacitor with the dielectric placed in between its plates. In either case the magnitude of the charge $\mathcal{Q}$ on the plates is the same.

a.

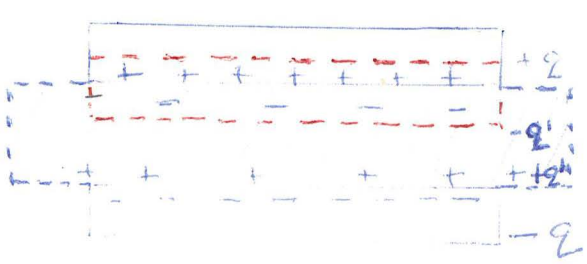


$$\epsilon_0 \oint \vec{E} \cdot d\vec{S} = \epsilon_0 E_0 A = \mathcal{Q}$$

or

$$E_0 = \frac{\mathcal{Q}}{\epsilon_0 A}$$

b.



Thickness of slab = d

So for case b. we write

$$\epsilon_0 \oint \vec{E} \cdot d\vec{S} = \mathcal{Q} - \mathcal{Q}',$$

The net charge within the red Gaussian surface where $-\mathcal{Q}'$ is the induced surface charge and $\mathcal{Q}$ is the free charge on the plates.

$$\epsilon_0 E A = \mathcal{Q} - \mathcal{Q}'$$

or

$$E = \frac{\mathcal{Q}}{\epsilon_0 A} - \frac{\mathcal{Q}'}{\epsilon_0 A}$$

But

$$E = \frac{E_0}{K} = \frac{\mathcal{Q}}{K \epsilon_0 A} \quad \cdots (A)$$

where

$$\frac{\mathcal{Q}}{K \epsilon_0 A} = \frac{\mathcal{Q}}{\epsilon_0 A} - \frac{\mathcal{Q}'}{\epsilon_0 A}$$

which gives

$$\mathcal{Q}' = \mathcal{Q}\left(1 - \frac{1}{K}\right)$$

This shows correctly that the induced surface charge q' is always less than the free charge q and is equal to zero if no dielectric is present, i.e., if $K = 1$.

So now we write Gauss's law as

$$\epsilon_0 \oint \vec{E} \cdot d\vec{S} = q - q'$$

and, substituting the expression for q' and rearranging we obtain

$$\epsilon_0 \oint K\vec{E} \cdot d\vec{S} = q$$

This relation although derived for a parallel plate capacitor, is true generally and is the form in which Gauss's law is usually written when dielectric is present.

4.5 The Electric Displacement

From the previous section we have the equation

$$\frac{q}{K\epsilon_0 A} = \frac{q}{\epsilon_0 A} - \frac{q'}{\epsilon_0 A}$$

which may be written in the form

$$\frac{q}{A} = \epsilon_0 \left(\frac{q}{K\epsilon_0 A} \right) + \frac{q'}{A}$$

in the parentheses

The first term on the right represents the electric field strength E in the dielectric (check). The last term is the induced surface charge per unit area. We call it the electric polarization P, or

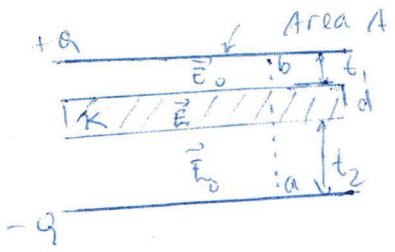
$$P = q'/A$$

P can also be defined in an equivalent way by multiplying the numerator and denominator in the above eq. by d, the thickness of the dielectric slab. Or

$$P = \frac{q'd}{Ad}$$

Now the numerator represents the product of the magnitude q' of the (equal and opposite) polarization charges by their separation d. Thus it is the induced electric dipole

Example: Determine the capacitance of a parallel-plate capacitor in which the region between the plates is partially filled with a dielectric slab of thickness d, as shown in the diagram.



Solution:

The electric field strength in the air gaps is $\vec{E}_0 = Q/\epsilon_0 A$. Also, $E = E_0/K = Q/K\epsilon_0 A$.

The potential difference between the plates can be expressed as

$$V = V_b - V_a = -\int_a^b \vec{E}\cdot d\vec{r}$$

Evaluating the integral along the dashed-line path shown in the diagram, we find

$$V = E_0\left(t_1 + \frac{d}{K} + t_2\right) = \frac{Q}{\epsilon_0 A}\left(t + \frac{d}{K}\right),$$

where $t = t_1 + t_2$ is the total thickness of the air gap. The capacitance is defined to be $C = Q/V$, so that

$$C = \frac{\epsilon_0 A}{t + d/K}.$$

... equivalent to three series capacitors of capacitance $\frac{\epsilon_0 A}{t_1}$, $K\frac{\epsilon_0 A}{d}$ and $\frac{\epsilon_0 A}{t_2}$.

moment of the dielectric slab. Since Ad is the volume of the slab, we see that the electric polarization can also be defined as the induced electric dipole moment per unit volume in the dielectric. This def. suggests that since the electric dipole moment is a vector the electric polarization is also a vector, its magnitude being P. The direction of $\bar{P}$ is from the negative induced charge to the positive induced charge, as for any dipole.

We can now write

$$\frac{q}{A} = \epsilon_0 E + P$$

The quantity on the right is called the electric displacement D, or

$$D = \epsilon_0 E + P$$

in which

$$D = q/A \qquad \cdots (E)$$

Since $\bar{E}$ and $\bar{P}$ are vectors, $\bar{D}$ must also be a vector so that one can write

$$\boxed{\bar{D} = \epsilon_0 \bar{E} + \bar{P}}$$

$\bar{D}$ and $\bar{P}$ are auxiliary vectors and they both can be expressed in terms of $\bar{E}$ alone. Thus,

$$\frac{q}{A} = K\epsilon_0 \left(\frac{q}{K\epsilon_0 A} \right)$$

Considering eq. (A) from sec. 4.4 and eq. (E) from this section and extending to a vector form we write

$$\boxed{\bar{D} = K\epsilon_0 \bar{E}}$$

We also write the polarization as

$$P = \frac{q'}{A} = \frac{q}{A}(1 - 1/K)$$

Since $q/A = D$, this eq. can be rewritten in vector form as

$$\boxed{\bar{P} = \epsilon_0 (K-1) \bar{E}}$$

Define $\chi_e = K - 1$, this is called the electric susceptibility of the dielectric

This shows clearly that in vacuum ($K=1$) the polarization vector is zero.

The last two eqs. in boxes hold for the so called isotropic materials.

Finally, the introduction of the electric displacement $\vec{D}$ enables us to write Gauss law for dielectrics as

$$\oint \vec{D} \cdot d\vec{S} = q$$

where q, as before, represents the free charges only, the induced surface charge being included in the left hand side of the eq.

Chapter 5 Electric Circuits

5-1 Electric Current and Current Density

We have seen in the previous sections that when an electric field is applied to a dielectric, a polarization of the dielectric results. But we have a diff. case for conductors. These materials have out free electrons that are in random motion like the molecules of a gas confined to a container. They have no net direction of motion. If an electric field is applied ~~applied~~ to such a material, the free charges (free electrons) are set in motion in a preferred direction and an electric current, instead of a polarization of the medium results. These charges are accelerated by the field and therefore gain energy.

Def. The current across an area is the net charge flowing across the area per unit time. Thus if a net charge q flows across an area in a time t, the current i across the area is

$$i = q/t$$

Unit

$$[\,i\,] = [\,q/t\,] = \frac{\text{Coulomb}}{\text{Second}} \equiv \text{Ampere}(A) \text{ in SI.}$$

If the rate of flow of charge is not constant with time, the current is given by the relation

$$i = \frac{dq}{dt}$$

For simplicity, the positive direction of a current is taken to be the direction of positive charge flow.

Consider a conducting wire of cross sectional area A and length l placed inside an external electric field $\vec{E}$ as shown below

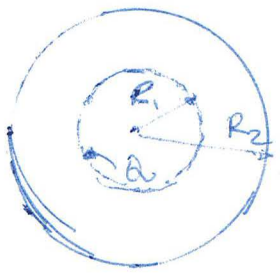
Example: A conducting sphere of radius R_1 carries a free charge Q and is surrounded by a concentric dielectric spherical shell with an outer radius R_2 and a dielectric const. K. This isolated combination is in air ($K \approx 1$).

a, Determine the displacement $\vec{D}$ everywhere.

b, " " electric field strength $\vec{E}$ everywhere.

c, What is the polarization $\vec{P}$ within the dielectric?

solution:



a, charge always resides on the surface of a conducting body so that inside the above conducting sphere, $r < R_1$, the field is zero: $\vec{E} = 0$, $\vec{D} = 0$. Using a spherical Gaussian surface of radius $r > R_1$, we have

$\oint_S \vec{D} \cdot d\vec{s} = Q$, only the free charge should be considered.

or,

$$4\pi r^2 D(r) = Q$$

(i.e)

$$\vec{D}(r) = \frac{Q}{4\pi r^2}\vec{e}_r, \quad r > R_1$$

b, In the region $R_1 < r < R_2$, we have

$$\vec{E} = \frac{\vec{D}}{K\epsilon_0} = \frac{Q}{4\pi\epsilon_0 K r^2}\vec{e}_r, \quad R_1 < r < R_2.$$

In the region $r > R_2$, (air gap, $K \approx 1$) there is no dielectric and hence

$$\vec{E} = \frac{\vec{D}}{\epsilon_0} = \frac{Q}{4\pi\epsilon_0 r^2}\vec{e}_r, \quad r > R_2.$$

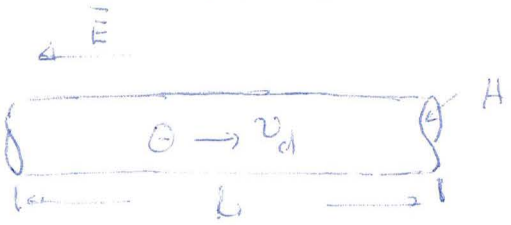
c, We have

$$\vec{D} = \epsilon_0\vec{E} + \vec{P}.$$

(i.e)

$$\vec{P} = \vec{D} - \epsilon_0\vec{E} = \frac{Q}{4\pi r^2}\vec{e}_r - \frac{Q}{4\pi K r^2}\vec{e}_r$$

$$= \frac{Q(K-1)}{4\pi r^2}\vec{e}_r, \text{ where the results in b are used.}$$



The length of the wire is l.

The electric current is a macroscopic quantity since it is associated with the gross motion of the charges within a conductor. On the other hand, a microscopic quantity that is related to the current, is the so called current density $\bar{j}$. If the current is distributed uniformly across the cross sectional area A of the wire, the magnitude of the current density is given by

$$j = i/A$$

$\bar{j}$ takes the direction in which a positive charge carrier would move. But the electrons move in a direction opposite to $\bar{j}$.

The general relation between i and $\bar{j}$ is written as

$$i = \int \bar{j} \cdot d\bar{s}$$

where $d\bar{s}$ is an element of a surface area and the integral is taken over the surface in question.

Let us now find the velocity called drift velocity v_d of the charge carriers in the wire in terms of the current density j. The no. of conduction electrons in the wire is nAl where n is the no. of conduction electrons per unit volume and Al is the volume of the wire. So the total charge q moving to the right is

$$q = (nAl)e$$

If such amount of charge moves a length l of the wire in time t, we have

$$t = l/v_d$$

But the current is given by

$$i = q/t = \frac{nAle}{l/v_d} = nev_dA$$

which gives

$$v_d = \frac{1}{ne}\frac{i}{A} = \frac{j}{ne}$$

Also,

$$j = nev_d .$$

Example: A copper wire of cross sectional area $3 \times 10^{-6}\ m^2$ carries a current of 10 A. Find the drift velocity of the electrons in this wire.

Solution:

To calculate the density of free electrons we assume that each atom contains one free electron. In that case, the density of free electrons, n, equals the density of atoms, given by $N_0 \rho / M$, where N_0 is Avogadro no., M is the atomic weight of copper, and ρ is its density. That is,

$$n = \frac{N_0 \rho}{M}$$

$$= \frac{6.02 \times 10^{23} \frac{\text{atoms}}{\text{mole}} \times 8.95\ \text{g/cm}^3\ (1\ \text{electron/atom})}{63.5\ \text{g/mole}}$$

$$= 8.48 \times 10^{22}\ \text{electrons/cm}^3 = 8.48 \times 10^{28}\ \text{electrons/m}^3.$$

Then,

$$i = jA = n e v_d A \quad \text{or} \quad v_d = \frac{i}{n e A}$$

(i.e.)

$$v_d = \frac{10\ A}{(8.48 \times 10^{28}\ m^{-3})(1.6 \times 10^{-19}\ C)(3 \times 10^{-6}\ m^2)} = 2.46 \times 10^{-4}\ m/s.$$

5.2 Resistance, ~~and~~ Resistivity and Conductivity & Electrical Conductivity; Ohm's Law

Ohm's law states as follows.

for a metallic conductor at const. temp. the ratio of the p.d. V between two points of the conductor to the electric current i through the conductor is constant.

This const. is called the electrical resistance R of the conductor between the two points. Thus ohm's law may be expressed as

$$\frac{V}{i} = R \quad \text{or} \quad i = \frac{V}{R}$$

unit

$$[R] = \left[\frac{V}{i}\right] = \frac{\text{volt}}{\text{ampere}} \equiv \text{ohm}\ (\Omega)$$

Consider now a cylindrical conductor of length l and cross section A. The current through it may be expressed as

$$i = jA$$

where j is the current density. The electric field (uniform) along the conductor is

$$E = V/l$$

so that the first eq. above together with this last eq. gives

$$\frac{El}{i} = R \quad \text{or} \quad i = \frac{El}{R}$$

And the relation between i and j becomes

$$j = \frac{i}{A} = \frac{El}{RA} = \left(\frac{l}{RA}\right)E$$

which means that the current density is proportional to the electric field. We then write

$$j = \sigma E$$

where $\sigma\ (= l/RA)$ is called the electrical conductivity of the material.

unit

$$[\sigma] = \left[\frac{l}{RA}\right] = \Omega^{-1}/m$$

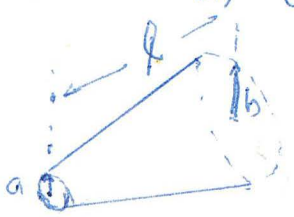
The reciprocal of the conductivity is called the resistivity ρ; i.e.,

$$\rho = 1/\sigma = \frac{RA}{l} \quad \text{or} \quad R = \rho\frac{l}{A}$$

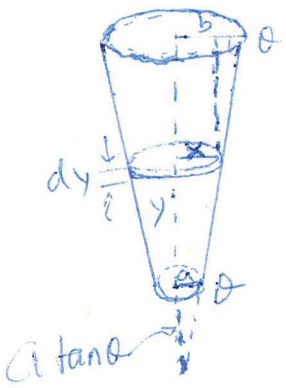
and

$$[\rho] = \Omega m$$

Example: A resistor is in the shape of a truncated right circular cone (fig. below). The end radii are a and b, the altitude is ℓ. If the taper is small, we may assume that the current density is uniform across any cross section. a) Calculate the resistance of this object.



Solution:



$$dR = \frac{\rho\, dy}{\pi x^2}$$

$$\frac{\ell + a\tan\theta}{b} = \frac{\ell - y}{b - x}$$

Since $\tan\theta = \frac{\ell - y}{b - x}$, we write

$$\frac{\ell + a\left(\frac{\ell - y}{b - x}\right)}{b} = \frac{\ell - y}{b - x} \quad \text{or} \quad \frac{\ell(b-x) + a(\ell - y)}{b(b-x)} = \frac{\ell - y}{b - x}$$

$\ell(b-x) + a(\ell-y) = b(\ell-y)$ Solving for x we find

$$x = b + \frac{a-b}{\ell}(\ell - y)$$

$$R = \frac{\rho}{\pi} \int_0^{\ell} \frac{dy}{\left[b + \frac{a-b}{\ell}(\ell-y)\right]^2}$$

Let $u = b + \frac{a-b}{\ell}(\ell - y)$, $du = \frac{b-a}{\ell} dy$ or $dy = \frac{\ell}{b-a} du$

$$\Rightarrow \quad R = \frac{\rho\ell}{\pi(b-a)} \int_a^b \frac{du}{u^2} = -\frac{\rho\ell}{\pi(b-a)} \frac{1}{u}\Big|_a^b = \frac{\rho\ell}{\pi(b-a)}\left(\frac{1}{a} - \frac{1}{b}\right)$$

or

$$R = \frac{\rho\ell}{\pi(b-a)} \frac{b-a}{ab} = \frac{\rho\ell}{\pi ab}$$

If the frustum taper is zero ($a = b$), we get

$$R = \frac{\rho\ell}{\pi a^2} = \rho\ell/A.$$

5.2 Joules Heat: Maintaining a Current in a conductor requires the expenditure of energy. This energy, however, increases the temperature of the conductor i.e., it is transferred into the heat energy of the conductor.

Suppose that the ends of a conductor are maintained at a const. p.d. V and let an amount of charge dq flow across the cross sectional area of the conductor in time dt. If i is the current, the work done is

$$dW = dqV = i\,dt\,V$$

Whence the corresponding power P is

$$P = \frac{dW}{dt} = iV$$

This is then the energy per unit time which is lost in the form of heat within the conductor. or

$$P = iV = i^2R = V^2/R$$

5.3 Electric Circuits

The main elements (components) within an electric circuit are:

1, an electromotive force.

Symbols $-|-$ or $-|||-$

The long line shows the part of the electromotive force seat which is at a higher potential and the short line shows the part at a lower potential. or they are called the positive and negative terminals of the source, resp.

2. a Resistor

Symbol $-\!\!\wedge\!\!\wedge\!\!\wedge-$

3. a Capacitor

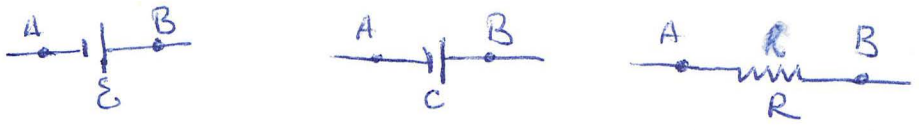
Symbol $-\!\!\rightarrow\!|-$

The straight line shows the positive plate of the capacitor and the curved line " " negative " " " " .

Another symbol is the following.

$\overline{\text{II}}^{+}$

Let us now discuss as to the sign of the p.d. when we traverse a circuit element in a certain direction.



(a) (b) (c)

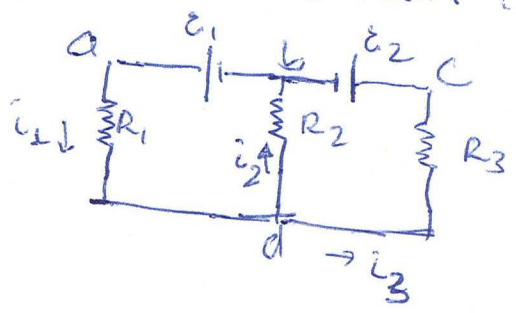
In (a) if a positive charge ~~is taken from~~ at point A has a low potential and if it is taken to point B it gains potential. Thus in going from the negative terminal to the positive terminal of a battery (source of emf) there is a rise in potential and this is taken to be positive. For the reverse direction it is taken to be negative.

In (b) an external agent must do work in taking a positive charge from point A to point B and hence there is a rise in potential. The reverse direction is accompanied by the drop of potential.

In (c) if the current is to the right, there will be a drop of potential when going from A to B as current flows from high to low potential. of course, the direction opposite to the current will be accompanied by a rise in potential.

34 Kirchhoff's Rules

Consider the circuit shown in the fig. below



If we assume that the directions of the currents are as given above, we have at junction d that

$$i_1 = i_2 + i_3$$

or

$$i_2 + i_3 - i_1 = 0$$

This eq. is generalized by the following statement.

At any junction the algebraic sum of the currents must be zero.

— Kirchhoff's first rule.

on the other hand, considering loop bcda we have

$$\mathcal{E}_2 + i_3 R_3 - i_2 R_2 = 0$$

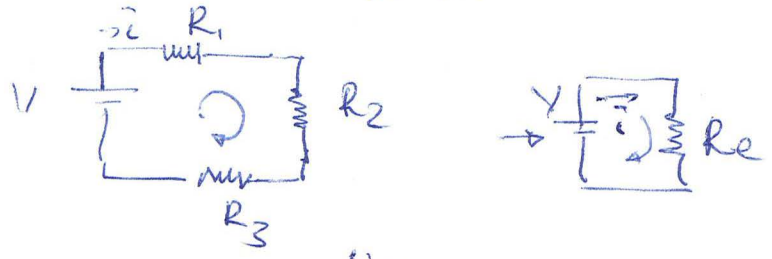
Which may be generalized by the statement that

> The sum of the voltage drops and rises across a closed loop is zero.

.. Kirchhoff's second rule.

Examples

1. Resistors in Series



Traversing the first loop cw and applying the 2nd rule

$$V - iR_1 - iR_2 - iR_3 = 0$$

Similarly, for the second loop

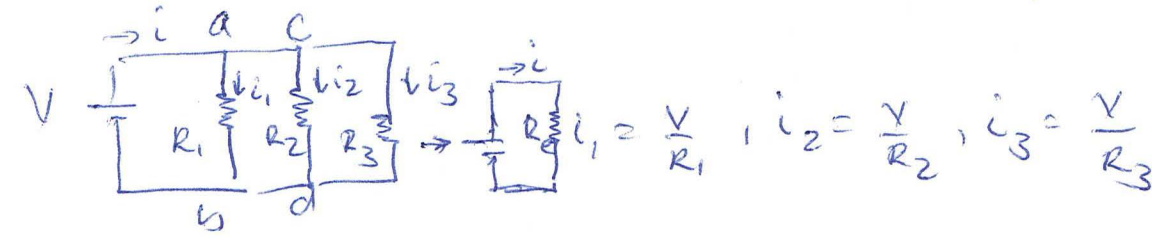
$$V - iR_e = 0$$

Comparing the two equations and solving for R_e we find

$$R_e = R_1 + R_2 + R_3$$

In general $R_e = \sum R_i$, Series Connection

2. Resistors in parallel



$i_1 = \frac{V}{R_1}$, $i_2 = \frac{V}{R_2}$, $i_3 = \frac{V}{R_3}$

Applying Kirchhoff's first rule for junction a

$$i_1 + i_2 + i_3 - i = 0$$

But $i = \frac{V}{R_e}$, in view of the second loop.

or

$$\frac{V}{R_1} + \frac{V}{R_2} + \frac{V}{R_3} - \frac{V}{R_e} = 0$$

Solving for R_e we get

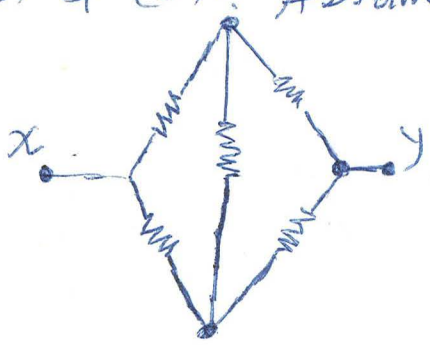
$$\frac{1}{R_e} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

In general,

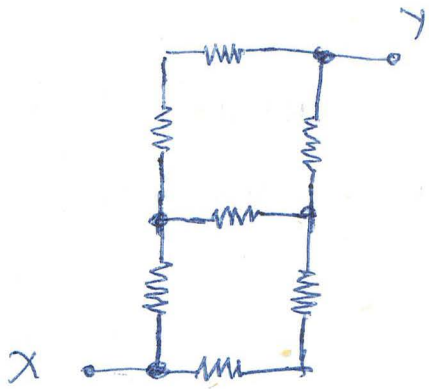
$$\frac{1}{R_e} = \sum \frac{1}{R_i} \text{ ; parallel connection.}$$

Examples:

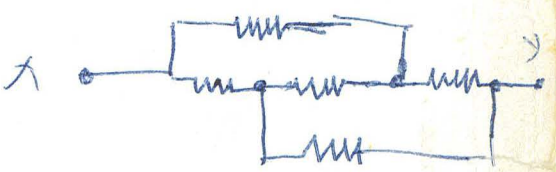
1. What is the equivalent resistance between the terminal points x and y of the circuits shown in fig. (a), (b) & (c)? Assume that the resistance of each is 10 Ω.



(a)



(b)



(c)

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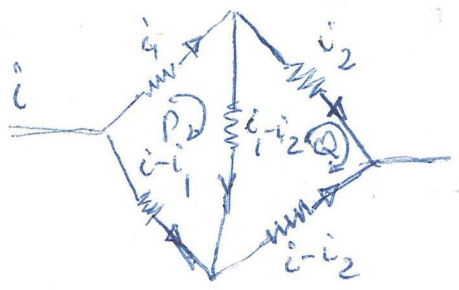
a,



The currents are indicated in the diagram by applying Kirchhoff's 1st rule.
Let us apply Kirchhoff's 2nd rule to loops P & Q.

P: $-10i_1 - 10(i_1 - i_2) + 10(i - i_1) = 0$

Q: $-10i_2 + 10(i - i_2) + 10(i_1 - i_2) = 0$

or,

P: $-30i_1 + 10i_2 = -10i$

Q: $10i_1 - 30i_2 = -10i$

Subtracting the 2nd eq. from the 1st we find $-40i_1 + 40i_2 = 0$ or $i_1 = i_2$

We then obtain,

$$-30i_1 + 10i_1 = -10i \quad \text{or,} \quad i_1 = i_2 = \frac{i}{2}.$$

The equivalent resistance can be obtained by using any route that takes the current from x to y.

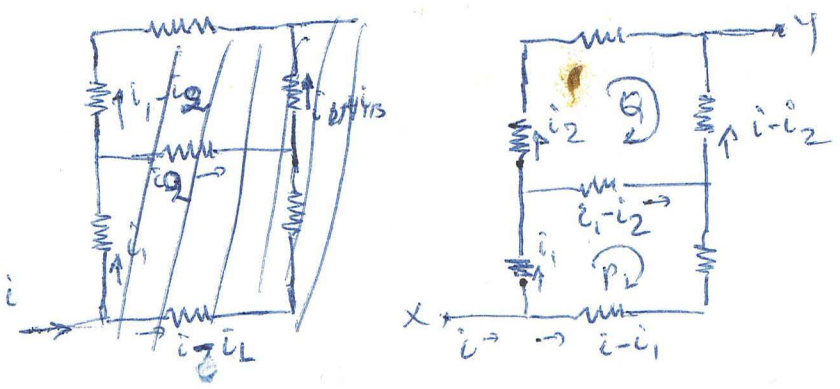
$$V_{xy} = i R_{xy} = 10\Omega \frac{i}{2} + 10\Omega \frac{i}{2} = (10\Omega)\, i$$

or,

$$R_{xy} = 10\,\Omega.$$

One can obtain this result by applying Symmetry Considerations.

b,



Applying Kirchhoff's 1st rule the currents are indicated in the diagram.

Apply Kirchhoff's 2nd rule to loops P & Q.

$$P: \quad -10 i_1 - 10(i_1 - i_2) + 20(i - i_1) = 0$$

$$Q: \quad -20 i_2 + 10(i - i_2) + 10(i_1 - i_2) = 0$$

or,

$$P: \quad -40 i_1 + 10 i_2 = -20 i \Big|_{-2} \Rightarrow -60 i_1 + 90 i_2 = 0$$

$$Q: \quad 10 i_1 - 40 i_2 = -10 i \Big|_{2} \quad \text{or,} \quad i_2 = \tfrac{2}{3} i_1$$

Then,

$$-40 i_1 + \frac{20}{3} i_1 = -20 i$$

$$100 i_1 = 60 i \quad \text{or} \quad i_1 = \tfrac{3}{5} i$$

$$\{i_1, i_2\} = \left\{ \tfrac{3}{5} i, \tfrac{2}{5} i \right\}.$$

Use one possible path that goes from x to y.

$$V_{xy} = i R_{xy} = 10\Omega \times \tfrac{3}{5} i + 20\Omega \tfrac{2}{5} i = \tfrac{14}{5} \Omega\, i$$

or,

$$R_{xy} = 14\,\Omega.$$

c, This connection is similar to the one in (a). Justify this assertion. Ans. 10 Ω.

Chapter 6 Magnetic Interaction

6-2 The Magnetic Field

There exist material bodies such as cobalt which attract pieces of iron. They are called magnets and this physical property of theirs is called magnetism. The magnetic[1] property of a body is found to be concentrated at small regions called magnetic poles. Experiments show that there[2] are two kinds of magnetic poles which may be designated by the letters N and S, corresponding to the north seeking and south seeking poles, respectively. If two magnetized bars are placed as shown below, the bars will either repel or attract each other, depending on whether like or unlike poles face each other. Thus from experiment

[3] the interaction between like magnetic poles is repulsive and the interaction between unlike magnetic poles is attractive.

Also, in 1820 Oersted discovered that a current in a wire can produce magnetic effects, namely it can change the orientation of a compass needle.

We define the space around a magnet or a current carrying wire as the site of a magnetic field, just as one defined the space near a charged body as the site of an electric field. The magnetic field vector designated $\bar{B}$ is called the magnetic induction. ~~It can be represented by lines of induction, just as the electric field was represented by lines of force.~~

– similarities of $\bar{B}$ with the electric field vector $\bar{E}$

a, there exist magnetic field lines called induction lines s.t.

i, the tangent to a line of induction at any point gives the direction of $\bar{B}$ at that point.

ii, the lines of induction are drawn so that the no. of lines per unit cross sectional area is proportional to the magnitude

a) The magnetic field vector $\bar{B}$.

b) Magnetic flux ϕ_B is defined analogously s.t.

$$\phi_B = \int \bar{B} \cdot d\bar{s}$$

(Taken over a closed or open surface).

- Differences of $\bar{B}$ from $\bar{E}$

a) The magnetic field exerts force only on moving charges

b) Ad for $\bar{E}$, the induction lines do not originate in one pole and terminate on the other. They rather form continuous closed loops.

6.2 Magnetic Force on a Moving charge

Interactions between magnetized bodies may be described by saying, in analogy with the gravitational and electrical cases, that a magnetized body produces a magnetic field in the space around it. The presence of a magnetic field in a region is tested by a small positive ~~test~~ charge which is set in motion in the region in question. Thus if a sideways deflecting force acts on it, we assert that a magnetic field exists.

It has been found out that the force $\bar{F}$ exerted by a magnetic field $\bar{B}$ on a moving charge q is perpendicular to the velocity $\bar{v}$ of the charge and to $\bar{B}$ itself. The magnitude of this force also depends on the magnitude of the charge. In mathematical form we put these experimental facts as follows.

$$\bar{F} = q\,\bar{v} \times \bar{B}$$

Where q is the charge in motion experiencing the force.

N.B $\bar{B}$ is also called the magnetic field strength.

On the other hand, when the charge q moves in a region where both electric and magnetic fields exist, the total force on the particle is the sum of the electric force $q\bar{E}$ and the magnetic force $q\bar{v}\times\bar{B}$. That is

$$\bar{F} = q(\bar{E} + \bar{v} \times \bar{B})$$

It is called the Lorentz force.

The direction of the magnetic force is indicated by

the right hand rule. It can be easily seen that the direction of the magnetic force on a negative charge is opposite to that on a positive charge.

If α is the angle between $\bar{v}$ and $\bar{B}$, the magnitude of $\bar{F}$ is

$$F = qvB\sin\alpha$$

The max. force is observed when $\alpha = \pi/2$ or $\bar{v} \perp \bar{B}$. Resulting in

$$F_{max} = qvB$$

unit

$$[B] = \left[\frac{F}{qv}\right] = \frac{N}{C \cdot m/s} = kg/Cs$$

This unit is called a tesla (T); i.e.,

$$1T = 1\,kg/Cs$$

So one tesla corresponds to the magnetic field that produces a force of one newton on a charge of one coulomb, moving perpendicular to the field at one meter per second.

Because the magnetic force $\bar{F} = q\bar{v}\times\bar{B}$ is perpendicular to the velocity, the work associated with the magnetic force is zero; therefore the magnetic force does not produce any change in the K.E. of the particle.

6.3 Magnetic Force on a Current Carrying Conductor

Motion of a charge in a Magnetic Field

Consider the motion of a charged particle in a uniform magnetic field; (i.e.) a magnetic field having the same intensity and direction at all points in space. For simplicity the case of a particle moving in a direction perpendicular to the magnetic field will be studied (Fig. below). Since the force is $\perp$ to

the magnetic field velocity. the effect of the force is to change the direction of the velocity without changing its magnitude; the result is a uniform circular motion. The acc. is then centripetal; and from the eq. for circular motion

$$F_c = \frac{mv^2}{r}$$

with F_c being the magnetic force, i.e.,

$$\frac{mv^2}{r} = qvB$$

or

$$r = \frac{mv}{qB}$$

which gives the radius of the circle described by the particle.

We know that $v = \omega r$ (ω - frequency of rotation) and substituting this expression in the last eq. above and solving for ω we obtain

$$\omega = \frac{qB}{m}$$

This eq. gives the magnitude of $\bar{\omega}$ but not its direction. However, the acc. in a uniform circular motion may be written in the form $\bar{a} = \bar{\omega} \times \bar{v}$. Therefore the eq. of motion $\bar{F} = m\bar{a}$ becomes

$$m\bar{\omega} \times \bar{v} = -q\bar{v} \times \bar{B}$$

or

$$\bar{\omega} \times \bar{v} = -\frac{q}{m}(\bar{B} \times \bar{v})$$

indicating that

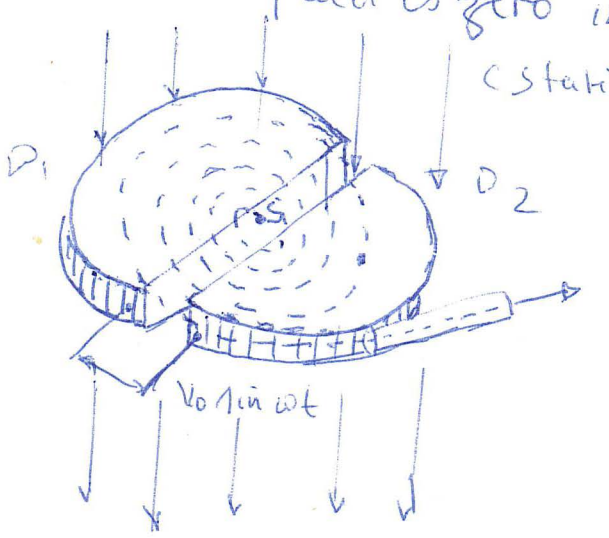
$$\bar{\omega} = -\frac{q}{m}\bar{B}$$

which gives $\bar{\omega}$ in magnitude and direction. One can see that $\bar{\omega}$ has the same direction as $\bar{B}$ for a negative charge and opposite for a positive charge. The angular velocity $\bar{\omega}$ of a charged particle in a uniform magnetic field is called the cyclotron frequency.

Example: The cyclotron

The cyclotron is used to accelerate charged particles, such as protons or deutrons, to high energies so that they can be used in atom-smashing experiments.

Essentially, a cyclotron (below) consists of a cylindrical cavity that is divided into two halves D_1 and D_2 (each called a dee), and it is placed in a uniform magnetic field parallel to its axis. The two cavities are electrically insulated from each other. An ion source is placed in the center of the space between the dees. ~~When the ions are positive they will be accelerated to the~~ An alternating p.d. of the order of 10^4 V is applied between the dees. When the ions are positive, they will be accelerated toward the negative dee. Once inside a dee the ion experiences no electrical force since the electric field is zero in the interior of a conductor.



Because of the magnetic field, the ions describe a circular orbit with a radius

$$r = \frac{mv}{qB}$$

and angular frequency equal to the cyclotron frequency of the particles given by

$$\omega = \frac{qB}{m}$$

The p.d. between the dees is made to oscillate with a frequency equal to $\omega/2\pi$ so that this p.d. is in resonance with the circular motion of the ions. After the particles has described half a revolution, the polarity of the dees is reversed; and when the ion

crosses the gap between them, the ion receives another small acc. The next half circle described then has a larger radius but the same angular velocity. The process repeats itself several times until the radius attains a max. value R, which is practically equal to the radius of the dees. The magnetic field at the edge of the dees is decreased sharply; and the particle moves tangentially, escaping through a convenient opening. The max. velocity is related to the radius R by,

$$R = \frac{m v_m}{qB} \quad \text{or} \quad v_m = \frac{q}{m} BR$$

The kinetic energy of the emerging particles is

$$E_K = \frac{1}{2} m v_m^2 = \frac{1}{2} \frac{q^2 B^2 R^2}{m}$$

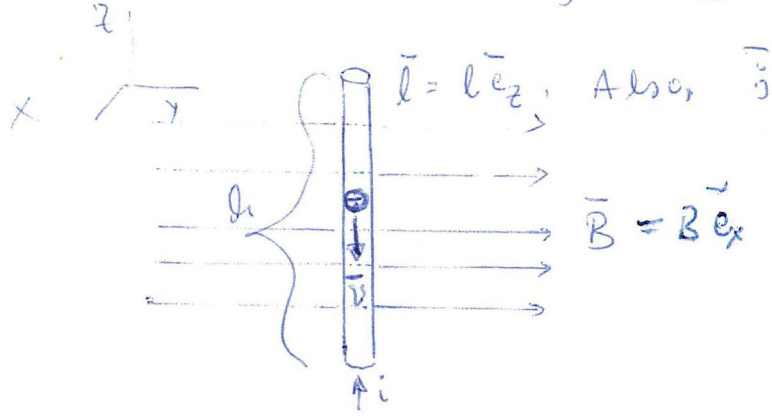
As the energy increases so does the velocity of the ion and this results in change of mass according to the relativistic relation

$$m = m_0 (1 - v^2/c^2)^{-1/2}$$

So the velocity has an upper bound c.

6.4 Magnetic Force on a Current

An electric current is a stream of electric charges moving in vacuum or through a conducting medium. Because a magnetic field exerts a sideways force on a moving charge, we expect that it will also exert a sideways force on a wire carrying a current. In the fig. below we have a wire of length l carrying a current i and placed in a uniform field $\bar{B}$ oriented to the right



If a total charge q crosses a cross sectional area of the wire, the magnetic force acting is

$$\bar{F} = q(\bar{v} \times \bar{B})$$

Where $\bar{v}$ is the drift velocity of the charge carriers q or $v = v_d$.

$$q = (nAl)e$$

$$\bar{v} = \bar{j}/(ne)$$

i.e.,

$$\bar{F} = nAle\,\frac{\bar{j}}{ne} \times \bar{B} = Al\,\vec{j} \times \bar{B} = Al\,j\bar{e}_z \times \bar{B}$$

But $Aj = i$ and $l\bar{e}_z = \bar{l}$ so that

$$\boxed{\bar{F} = i\,\bar{l} \times \bar{B}}$$

Direction of $\bar{F}$ is given by the RHR.

If we consider a differential element of length $d\bar{l}$ of a conductor of length $d\bar{l}$, the force $d\bar{F}$ acting on it can be found, by analogy with the above eq. from

$$d\bar{F} = i\,d\bar{l} \times \bar{B}.$$

By integrating this formula we can find the force $\bar{F}$ acting on a nonlinear conductor, or

$$\bar{F} = \int_{leng.} i\,d\bar{l} \times \bar{B}$$

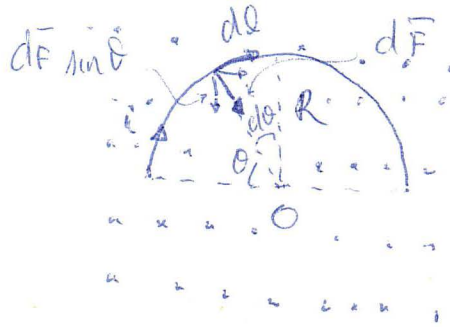
For a volume distribution of current we have

$$i\,d\bar{l} \times \bar{B} = A\,j\,d\bar{l} \times \bar{B} = A\,dl \times j \times \bar{B} = dv\,j \times \bar{B}$$

so that

$$\bar{F} = \int_{vol.} dv\,\bar{j} \times \bar{B}$$

Example #10 A wire bent as shown below carries a current i and is placed in a uniform magnetic induction $\bar{B}$ that emerges from the plane of the figure. Calculate the force acting on the wire.



The segment $d\bar{l}$ of the wire experiences a magnetic force of magnitude

$$dF = i\,dl\,B\sin 90^\circ = iB\,dl = iBR\,d\theta$$

$d\bar{F}$ is directed radially toward the center of the arc. only the downward comp. of this force is effective, the horizontal being canceled by an oppositely directed comp. associated with the corresponding arc on the opposite side of O. Thus the total force on the semi circle of wire about O points down and is

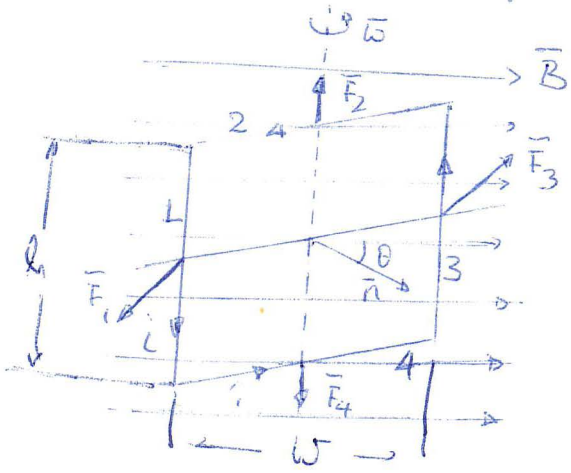
$$F = \int_0^{\pi} dF\sin\theta = iBR\int_0^{\pi} \sin\theta\,d\theta = iBR(-\cos\theta)_0^{\pi}$$

or

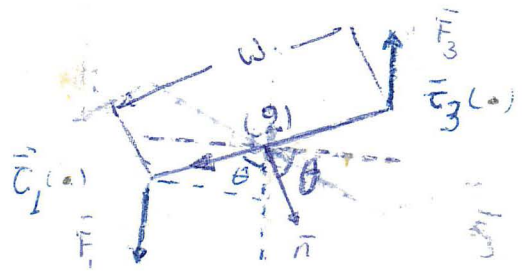
$$F = 2iBR$$

6.5 Torque on a Current Loop

The fig. below shows a rectangular loop of wire of length L and width w carrying a current i in a uniform field of induction $\bar{B}$. The normal $\bar{n}$ to the plane of the loop, determined by the right hand rule in the sense of direction of the current, makes an angle θ with the field vector $\bar{B}$ and two sides (2) and (3) of the loop are $\perp$ to this field.



But if we look down from side (2) we can have the figure shown below.



One can see from the figure that the angle between side 2 (which is in the direction of the current) and the field $\bar{B}$ is $(90° + \theta)$ whereas that between side 4 and $\bar{B}$ is $90° - \theta$. Thus,

$$\left.\begin{aligned} F_2 &= iwB\sin(90°+\theta) = iwB\cos\theta \\ F_4 &= iwB\sin(90°-\theta) = iwB\cos\theta \end{aligned}\right\} \Rightarrow F_2 = F_4$$

But applying the right hand rule one can see that $\bar{F}_2$ is directed upward resulting in a zero net torque i.e., $\bar{F}_2$ and $\bar{F}_4$ do not produce a rotational motion of the loop (same line of action). $\bar{F}_4$ is directed downward

On the other hand, $\bar{F}_1$ and $\bar{F}_3$ don't have the same line of action, so that they produce rotation of the loop about the vertical line. They also have the same magnitude.

$$F_1 = F_3 = iLB$$

and the net torque that tends to orient the loop in the field is

$$\tau = F_1 \cdot \frac{w}{2}\sin\theta + F_3 \frac{w}{2}\sin\theta = iLwB\sin\theta$$

or, if

$$M = iLw \quad (A = Lw)$$

In vector form we write

$$\bar{M} = iA\,\bar{n}$$

so that

$$\bar{\tau} = \bar{M} \times \bar{B}$$

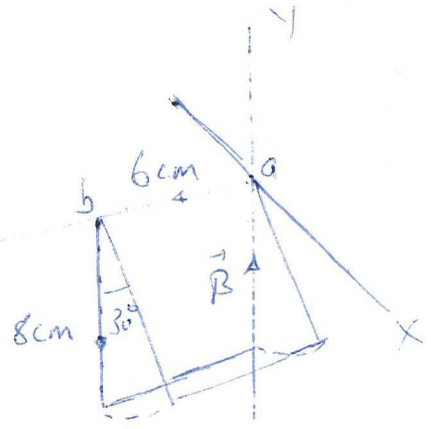
$\bar{M}$ is called the magnetic dipole moment of the current loop. If there are N turns (coils) we write

$$\bar{M} = NiA\,\bar{n}.$$

If the loop is oriented at an angle θ with the field, work W is required to change this orientation. This is stored as the P.E. of the loop. If the initial position is $\theta_0 = 90°$ and the final is θ, then

$$W = U = \int_{90}^{\theta} MB\sin\theta\, d\theta = -MB\cos\theta \quad \text{or} \quad U = -\bar{M}\cdot\bar{B}$$

Example:
The rectangular loop of wire in the figure below has a mass of 0.1 g per centimeter of length, and is pivoted about side ab on a frictionless axis. The current in the wire is 10A in the direction shown.
a, Find the magnitude and sense of the magnetic field, parallel to the y-axis, that will cause the loop to swing up until its plane makes an angle of 30° with the yz plane.
b. Discuss the case where the field is parallel to the x-axis.



Solution:
a, Total mass of the loop = $0.1 \frac{g}{cm} \times 2(14)cm = 2.8g$
Weight, $W = 2.8 \times 10^{-3} kg \times 10 m/s^2 = 2.8 \times 10^{-2} N$.
Assuming The Center of gravity of the loop is at the geometric Center of the loop. This force gives a torque so as to bring the loop back to its initial (vertical) position.
If $l = 8cm$, we have

$$\tau_W = W \frac{l}{2} \sin 30°$$

which is the torque due to the weight of the loop. Moreover this torque counterbalances the torque due to the magnetic field. or

$$W \frac{l}{2} \sin 30° = i l B \sin(90° - 30°)$$

or,

$$B = \frac{W \sin 30°}{2 i \sin 60°} = \frac{2.8 \times 10^{-2} N \times 1/2}{2 \times 6 \times 10^{-2} \times \sqrt{3}/2 \times 10} = \frac{1.4}{60\sqrt{3}} T = 0.013 T$$

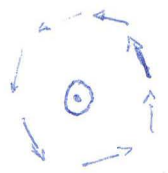
b. If the $\vec{B}$ field is parallel to the x-axis, it is then parallel to the magnetic dipole moment of the loop, and it will not provide any torque so as to swing the loop through an angle.

chapter 7 Magnetic Fields Produced by Current-Carrying Wires

7.1 The Biot-Savart Law

It has been found out from experiments that when an electric charge is in motion- that is when an electric current exists, there exists a magnetic field in the vicinity of the current. To put it another way. An electric current produces a magnetic field.

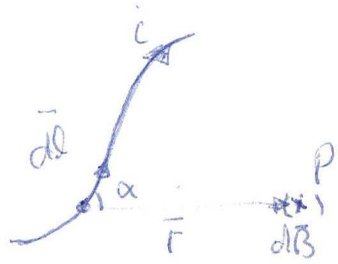
This can best be seen by observing the way a system of compass needles (small magnets) behave when placed in the vicinity of a current carrying wire. Now, if we do this experiment, you can find that the needles point so as to suggest that the magnetic ~~induction~~ lines form closed ~~closed~~ circles around the wire. If the



sense of the electric current through the wire is reversed in this experiment, so is the sense of the magnetic field lines. These experimental results can be described by the right-hand rule which is stated as follows: place

> place your right hand so that the thumb points in the sense of the current flow in the wire. The fingers then curl in the sense of the magnetic field lines encircling the wire.

Having discussed the qualitative properties of the magnetic field due to a current carrying wire, let us come to the quantitative determination of this field. Assume that we have a wire which carries a current i as shown below we would like to determine $\bar{B}$ at point P



that is located at a dist. r from the infinitesimal current element $d\bar{\ell}$. Experiments show that the contribution of this element to the magnetic field is $d\bar{B}$ given by

$$d\bar{B} = \frac{\mu_0}{4\pi} \frac{i\, d\bar{\ell} \times \bar{r}}{r^3}$$

So the resultant field $\bar{B}$ at the indicated point is obtained by adding (integrating) such contributions vectorially.

$$\bar{B} = \int d\bar{B} = \frac{\mu_0}{4\pi} \int \frac{i\, d\bar{\ell} \times \bar{r}}{r^3}$$

This integration can be done comp. by comp. so that

$$B_x = \int (d\bar{B})_x, \quad B_y = \int (d\bar{B})_y, \quad B_z = \int (d\bar{B})_z$$

μ_0 is a constant. The direction of $d\bar{B}$ is determined by the right hand rule and $d\bar{B} \perp d\bar{\ell}, \bar{r}$. Its magnitude is

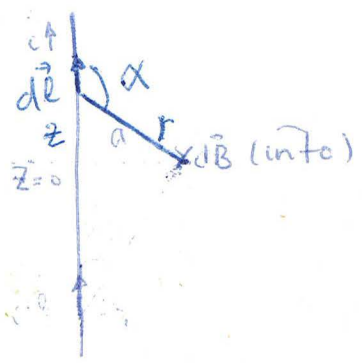
$$dB = \frac{\mu_0}{4\pi} \frac{i\, d\ell \sin\alpha}{r^2}, \quad \alpha \measuredangle (d\bar{\ell}, \bar{r})$$

μ_0 is a constant called the permeability of free space. The value of μ_0 (numerical) that is determined by experiment is $\mu_0 = 4\pi \times 10^{-7}$ whereas the unit in SI is as given below.

$$[\mu_0] = \left[\frac{Br}{i}\right] = \frac{T \cdot m}{A}$$

Examples:

1. Calculate the magnetic field near a long current carrying wire.



$$dB = \frac{\mu_0}{4\pi} \frac{i\,dl\,\sin\alpha}{r^2}$$

Let us choose α as a variable of integration.

$$r = \frac{a}{\sin(\pi-\alpha)} = \frac{a}{\sin\alpha}$$

Also,

$$dl = +dz$$

since dl lies along the z-axis.

$$\frac{a}{+z} = +\tan(\pi-\alpha) \quad \text{or} \quad z = -a\cot\alpha$$

$$\Rightarrow \quad dz = -(-a\,\mathrm{cosec}^2\alpha)\,d\alpha = a\,\mathrm{cosec}^2\alpha\,d\alpha$$

(i.e.)

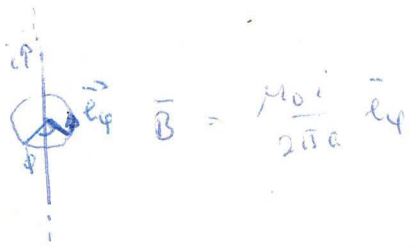
$$dB = \frac{\mu_0}{4\pi} \; i\,\frac{a}{\sin^2\alpha}\cdot\sin\alpha\cdot\frac{d\alpha}{a^2/\sin^2\alpha} = \frac{\mu_0 i}{4\pi a}\sin\alpha\,d\alpha$$

$\alpha \in [0, \pi]$

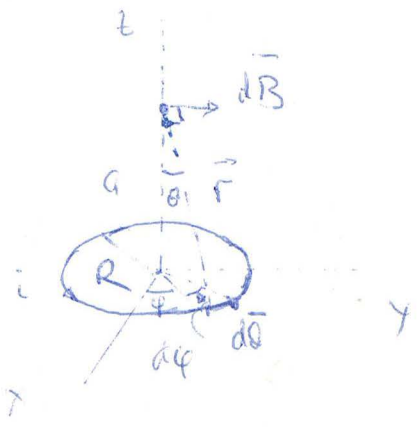
$$B = \frac{\mu_0 i}{4\pi a}\int_0^{\pi}\sin\alpha\,d\alpha = -\frac{\mu_0 i}{4\pi a}\cos\alpha\Big|_0^{\pi}$$

or

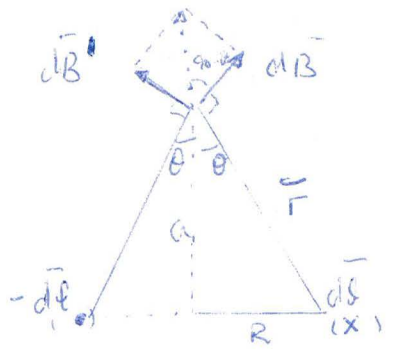
$$B = \frac{\mu_0 i}{2\pi a}$$



2. Find $\vec{B}$ due to a circular current loop at a point lying on its axis.



To see the situation more clearly we can draw the picture shown below.



$$dB = \frac{\mu_0 i}{4\pi}\frac{d\ell \sin\alpha}{r^2} = \frac{\mu_0 i}{4\pi}\frac{d\ell}{r^2}, \quad \sin\alpha = 90^\circ$$

This is the contribution due to the element on the right. However, the corresponding element to the left also gives a contribution of identical the same magnitude and both of them resultant are in such a way as to give an magnetic field parallel to the axis of the loop. or

$$dB_{\parallel} = dB\cos(90-\theta) = dB\sin\theta$$

From the picture

$$\sin\theta = \frac{R}{r} = \frac{R}{\sqrt{R^2+a^2}}$$

$$dB_{\parallel} = \frac{\mu_0 i}{4\pi}\frac{d\ell}{r^2}\cdot\frac{R}{r} = \frac{\mu_0 i R}{4\pi r^3}\,d\ell$$

and

$$B_{\parallel} = \frac{\mu_0 i R}{4\pi r^3}\int_C d\ell = \frac{\mu_0 i R}{4\pi r^3}\int_0^{2\pi} R\,d\varphi$$

or

$$B_{\parallel} = \frac{\mu_0 i R}{4\pi r^3}\cdot 2\pi R = \frac{\mu_0 i}{2\pi r^3}\pi R^2$$

Since πR^2 is the area A of the loop we also write

$$B_{\parallel} = \frac{\mu_0 i A}{2\pi r^3} = \frac{\mu_0 M}{2\pi r^3}$$

In vector form we write

$$\bar{B}_{\parallel} = \frac{\mu_0}{2\pi r^3}\bar{M} = \frac{\mu_0}{2\pi\left[a^2+R^2\right]^{3/2}}\bar{M}; \quad \text{if } a \gg R \quad \bar{B}_{\parallel} = \frac{\mu_0 \bar{M}}{2\pi a^3}$$

Since the field $\bar{B}_{\parallel}$ of the loop at a large distance on the axis of the loop is similar (in form) to that due to corresponding field $\bar{E}$ due to an electric dipole, we call the current loop a magnetic dipole.

Note that in a uniform field $\bar{B}$, the

(i) torque acting on the loop is given by

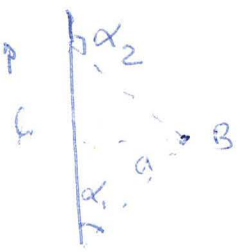
$$\bar{\tau} = \bar{M} \times \bar{B}$$

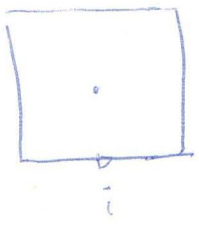
(ii) p.E. it acquires is

$$U = -\bar{B} \cdot \bar{M} = -\vec{M} \cdot \vec{B}$$

Exercise

Find the magnetic field due to each of the following current carrying wires.

(i) 

(ii) 

At the center of the square

(iii)

(iv) (Regular hexagon)
At the center.

7.2 Ampere's circuital law

The Biot-Savart law is the magnetic analogue of the electric coulomb's law. Like coulomb's law, it is perfectly general. It can be used to find the magnetic field vector B at any point in the vicinity of any steady electric-current configuration, no matter how twisted the path the current may be and even if there are several independent currents. But just as in the case of coulombs law, the calculation can become quite difficult.

Continuing the reasoning by analogy with the electric case, we ~~write down~~ an equation which has resemblance to gauss's law for the electric field. Or

$$\int_{\text{closed geometric figure}} \vec{B} \cdot d(\text{geometric quantity}) \propto i$$

To make the general mathematical relation of the above proportionality consider the result from example 1. we have

$$\vec{B} = \frac{\mu_0 i}{2\pi a} \vec{e}_\phi$$

which may give

$$B(2\pi a) = \mu_0 i$$

The quantity $2\pi a$, is simply the circumference of the circular path surrounding the wire at a dist. a.

On the other hand, if we consider ~~this~~ circular path of radius a about the wire, divide ~~this path~~ it into small elements $d\bar{\ell}$ and take the sum of $\bar{B}d\bar{\ell}$ (integrate) the quantity $\bar{B}d\bar{\ell}$ we will find a similar result, i.e.,

$$\oint_C \bar{B}\cdot d\bar{\ell} = \mu_0 i \quad \oint B\,d\ell \text{ since } \bar{B} = B\bar{e}_\varphi,\ d\bar{\ell} = d\ell\,\bar{e}_\varphi$$

$$= B\int_0^{2\pi} a\,d\varphi$$

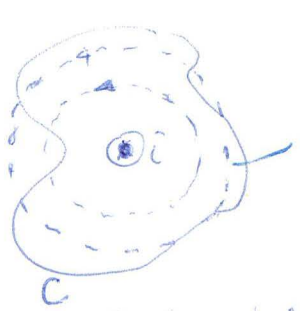
$$= B\,2\pi a = \mu_0 i$$

Or

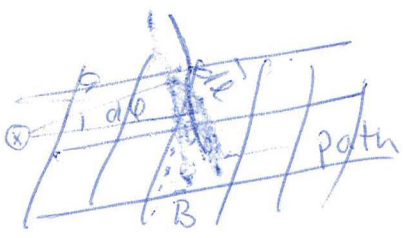
$$\oint_C \vec{B}\cdot d\bar{\ell} = \mu_0 i$$

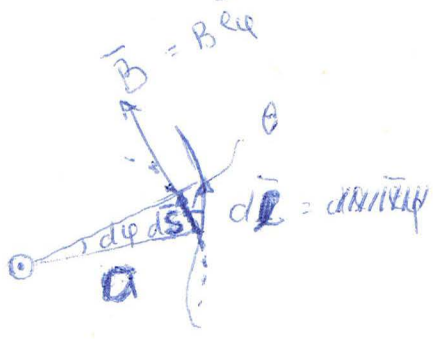
So this eq. gives the desired result as suggested by the proportionality.

Next consider an arbitrary closed path C surrounding the current i of the long wire.



More clearly we draw the fig. below.





Now, $\bar{B}\cdot d\bar{\ell} = \frac{\mu_0 i}{2\pi a}\bar{e}_\varphi \cdot d\bar{\ell}$; $\bar{e}_\varphi = \frac{\mu_0 i}{2\pi a} d\ell\cos\theta = \frac{\mu_0 i}{2\pi a} dS = \frac{\mu_0 i}{2\pi} d\varphi$ ($dS = a\,d\varphi$)

integrating gives

$$\oint_C \vec{B}\cdot d\bar{\ell} = \frac{\mu_0 i}{2\pi}\int_0^{2\pi} d\varphi = \frac{\mu_0 i}{2\pi}\,2\pi$$

or

$$\oint_C \vec{B}\cdot d\bar{\ell} = \mu_0 i$$, true for arbitrary path as well.

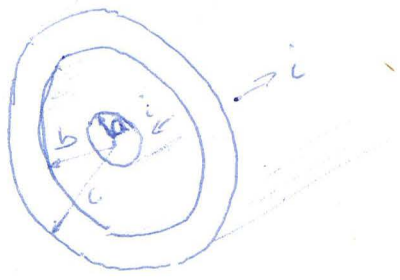
This is called the magnetic circulation and it is the Amperes law which states

> The magnetic circulation around a closed curve is equal to μ_0 times the electric current penetrating it.

It helps to calculate the magnetic field due to a current i. Note Examples that i must be the net current flowing through the closed curve. Also it is useful if only the prob has axial symm.

Examples:

1. For the fig. below find $\bar{B}$ at distance (i) $r < a$, (ii) $a < r < b$, (iii) $b < r < c$ and (iv) $r > c$. Assume, the currents are uniformly distributed over the two concentric conductors.



Solution

(i) Let us take the direction of the current 'i' to be +ve.

$\oint_C \vec{B} \cdot d\vec{l} = \mu_0 i'$, i' is the current within the circular path of radius, $r < a$.

$B\, 2\pi r = \mu_0 i'$

$B = \frac{\mu_0}{2\pi} \frac{i'}{r}$

But $\frac{i'}{i} = \frac{j \pi r^2}{j \pi a^2} = \frac{r^2}{a^2} \Rightarrow i' = i \frac{r^2}{a^2}$

(i.e.,) $B = \frac{\mu_0}{2\pi} \frac{ir}{a^2}$

(ii) $a < r < b$

$\oint_C \vec{B} \cdot d\vec{l} = \mu_0 i$

$$B\,2\pi r = \mu_0 i$$

or

$$B = \frac{\mu_0}{2\pi}\frac{i}{r}$$

(III) $b < r < c$

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 (i - i^*)$$

$$B\,2\pi r = \mu_0 (i - i^*)$$

But

$$i^* = i\,\frac{\pi(r^2-b^2)}{\pi(c^2-b^2)} = i\,\frac{(r^2-b^2)}{c^2-b^2}$$

or

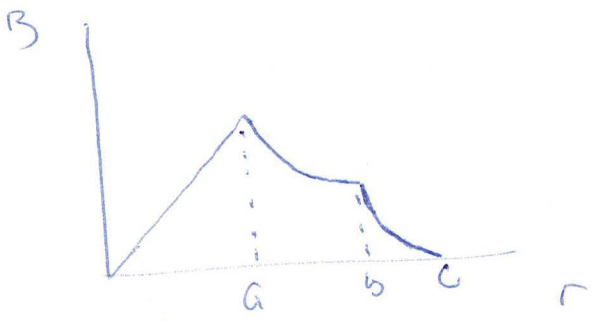
$$B = \frac{\mu_0}{2\pi r}\left(i - i\frac{(r^2-b^2)}{c^2-b^2}\right) = \frac{\mu_0}{2\pi}\,\frac{i(c^2-r^2)}{r(c^2-b^2)}$$

(iv) $r > c$

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 (i - i) = 0$$

$$\rightarrow \quad B = 0$$

Graph:



In general we may write Amperes law

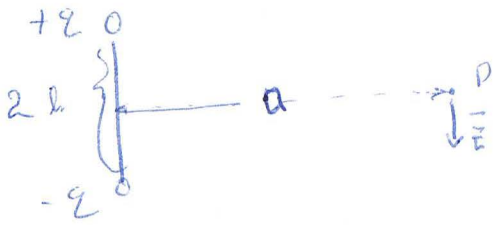
$$\oint_C \vec{B}\cdot d\vec{l} = \mu_0 \int_S \vec{j}\cdot d\vec{s}$$

where the integral on the right is to be evaluated over any surface S for which the closed loop is a boundary.

7.3 Magnetic Dipoles

So far we have studied magnetic fields set up by long straight wires, broken wires and current loops. But now let us investigate the magnetic analogue of an electric dipole.

Consider the electric dipole shown in the fig. below.



The electric field at P, directed downward, has a magnitude given by

$$E = \frac{1}{4\pi\epsilon_0} \frac{p}{(l^2+a^2)^{3/2}}$$

where p is the electric dipole moment, [Halliday] $p = 2lq$.

At large distances ($a \gg l$), we have

$$E = \frac{1}{4\pi\epsilon_0} \frac{p}{a^3}$$

[Example 3, chap. 24 Halliday].

We have also seen that, in an external electric field $\vec{E}$, the electric dipole is acted upon by a net torque

$$\vec{\tau} = \vec{p} \times \vec{E}$$

and acquires a $p.E$.

$$U = -\vec{p} \cdot \vec{E}$$

But similar expressions hold for a rectangular current loop placed within a uniform external magnetic field $\vec{B}$, (i.e.)

and

$$\vec{\tau} = \vec{M} \times \vec{B}$$

$$U = -\vec{M} \cdot \vec{B}$$

Further more, from example 2 in sec 7.7 the magnetic field of a circular current loop at a point a dist. from its center is given by

$$B = \frac{\mu_0}{2\pi} \frac{M}{(R^2+a^2)^{3/2}}$$

(Sec. 6.5)

This similarity of the torque and magnetic p.E. of a rectangular loop with those of an electric dipole suggests that the rectangular loop may be regarded as a magnetic dipole.

Further, from the result of example 2 in sec. 7.1 the magnetic field of a circular loop at a point a dist. a from its center is given by

$$B = \frac{\mu_0}{2\pi} \frac{M}{(R^2+a^2)^{3/2}}$$

where R is the radius of the loop. If $a \gg R$, we have the approximate expression

$$B = \frac{\mu_0}{2\pi} \frac{M}{a^3}$$

Again we observe similarity between this result and the corresponding expression for the electric field of an electric dipole. So a circular current loop is considered as an magnetic dipole, too.

7.4 Gauss's Law for the Magnetic Field

Similar to the electric flux, the magnetic flux over a surface, closed or not, is given by

$$\Phi_B = \int_{(\text{surface},S)} \vec{B} \cdot d\vec{S}$$

This has been given at the very begining of chapter six

unit

$$[\Phi_B] = [BA] = Tm^2 \equiv \text{Weber (Wb)}.$$

Since there are no isolated magnetic masses or poles (none have been yet observed), the lines of force of the magnetic field $\vec{B}$ are closed. Therefore

> The total magnetic flux through a closed surface is always zero.

That is, the inward flux through a closed surface is equal to the outward flux. Thus

$$\oint_S \vec{B} \cdot d\vec{S} = 0, \quad \text{closed surface.}$$

which is Gauss's law for the magnetic field.

Exercise

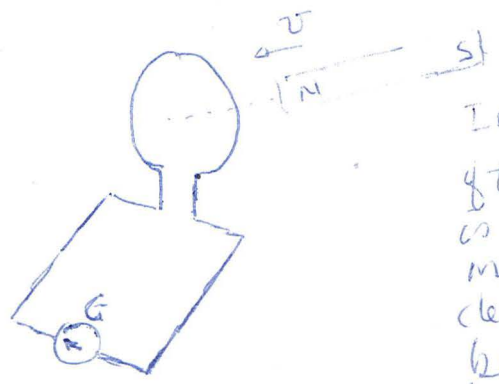
Show that the magnetic flux due to the long current carrying wire over the surface of the cylinder in the fig. is zero. (The wire lies on the axis of the cylinder.)

Chapter 8 Electromagnetic Induction

8.1 Faraday's Law of Induction

Let us do the following two Experiments

1. A coil in relative motion with a bar magnet.

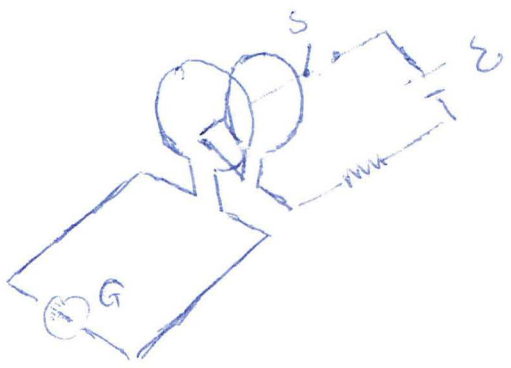


G - Galvanometer

In the absence of the bar magnet the pointer of the galvanometer does not deflect since there is no emf in the circuit; but if the magnet moves toward the coil, the galvanometer deflects, showing that a current has been set up in the coil. If the magnet is held stationary wrt the coil, the galvanometer does not deflect. If the magnet is moved away from the coil, the galvanometer again deflects, but in the opposite direction, which means the current in the coil is in opposite direction. If we use the South pole end of a magnet instead of the north pole end, the experiment works as described but the deflections are reversed. So what matters is the relative motion.

The current that appears in this experiment is called an induced current and we say it is set up by an induced electromotive force.

2. Two coils in relative motion near each other



The coils are at rest relative to each other. When the switch S is closed, thus setting up a steady current in the right coil, the galvanometer deflects momentarily; when the switch is opened thus interrupting this current, the galvanometer again deflects momentarily, but in the opposite direction. Here again an induced emf exists in the left coil whenever the current in the right coil is changing.

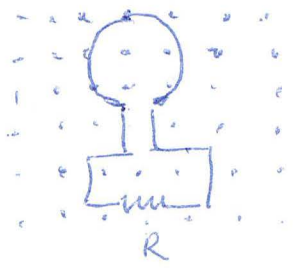
According to Faraday, it is the change in the magnetic flux due to Φ_B the bar magnet or the changing current in the right coil that

Examples:

1. In the fig. below the magnetic flux through the loop shown increases according to the relation

$$\phi_B = 6t^2 + 7t \quad (\phi_m - \text{milliweber}, t - \text{second})$$

a, what is the magnitude of the emf induced in the loop when $t = 2$ s? b, What is the direction of the current through R.



a, $|\mathcal{E}_i|_{t=2} = \left| -\frac{d}{dt}\phi_B \right|_{t=2}$

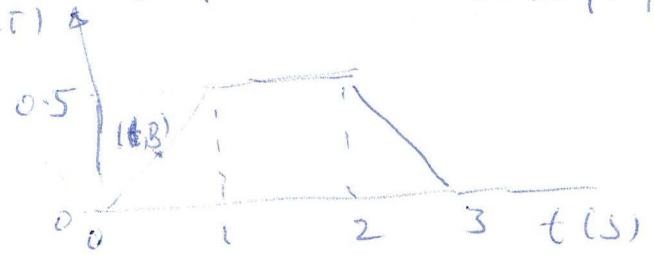
$= |12t + 7|_{t=2}$

$= 24 + 7 = 31$ mV

b, Since i should oppose the increase in ϕ_B it is to the left through R

2. Take:

The B field through a one turn loop of wire 10 cm in radius and 10 Ω in resistance changes with time as shown in the fig. below. plot a. The emf in the loop and b. the rate of thermal energy production in the loop. The uniform $\vec{B}$ is ⊥ to the plane of the loop.



a, Area of loop $= \pi R^2 = 3.14\,(0.1\text{m})^2 = 3.14 \times 10^{-2}\ m^2$

At any time since $\vec{B}$ is ⊥ to the plane of the loop

$$\phi_B = \int \vec{B} \cdot d\vec{S} = B \int dS = B\,\pi R^2$$

$$\Rightarrow \quad V_i = -\frac{d\phi_B}{dt} = -\pi R^2 \frac{dB}{dt}$$

- $t \in [0, 1]$,

$\frac{0.5 - 0}{1 - 0} = \frac{B - 0}{t - 0} \Rightarrow B = \frac{1}{2}t$, Tesla and $V_i = -\frac{d\phi_B}{dt} = -1.57 \times 10^{-2}$ V

$P_i = V_i / R \approx 2.5 \times 10^{-5}$ W

- $t \in [1, 2]$

$B \equiv$ constant $= 0.5$ Tesla and $V_i = -\frac{d\phi_B}{dt} = 0$, $P_i = 0$

- $t \in [2, 3]$

is responsible for the presence of induced emf in the left coil. Faraday then states,

> the induced emf V_i in a circuit is equal to the negative rate at which flux through the circuit is changing

or,

$$V_i = -\frac{d\Phi_B}{dt}$$

The minus sign here indicates that V_i is in such a direction as to oppose the change of the flux that has produced it. This tendency is called Lenz's law and it is the consequence of conservation of energy.

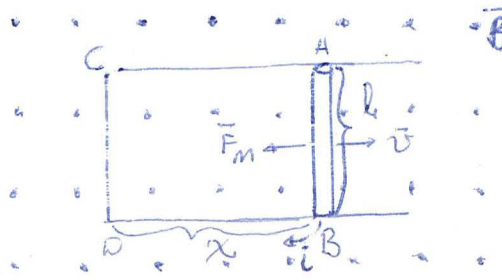
In fact if the sign of the induced emf V_i is the same as that of $d\Phi_B/dt$, the magnetic field produced by the current generated by V_i tends to change Φ_B in the same sense and thereby contributes to an increased value of V_i, and so on. Thus a small change in Φ_B would initiate a continuous change so that the small amount of energy used to change Φ_B initially would give rise to a large change in the magnetic energy of a system = a violation of the conservation of energy.

Note The sense of the induced emf is the same as the sense of the current it drives.

8.2 Motional Electromotive Force

The general idea is that a change in a magnetic flux (a time varying magnetic field) generates an induced emf. However, it is necessary to investigate whether the same results occur when the change in flux is due to a motion or deformation of the circuit without $\vec{B}$ necessarily changing with time.

Let us consider the arrangement of conductors shown in the fig below. They are placed in a uniform $\vec{B}$ field, directed out of the plane of the figure.



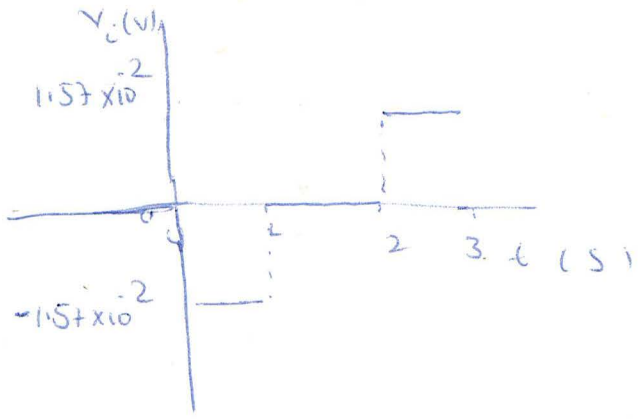
The vertical conductor AB is pulled to the right at a velocity $\vec{v}$, always keeping being in contact with the other conductor at the bottom and top of it

Suppose that the distance moved by the vertical conductor AB is x. Then the flux through the plane surface ABDCA is given by

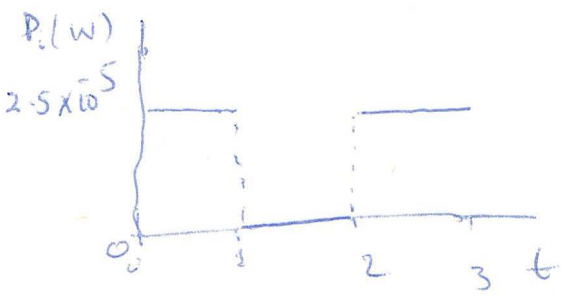
$$\phi_B = \int_S \vec{B}\cdot d\vec{S} = Blx$$

$$\frac{0.5-0}{2-3} = \frac{B-0}{t-3} \Rightarrow B = -\frac{1}{2}(t-3)\ \text{Tesla}$$

and $V_i = -\frac{d\phi_B}{dt} = 3.14\times10^{-2}\cdot\frac{1}{2}V = 1.57\times10^{-2}\ V$, $P_i = 2.5\times10^{-5}\ W$



b,



The change of flux per unit time is thus

$$-\frac{d\Phi_B}{dt} = -\frac{d}{dt}(Blx) = Bl\left(-\frac{dx}{dt}\right)$$

Since $-dx/dt = v$, we have

$$-\frac{d\Phi_B}{dt} = V_i = Blv$$

An induced emf such as this produced by pulling a conductor through a magnetic field is sometimes called a motional emf.

The emf Blv sets up a current in the circuit ABDCA given by

$$i = \frac{V_i}{R} = \frac{Blv}{R}$$

where R is the total resistance of the circuit.

From Lenz's law, this current (and hence V_i) must be clockwise; it opposes the "change" (the increase in Φ_B) by setting up a field opposite to the external field within the circuit. So in order to move the conductor AB to the right, an external agent must exert a force $\vec{F}$ that has the same magnitude as the leftward magnetic force F_m. or

$$F = F_m = ilB\sin 90^\circ = \frac{B^2l^2v}{R}$$

The agent that pulls the conductor AB must do work at the steady rate of

$$P = Fv = \frac{B^2l^2v^2}{R}$$

From the principle of the conservation of energy, Joule heat must appear in the circuit at this same rate. Thus,

$$P_s = i^2R = \left(\frac{Blv}{R}\right)^2 R = \frac{B^2l^2v^2}{R} = P$$

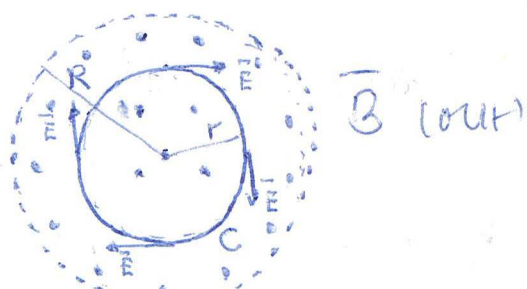
which gives a quantitative illustration of the conservation of mechanical energy (the work done by an external agent) into electrical energy (the induced emf) and finally into thermal energy (the Joule heating).

8.3 Induced Electric Field

So far we have considered emfs induced by the relative

motion of magnets and coils, as in the first experiment of sec. 8.1. In this section we assume that there is no physical motion of gross objects but that the magnetic field may vary with time, much as the second experiment of that section. If we place a conducting loop in such a time varying field, the flux through the loop will change and an induced emf will appear in the loop. This emf will set the charge carriers in motion, that is, it will induce a current.

We can say, equally well, that the changing flux of $\bar{B}$ sets up an induced emf electric field $\bar{E}$ around the loop. This induced electric field will then exert a force $\bar{F}$ on each charge carrier q given by $\bar{F} = q\bar{E}$. Let us illustrate our discussion by considering a circular loop of wire of arbitrary radius r, that is placed within a uniform magnetic field $\bar{B}$ which increases in magnitude at a constant rate dB/dt with time at every point in the region indicated by the dotted circle (fig. below).



The loop above encloses a changing magnetic flux ϕ_B which creates an induced emf given by

$$V_i = -\frac{d\phi_B}{dt}$$

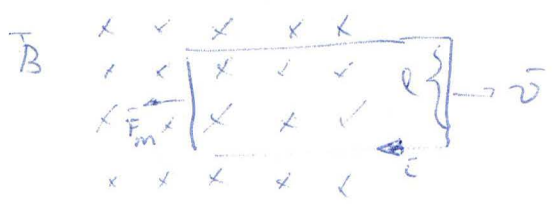
Lenz's law indicates that the current flows clockwise through the loop. The fact that the current density is the direction of the electric field and the consideration of symmetry imply that the induced electric field $\bar{E}$ must be tangent to the loop in the direction shown above. Thus the lines of force form concentric circles.

Next imagine that a positive charge q_0 is moving around the loop of radius r. The work W done on this charge per revolution is

$$W = qV_i$$

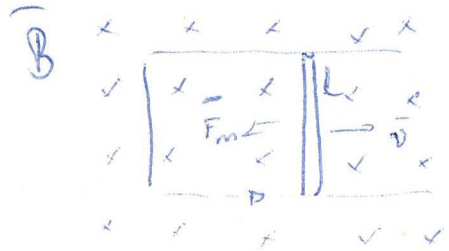
Examples:

1.



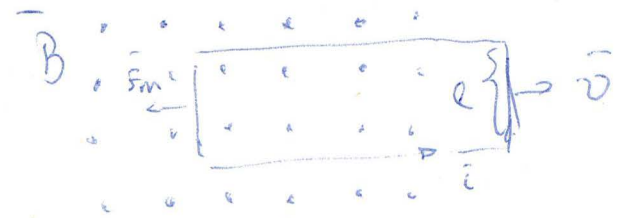
i - cw because the field due to i must add up to the original field B so as to oppose the decrease in flux.

2.



i ccw because now the flux is increasing and i should counter-balance this effect.

3.

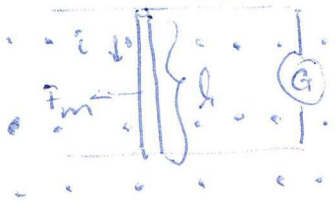


i - ccw because the flux is decreasing, and it should oppose this change.

In any of these cases, $v_i = B\ell v$, $i = \frac{B\ell v}{R}$, $P = \frac{B^2\ell^2 v^2}{R}$

R being the resistance of each circuit

4.



The conductor of mass m ~~moves to the left~~ slides without friction on two horizontal rails. $\bar{B}$ is vertical and G supplies a const current i. Find the velocity of the conductor as a function of time, assuming it to be at rest at $t = 0$.

$$F_m = i\ell B = m\frac{dv}{dt}, \quad dv = \frac{i\ell B}{m}dt$$

$$\Rightarrow \quad v(t) = \frac{i\ell B}{m}t + c$$

But $v(0) = 0 \Rightarrow c = 0$.

$$\therefore \quad v(t) = \frac{i\ell B}{m}t, \text{ to the left.}$$

The change of flux per unit time is thus

$$-\frac{d\Phi_B}{dt} = -\frac{d}{dt}(Blx) = Bl\left(-\frac{dx}{dt}\right)$$

Since $-dx/dt = v$, we have

$$-\frac{d\Phi_B}{dt} = V_i = Blv$$

An induced emf such as this produced by pulling a conductor through a magnetic field is sometimes called a motional emf.

The emf Blv sets up a current in the circuit ABDCA given by

$$i = \frac{V_i}{R} = \frac{Blv}{R}$$

where R is the total resistance of the circuit.

From Lenz's law, this current (and hence V_i) must be clockwise; it opposes the "change" (the increase in Φ_B) by setting up a field opposite to the external field within the circuit. So in order to move the conductor AB to the right, an external agent must exert a force $\vec{F}$ that has the same magnitude as the leftward magnetic force F_m. or

$$F = F_m = ilB\sin 90^\circ = \frac{B^2l^2v}{R}$$

The agent that pulls the conductor AB must do work at the steady rate of

$$P = Fv = \frac{B^2l^2v^2}{R}$$

From the principle of the conservation of energy, Joule heat must appear in the circuit at this same rate. Thus,

$$P_J = i^2R = \left(\frac{Blv}{R}\right)^2 R = \frac{B^2l^2v^2}{R} = P$$

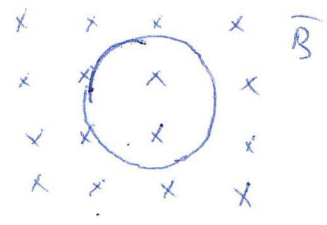
which gives a quantitative illustration of the conservation of mechanical energy (the work done by an external agent) into electrical energy (the induced emf) and finally into thermal energy (the Joule heating).

8.3 Induced Electric Field

So far we have considered emfs induced by the relative

Examples:

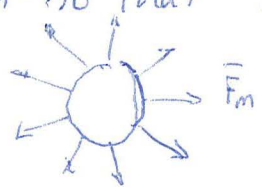
1. A circular loop of wire of area A and electrical resistance R is placed in a spatially uniform magnetic field B directed into the board and ⊥ to the plane of the loop as shown in the fig. below. The magnetic field is gradually reduced from an initial value of B_0, in such a way that the magnetic field strength as a function of time is $B(t) = B_0 e^{-\alpha t}$. a, Indicate on a diagram the direction of the induced current. Explain your choice. b, Do the electromagnetic forces on this current tend to make the loop expand or contract? Explain. c, Determine an expression, in terms of B_0, A and R, that describes the total quantity of charge that flows past a point in the loop during the time the magnetic field is reduced from B_0 to zero. d, Determine an expression for the amount of energy dissipated as heat in the loop, in terms of B_0, A, R, and α, during the time the magnetic field is reduced from B_0 to zero.



Solution

a, i is cw because the flux due to B is decreasing and the magnetic field associated with this current must oppose this effect.

b, The right hand rule shows that these forces act in all directions radially outward so that they tend to make it loop expand.



c,

$$\Phi_m = \int \vec{B} \cdot d\vec{s} = BA \quad (\vec{B} \text{ does not depend on position and has the same direction as } d\vec{s})$$

$$\Rightarrow \mathcal{V}_i = -\frac{d\Phi_m}{dt} = -A\frac{dB}{dt} = \alpha A B_0 e^{-\alpha t}$$

$$\Rightarrow i = \frac{dQ}{dt} = \frac{\mathcal{V}_i}{R} = \frac{\alpha A B_0}{R} e^{-\alpha t}$$

$$\int_0^Q dQ = \frac{\alpha A B_0}{R} \int_0^\infty e^{-\alpha t} dt = \frac{\alpha A B_0}{R}\left(-\frac{e^{-\alpha t}}{\alpha}\right)_0^\infty$$

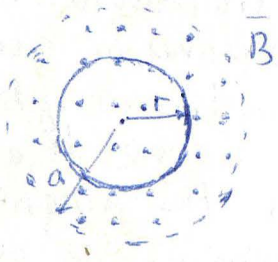
or $$Q = AB_0/R$$

a, $\frac{dU}{dt} = iV_i = \left(\frac{\alpha A B_0}{R} e^{-\alpha t}\right)\left(\alpha A B_0 e^{-\alpha t}\right) = \frac{\alpha^2 A^2 B_0^2}{R} e^{-2\alpha t}$

$\Rightarrow U = \frac{\alpha^2 A^2 B_0^2}{R} \int_0^\infty e^{-2\alpha t}\, dt = \frac{\alpha^2 A^2 B_0^2}{R} \left(\frac{e^{-2\alpha t}}{-2\alpha}\right)_0^\infty$

or $U = \alpha A^2 B_0^2 / 2R$

2.



In the cylindrical region of radius a, the magnetic field increases at a constant rate s.t. $B = B_0 + Ct$ where B_0 and C are constants and t is time. A circular conducting loop of radius r and resistance R is placed ⊥ to the magnetic field.

a, Indicate on a diagram the direction of the induced current in the loop. Explain your choice.

b, Derive an expression for the induced current in the loop.

c, " " " " " magnitude of the induced electric field at any radius $r < a$. c, Derive an expression for the magnitude of the induced electric field at any radius $r \geq a$.

Soln.

a. ccw since it has to oppose the increase in the flux of the increasing magnetic field.

b,

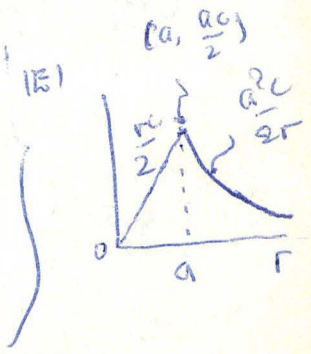
$\Phi_m = \int \vec{B} \cdot d\vec{s} = -BA$ since $\vec{B} \parallel \vec{A}$ and $\vec{B}$ is spatially const.
$= -B \cdot \pi r^2$

$V_i = -\frac{d\Phi_m}{dt} = +\pi r^2 \frac{dB}{dt} = +\pi r^2 C$

The magnitude of the induced current is given by

$i = \frac{V_i}{R} = +\frac{\pi r^2 C}{R}$

~~with magnitude, $i = \pi r^2 C / R$~~



c, $\oint_c \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \int_s \vec{B} \cdot d\vec{s} = V_i = -\pi r^2 C$, $r < a$

$E \cdot 2\pi r = +\pi r^2 C$

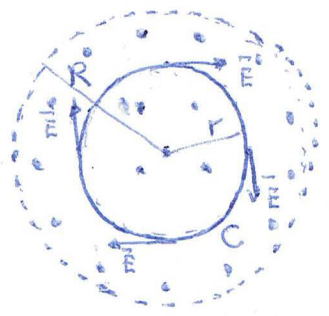
or $E = +rC/2$, ~~$|E| = rC/2$~~

d, $\oint \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \int \vec{B} \cdot d\vec{s} = +\frac{dB}{dt} \pi a^2$ since B/Hp for $r \geq a$.

i.e., $E \cdot 2\pi r = +\pi a^2 \frac{dB}{dt}$ or $E = +\frac{1}{2} \frac{a^2}{r} \cdot C$, ~~$|E| = \frac{a^2 C}{2r}$~~

motion of magnets and coils, as in the first experiment of sec. 8.1. In this section we assume that there is no physical motion of gross objects but that the magnetic field may vary with time, much as the second experiment of that section. If we place a conducting loop in such a time varying field, the flux through the loop will change and an induced emf will appear in the loop. This emf will set the charge carriers in motion, that is, it will induce a current.

We can say, equally well, that the changing flux of $\bar{B}$ sets up an induced emf electric field $\bar{E}$ around the loop. This induced electric field will then exert a force $\bar{F}$ on each charge carrier q given by $\bar{F} = q\bar{E}$. Let us illustrate our discussion by considering a circular loop of wire of arbitrary radius r, that is placed within a uniform magnetic field $\bar{B}$ which increases in magnitude at a constant rate dB/dt with time at every point in the region indicated by the dotted circle (fig. below).



The loop above encloses a changing magnetic flux Φ_B which creates an induced emf given by

$$V_i = -\frac{d\Phi_B}{dt}$$

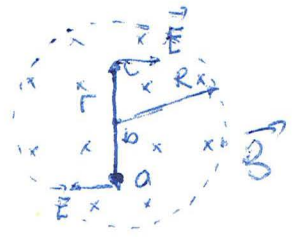
Lenz's law indicates that the current flows clockwise through the loop. The fact that the current density is in the direction of the electric field and the consideration of symmetry imply that the induced electric field $\bar{E}$ must be tangent to the loop in the direction shown above. Thus the lines of force form concentric circles.

Next imagine that a positive charge q_0 is moving around the loop of radius r. The work W done on this charge per revolution is

$$W = qV_i$$

Example:

The figure below shows a uniform magnetic field $\vec{B}$ confined to a cylindrical volume of radius R. $\vec{B}$ is decreasing in magnitude at a const. rate of 10 mT/s. What is the instantaneous acc. (direction and magnitude) experienced by an electron placed at a, at b, and at c? Assume r = 5 cm.



Solution:

Considering a circular region of radius r we have

$$\oint_c \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\int_s \vec{B}\cdot d\vec{s} \quad , \quad (\vec{B}\cdot d\vec{s}) = 180^\circ$$

$$E \cdot 2\pi r = +\frac{d}{dt}(\pi r^2 B) = \pi r^2 \frac{dB}{dt}$$

or,

$$E = \frac{r}{2}\frac{dB}{dt} \ ,$$

And the force acting on the electron at a is ~~directed~~ given by

$$F_a = -e(-E) = +\frac{er}{2}\frac{dB}{dt} = m_e a$$

so that the acc. at a is $a = F_a/m_e = +(er/2m_e)\frac{dB}{dt}$. Or,

$$a_a = \frac{F_a}{m_e} = +1.6\times10^{-19}\,C \times \frac{5\times10^{-2}}{2}\,m \times (+10\times10^{-3}\,T/s) \times \frac{1}{9.1\times10^{-31}\,kg}$$

$$= 4.4\times10^{7}\ m/s^2.$$

Then the acc. is directed to the right.

Similarly, the force at point c is

$$F_c = -e(+E) = -eE = -\frac{er}{2}\frac{dB}{dt}.$$

Then,

$$a_c = \frac{F_c}{m_e} = \frac{er}{2m_e}\frac{dB}{dt},$$

or,

$$a_c = -1.6\times10^{-19}\,C \times \frac{5\times10^{-2}\,m}{2}\,(+10\times10^{-3}\,T/s) \times \frac{1}{9.1\times10^{-31}\,kg}$$

$$= -4.4\times10^{7}\ m/s^2.$$

This acc. is directed to the left.

At point b. $F_b = 0$ since $r = 0$ and hence $a_b = 0$.

In terms of the force exerted and the dist. moved we have

$$W = qE \cdot 2\pi r$$

i.e.,

$$qV_i = qE\,2\pi r$$

or

$$V_i = E \cdot 2\pi r$$

In general, since $E \cdot 2\pi r$ is the circulation of the induced electric field we write

$$V_i = \oint_C \vec{E} \cdot d\vec{l}$$

But

$$V_i = -\frac{d\phi_B}{dt} = -\frac{d}{dt}\int_S \vec{B} \cdot d\vec{s}$$

whence

$$\oint_C \vec{E} \cdot d\vec{l} = -\frac{d}{dt}\int_S \vec{B} \cdot d\vec{s}$$

The left integral here is not zero as might be expected from our discussion of this integral for the electric field associated with stationary charges. So this fact tells us that electric fields associated with changing magnetic fields are non conservative.

Now assume that the loop and its surface elements do not change with time. Then the operations of taking the derivative wrt time and integrating over the area of the loop can be interchanged, giving

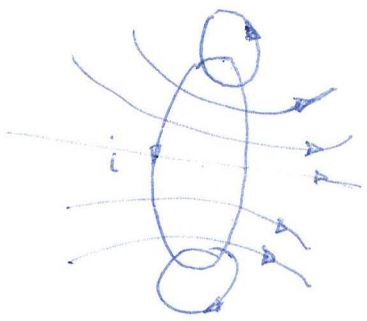
$$\oint_C \vec{E} \cdot d\vec{l} = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s}$$

The partial derivative is taken because, the field vector $\vec{B}$ may be a function of position as well as time.

chapter 9 Inductance

9.1 Self-Induction

Consider a circuit carrying a current i (below). According to Ampere's law, the current produces a magnetic field that is proportional to i at each point. The magnetic flux through the circuit produced by its own magnetic field is called the self flux. This magnetic flux is



then proportional to the magne current i and may be written as

$$\phi_B = Li$$

The coefficient L depends on the geometric shape of the conductor and is called the self-inductance of the circuit.

unit

$$[L] = \left[\frac{\phi_B}{i}\right] = \frac{T\cdot m^2}{A} = \frac{\text{Weber}}{\text{Ampere}} \left(\frac{Wb}{A}\right)$$

And the unit Weber/Ampere is called The Henry (H).

Suppose that the electric current in a circuit is not constant. For example the current may be varied by varying either the emf applied to the circuit or the electric resistance of the circuit, or by varying both. When the current i changes with time, the magnetic flux through the circuit also changes; and according to Faraday's law of induction, an emf is induced in the circuit. This called self-induction. So we write

$$V_i = -\frac{d\phi_B}{dt} = -L\frac{di}{dt}$$

once again the minus sign indicates that V_i is opposed

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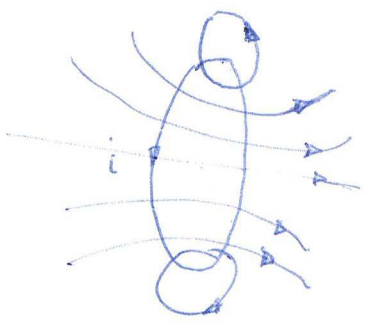

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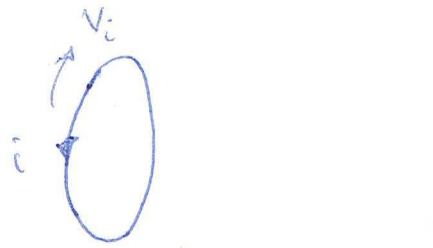
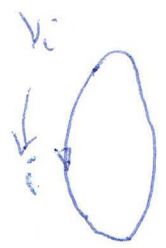
$$V_i = -\frac{d\phi_B}{dt} = -L\frac{di}{dt}$$

once again the minus sign indicates that V_i is opposed

to the change in current. Thus if the current increases di/dt is positive and v_i is opposed to the current fig. (a) below. If the current decreases, di/dt is negative and v_i acts in the same direction as the current fig. (b) below.



(a) i increases

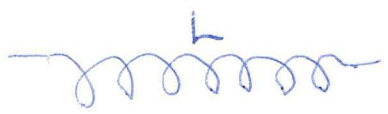


(b) i decreases

If the shape of the circuit changes with time, so does L and we write

$$v_i = -\frac{d}{dt}(Li)$$

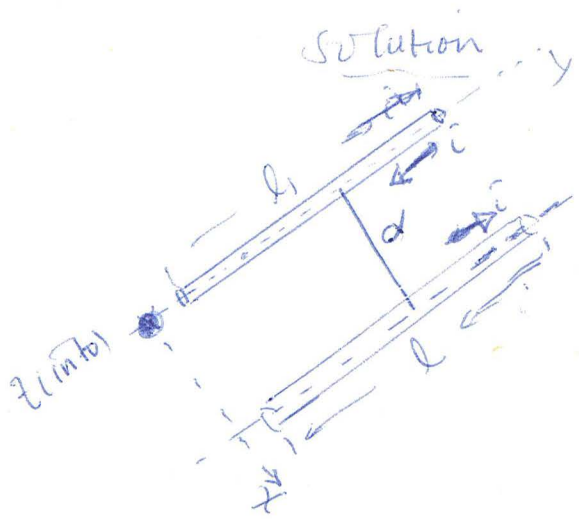
If a conductor in a circuit has an inductance, it is indicated by the symbol shown below.



Example: Two long parallel wires whose centers are a dist. d apart carry equal currents in opposite directions. Show that neglecting the flux within the wires themselves, the inductance of a length ℓ of such a pair of wires is given by (the radii of the wires is a)

$$L = \frac{\mu_0 \ell}{\pi} \ln\left(\frac{d-a}{a}\right)$$

Solution



$$\int \vec{B}\cdot d\vec{\ell} = \mu_0 i$$

$$B 2\pi x = \mu_0 i$$

$$\Rightarrow B = \frac{\mu_0}{2\pi}\frac{i}{x}$$

The flux in the region between the conductors (and neglecting the flux on the conductors) is

$$\Phi_B = \int_S \vec{B}\cdot d\vec{S} + \int_S \vec{B}\cdot d\vec{S}$$

due to both wires. We have added the contributions of both wires since $\vec{B}$ in each case has the same direction.

Assume that the positive direction of the z-axis is out of the plane of the figure. Thus,

$$\vec{B} = B\vec{e}_z \cdot d\vec{S} = l\,dx\,\vec{e}_z .$$

in which case we have considered a length l of the region between the pair of wires. Next we write

$$\Phi_B = 2\int \vec{B}\cdot d\vec{S}$$

$$= 2\cdot\frac{\mu_0 i l}{2\pi}\int_a^{d-a}\frac{dx}{x} = \frac{\mu_0 i l}{\pi}\ln\frac{d-a}{a}$$

or

$$\Phi_B = Li = \frac{\mu_0 i l}{\pi}\ln\frac{d-a}{a}$$

(i.e.)

$$L = \frac{\mu_0 l}{\pi}\ln\frac{d-a}{a}$$

One can see from this result that L indeed depends on geometry.

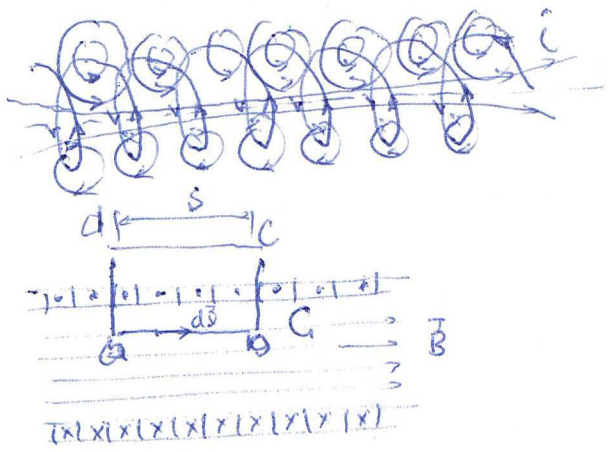
Exercise: Show that the self inductance L of a a section of length l of a long solenoid is given by

$$L = \mu_0 n^2 l A$$

where n is the no. of turns per unit length and A is the cross sectional area of the solenoid.

9.2 Mutual Induction

From Faraday's 2nd experiment we recall that, if two coils (circuits) are close together, as in the fig. of last section, a steady current i in one coil will set



$$\oint_C \vec{B}\cdot d\vec{l} = \mu_0 i'$$

$$\Rightarrow \int_a^b \vec{B}\cdot d\vec{l} + \int_b^c \vec{B}\cdot d\vec{l} + \int_c^d \vec{B}\cdot d\vec{l} + \int_d^a \vec{B}\cdot d\vec{l} = \mu_0 i'$$

($\int_b^c$: $\vec{B}\perp d\vec{l}$, so $\vec{B}\cdot d\vec{l} = 0$; $\int_c^d$: $\vec{B}\approx 0$; $\int_d^a$: $\vec{B}\perp d\vec{l}$)

$$\int_0^b \vec{B}\cdot d\vec{l} = \mu_0 i'$$

$$B\,s = \mu_0 i'$$

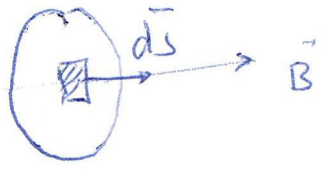
Suppose there are n turns per unit length. So the total current enclosed will be

$$i' = n s i$$

$$B\,s = \mu_0 n s i$$

or $$B = \mu_0 n i$$

Consider one circular turn of the solenoid



$$\phi_1 = \int \vec{B}\cdot d\vec{s} = BA = \mu_0 n i A$$

A is the area of this cross section

But for a total no. N turns of the solenoid, the total flux will be

$$\phi_T = N\phi_1 = N\mu_0 n i A = Li$$

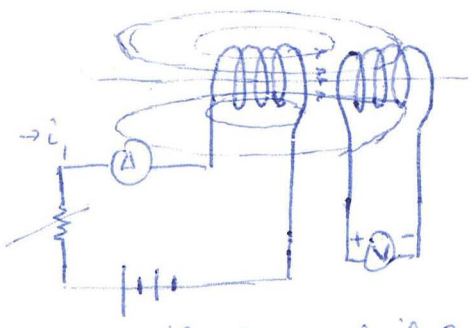
$$\Rightarrow L = \mu_0 N n A$$

If l is the length of the solenoid, $N = nl$. Thus,

$$L = \mu_0 n^2 l A.$$

up a magnetic flux ϕ linking the other coil. This process is called mutual induction so as to suggest the mutual interaction of the two coils and to distinguish it from self-induction, in which only one coil is involved.

In the fig. below we have two coils set up in a slight modification of the Faraday experiment. When a



steady current i_1 flows through coil 1, some of the ~~flux~~ associated magnetic flux penetrates the area enclosed by coil 2. It is this part of the magnetic flux that links the two coils; call it ϕ_{12}. Now,

$$\phi_{12} \propto i_1$$

If the current i_1 changes at a rate di_1/dt, the magnetic flux penetrating coil 2 will vary at a rate $d\phi_{12}/dt$. According to Faraday's law, the voltmeter across coil 2 will read an induced emf

$$V_2 = -\frac{d\phi_{12}}{dt} \propto -\frac{di_1}{dt}$$

i.e.) the emf induced in coil 2 is proportional to the rate of change of current flowing through coil 1. The proportionality constant is called the mutual inductance M_{12}

$$V_2 = -M_{12}\frac{di_1}{dt}$$

Similarly let us now interchange the roles of coils 1 and 2. That is, if we put a battery (source of emf) in coil 2 and set up a current i_2, this current will produce a magnetic flux ϕ_{21} at the site of coil 1. If we vary i_2 with time, we have, by the same argument given above,

$$V_1 = -M_{21}\frac{di_2}{dt}$$

Which is similar to the corresponding expression for V_2. The proportionality constants M_{12} and M_{21} are found to be the same, i.e.,

$$M_{12} = M_{21} = M$$

and we write,

$$V_2 = -M\frac{di_1}{dt} \quad \text{and} \quad V_1 = -M\frac{di_2}{dt}$$

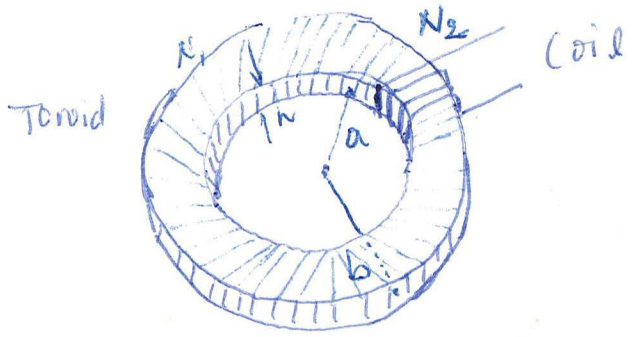
The SI unit of M as for L is Henry (H).

Once again M depends on geometry as the example below suggests.

Examples

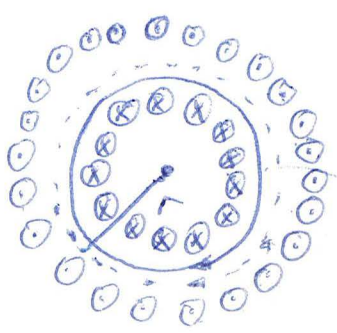
1. The fig. below shows a coil of N_2 turns linked to a toroid of N_1 turns. The inner toroid radius is a, its outer radius is b, and its height is h. Show that the coeff. of mutual inductance M for the toroid-coil combination is

$$M = \frac{\mu_0 N_1 N_2 h}{2\pi} \ln\frac{b}{a}$$



Solution

The cross section of the toroid is shown below.



$\vec{B}$ field

Loop of Ampere's law, C.

Applying Ampere's law

$$\oint_C \vec{B}\cdot d\vec{\ell} = \mu_0 i', \quad (\ i' \text{ current enclosed by the loop})$$

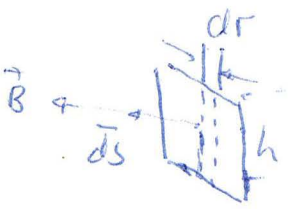
and $i' = N_1 i$, i is the current within (N turns)

From symmetry B is const on the loop. Thus

$$B\, 2\pi r = \mu_0 N_1 i$$

$$\Rightarrow \quad B = \frac{\mu_0 N_1}{2\pi}\frac{i}{r}$$

Consider a single turn of the coil; the cross-section is as shown below.



$$ds = h\,dr$$

So the flux over this turn is

$$\phi_1 = \frac{\mu_0 N_1 i h}{2\pi}\int_a^b \frac{dr}{r} = \frac{\mu_0 N_1 i h}{2\pi}\ln\frac{b}{a}$$

Since there are N_2 turns, the total flux over the coil is

$$\phi_T = N_2\phi_1 = \frac{\mu_0 N_2 N_1 i h}{2\pi}\ln\frac{b}{a}$$

But we can write

$$\phi_T = M i$$

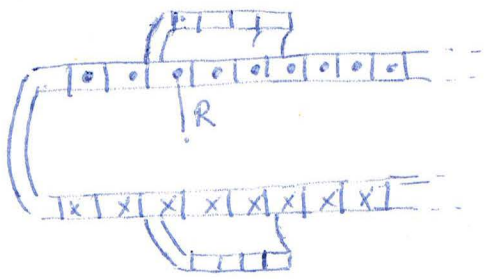
i.e., the flux over the coil is due to the magnetic field set up by the current in the toroid. Now comparison gives

$$M = \frac{\mu_0 N_1 N_2 h}{2\pi}\ln\frac{b}{a}$$

2. A coil of N turns is placed around a long solenoid of radius R and n turns per unit

length as in the figure below. Show that the coeff. of mutual inductance for the coil-solenoid combination is given by

$$M = \mu_0 \pi R^2 n N$$

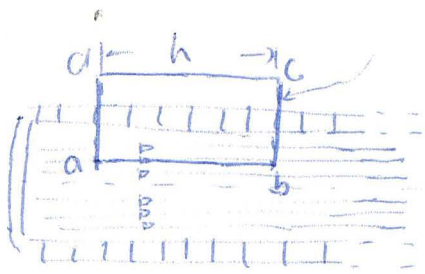


A current i flows through each turn of the solenoid. The no. of turns per unit length is n (i.e., $n = N/\ell$).

Solution

Find $\vec{B}$ for the solenoid.

Applying the right hand rule one can see that $\vec{B}$ is parallel to the axis of the solenoid for the region inside the solenoid. Moreover, for a long solenoid $\vec{B}$ is approximately (or practically) zero for points outside the solenoid.



Loop of Ampere's law

$$\oint \vec{B} \cdot d\vec{\ell} = \mu_0 i'$$

$$\int_a^b \vec{B} \cdot d\vec{\ell} + \int_b^c \vec{B} \cdot d\vec{\ell} + \int_c^d \vec{B} \cdot d\vec{\ell} + \int_d^a \vec{B} \cdot d\vec{\ell} = \mu_0 i'$$

The 3rd integral is also zero since $\vec{B}$ is zero at external points

$$\int_a^b B\, d\ell = \mu_0 i' = \mu_0 (nh)\, i$$

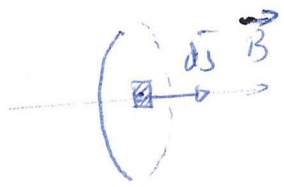
where nh is the no. of turns inside the rectangular path. So

$$Bh = \mu_0 nh i$$

or

$$B = \mu_0 n i$$

Let us find the flux over a single turn of the coil.



$$\phi_1 = \int \vec{B} \cdot d\vec{S} = \int B\,dS = B\,\pi R^2$$

flux over this turn is the same as the flux over the cross sectional area of the solenoid since $B \approx 0$ at external points.

The turn is circular with radius R of the solenoid

Total flux will be

$$\phi_T = N\phi_1 = NB\pi R^2$$

Then.

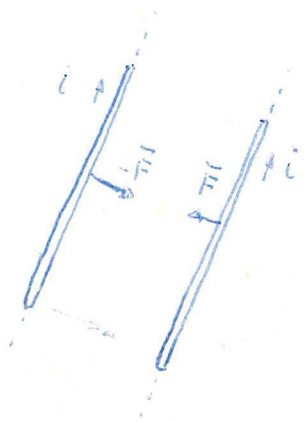
$$Mi = N \cdot \mu_0 n i \pi R^2$$

or

$$M = \mu_0 \pi R^2 n N$$

9.3 Energy Stored in a Magnetic Field

We now derive an expression for the density of energy u_B in a magnetic field. Consider a length l of a portion near the center of a very long solenoid; Al is the volume associated with this length. The stored energy must is stored within a magnetic field. For example, if we consider two long, rigid, parallel wires (fig below) carrying current in the same direction, we can see that an external agent must do work to pull them apart. We can get this stored energy back at any time by letting the wires move back to their original positions



To derive a quantitative expression for the energy stored in the magnetic field, consider the figure below which shows a circuit consisting of an inductor of self-inductance L & a resistor of resistance R that are connected in series with a source of emf V.

Applying Kirchhoff's loop rule we find

$$V - iR - L\frac{di}{dt} = 0.$$

or

$$V = iR + L\frac{di}{dt}$$

This is a differential eq. that describes the growth of current in the circuit. multiplying both sides of the eq by i, we obtain

$$Vi = i^2R + Li\,di/dt$$

- If a charge dq passes through the source of emf V in time dt, the source does work $dW = V dq$ on this charge. So $Vi (= V\,dq/dt)$ represents the rate at which the source delivers energy to the circuit.

- The second term represents the rate at which energy appears as thermal energy in the resistors.

- Since this eq. is a statement of the conservation of energy, the last term on the right must represent the rate dU_B/dt at which energy is stored in the magnetic field, or

$$\frac{dU_B}{dt} = Li\frac{di}{dt} \Rightarrow dU_B = Li\,di$$

Integrating:

$$U_B = \int_0^{U_B} dU_B = \int_0^i Li\,di = \tfrac{1}{2}Li^2$$

which is the total magnetic energy stored by an inductor L carrying a current i. This expression is similar to the corresponding expression of the electric energy stored within the electric field of a capacitor of charge Q and capacitance C, namely,

$$U_E = \frac{1}{2}\frac{Q^2}{C}$$

Next let us derive an expression for the density of energy u_B in a magnetic field. Consider a portion of a length ℓ near the center of a very long solenoid; $A\ell$ is the volume associated with this length. The stored energy must lie entirely within this volume because the magnetic field outside such a solenoid is essentially zero. Moreover, the stored energy must be uniformly distributed throughout the volume of the solenoid because the magnetic field is uniform everywhere inside. So we write

$$u_B = U_B/A\ell = \tfrac{1}{2}Li^2/A\ell$$

But for a solenoid of length ℓ,

$$L = \mu_0 n^2 \ell A$$

whence

$$U_B = \frac{1}{2}\frac{(\mu_0 ni)^2 \ell A}{\mu_0 A \ell}$$

i.e.,

$$u_B = \frac{B^2}{2\mu_0}$$

Although this eq. is derived for a special case, the solenoid, it holds for all magnetic field configurations. This eq. can be compared with the analogous eq. of the electric field.

$$u_E = \tfrac{1}{2}\epsilon_0 E^2$$

which gives the energy density (in vacuum) at any point in an electric field. Note that both u_B and u_E are proportional to the square of the appropriate field quantity, $\vec{B}$ or $\vec{E}$.

Examples:

1. (a) Find an expression for the energy density as a function of the radius ~~for an toroid~~ (dist. from the center) for a toroid.
(b) Find an expression for the total energy stored in the field of a toroid.

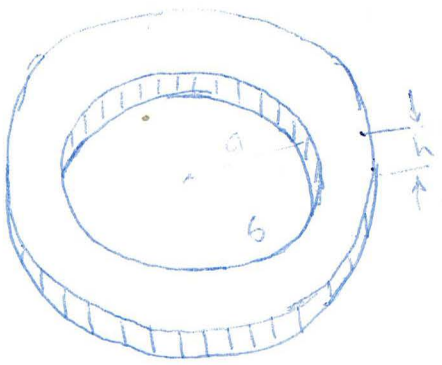
a. From example 1 Sec. (9.2), the magnetic field within a toroid is given by the expression

$$B = \frac{\mu_0 N i}{2\pi r}$$

where N is the no. of turns of the toroid. So the energy density is given by

$$u_B = \frac{B^2}{2\mu_0} = \frac{\mu_0 N^2 i^2}{8\pi^2 r^2}$$

b,



Consider a toroid of inner radius a, outer radius b and height h.

At a dist. r from the center of the toroid consider a circular loop whose thickness is dr. Its volume of course its height is the same as that of the toroid, h. So the volume of this element is given by

$$dV = (2\pi r\, dr)\, h$$

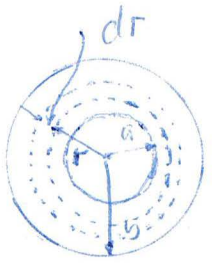
(i.e.) the magnetic energy here is

$$dU_B = \frac{\mu_0 N^2 i^2}{8\pi^2 r^2} \cdot 2\pi r h\, dr = \frac{\mu_0 N^2 i^2}{4\pi r} h\, dr$$

or within the toroid as a whole

$$U_B = \frac{\mu_0 N^2 i^2 h}{4\pi} \int_a^b \frac{dr}{r} = \frac{\mu_0 N^2 i^2 h}{4\pi} \ln \frac{b}{a}$$

2. A long coaxial cable (fig. below) consists of two concentric cylinders with radii a and b. Its central conductor carries a steady current i, the outer conductor providing the return path. a, Calculate the energy stored in the magnetic field between the conductors for a length ℓ of such a cable. b, What is the self inductance of a length ℓ of this coaxial cable?



a, From Ampere's law, in the space between the two conductors

$$\oint_c \vec{B} \cdot d\vec{l} = \mu_0 i$$

which yields

$$B \cdot 2\pi r = \mu_0 i$$

or

$$B = \frac{\mu_0 i}{2\pi r}$$

For points in this region,

$$u_B = \frac{B^2}{2\mu_0} = \frac{\mu_0 i^2}{8\pi^2 r^2}$$

For a cylindrical volume element

$$dV = 2\pi r \ell \, dr$$

the energy contained in

$$dU_B = u_B dV = \frac{\mu_0 i^2 \ell}{4\pi} \frac{dr}{r}$$

So the total energy ~~contributions~~ stored in the magnetic field in the space between the conductors is

$$U_B = \frac{\mu_0 i^2 \ell}{4\pi} \int_a^b \frac{dr}{r} = \frac{\mu_0 i^2 \ell}{4\pi} \ln \frac{b}{a}$$

No energy is contained in the space outside the conductor because the magnetic field is zero there as may be verified from Ampere's law.

b,

$$U_B = \tfrac{1}{2} L i^2 \Leftrightarrow L = \frac{2U_B}{i^2}$$

or

$$L = \frac{\mu_0 \ell}{2\pi} \ln \frac{b}{a}$$

One can also calculate L from the relation $N\Phi_B = Li$.

3. Show that the self-inductance for a length ℓ of a long wire associated with the flux inside the wire only is $\mu_0 \ell / 8\pi$, independent of the wire diameter.

Solution

Consider a portion of a length ℓ of a long wire.

If a is the radius of the wire, for points lying within the region of the wire s.t. $r < a$

$$\oint \vec{B} \cdot d\vec{\ell} = \mu_0 i'$$

$$B \cdot 2\pi r = \mu_0 i'$$

$$\Rightarrow B = \frac{\mu_0 i'}{2\pi r} = \frac{\mu_0 i r}{2\pi a^2}$$

$$u_B = \frac{B^2}{2\mu_0} = \frac{\mu_0 i^2 r^2}{8\pi^2 a^4}$$

For a cylindrical volume element of length ℓ.

$$dV = (2\pi r dr)\ell$$

$$\Rightarrow dU_B = \frac{\mu_0 i^2 \ell r^3}{4\pi a^4} dr$$

$$U_B = \frac{\mu_0 i^2 \ell}{4\pi a^4} \int_0^a r^3 dr = \frac{\mu_0 i^2 \ell a^4}{16\pi a^4} = \frac{\mu_0 i^2 \ell}{16\pi}$$

But

$$U_B = \tfrac{1}{2} L i^2 = \frac{\mu_0 i^2 \ell}{16\pi}$$

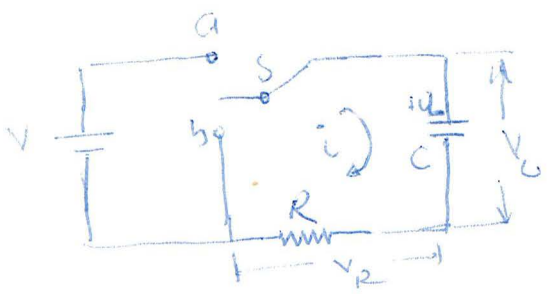
i.e.,

$$L = \frac{\mu_0 \ell}{8\pi}$$

Chapter 10 Circuits with Time-varying Currents

10.1. RC circuits

Charging a capacitor: Consider first the process of charging a capacitor in the circuit shown in the figure below.



We begin with zero charge on the capacitor. At time $t=0$ the switch S is closed to position a and a time dependent current $i(t)$ begins to flow. The charge q on the capacitor increases in accordance with the relation

$$i = \frac{dq}{dt}$$

Let us traverse the loop clockwise and apply Kirchhoff's second rule so as to obtain

$$V - V_C - V_R = 0 \Rightarrow V = V_C + V_R$$

or

$$V = iR + \frac{q}{C}$$

In differential form we write

$$V = R\frac{dq}{dt} + \frac{q}{C}$$

By arranging this can be written as

$$\frac{dq}{q - VC} = -\frac{1}{RC}dt$$

i.e.,

$$\ln(q - VC) = -\frac{t}{RC} + \ln C_1$$

or

$$q(t) = VC + C_1 e^{-t/RC}$$

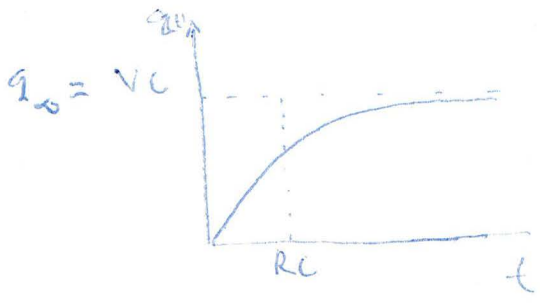
C_1 is a constant of integration that is determined by the initial condition $q = 0$ at $t = 0$. We find $C_1 = -VC$ so that

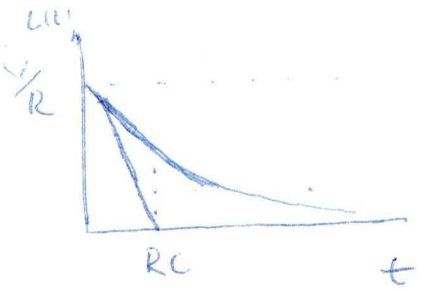
$$q(t) = VC\left(1 - e^{-t/RC}\right)$$

By differentiating $q(t)$ we obtain the expression for the time dependence of the current $i(t)$ in the circuit. Thus

$$i(t) = \frac{dq(t)}{dt} = \frac{V}{R} e^{-t/RC}$$

Plots of the two functions





The product RC has the dimensions of time and is referred to as the time constant τ_c of the circuit. For typical values, $R = 100\ \Omega$ and $C = 10^{-9}$ F, we find $\tau = RC = 10^{-7}$ s. The characteristic length associated with this circuit is $c\tau = 30$ m.

The 2nd graph of the eq. for $i(t)$ shows that the initial rate of charging of the capacitor is $d\mathcal{E}/dt = V/R$ and that the rate decreases with time as the capacitor approaches the fully charged condition, $q_\infty = VC$. This charge is reached strictly only after an infinite time. However, for all practical purposes, the capacitor may be considered to be fully charged after a time of approximately 5τ, for then $q/VC = 1 - e^{-5} = 0.993$. Similarly, after a time of 5τ, the charging current is essentially zero.

Discharging a Capacitor: After the capacitor in the above figure has been fully charged, the switch is moved to position b, and the capacitor begins to discharge through the resistance R. The differential eq. for the discharging condition can be obtained by simply setting $V = 0$ in the loop eq. The direction for positive current

flow and the charge polarity are as shown in that same circuit diagram above. Then for Kirchhoff's rule around a clockwise path, we have

$$-iR + \frac{q}{C} = 0 \Rightarrow iR + \frac{q}{C} = 0$$

or

$$i = +\frac{dq}{dt} = -\frac{q}{RC}$$

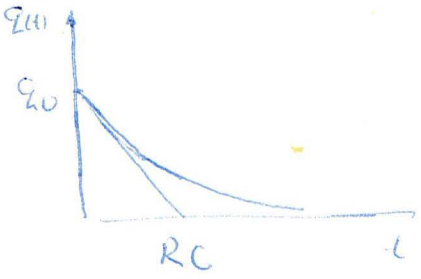
As expected $i < 0$, which actually means that the current flows in the counter clockwise direction. Applying the initial condition $q = q_0$ at $t = 0$, we solve the above differential eq. to obtain

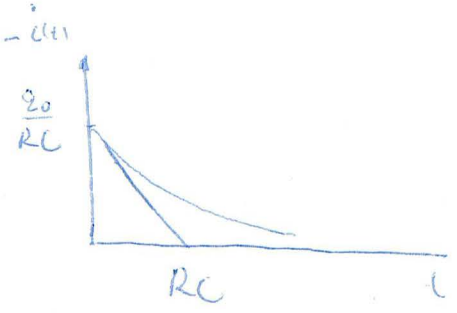
$$q(t) = q_0 e^{-t/RC}$$

and the discharge current is

$$i(t) = \frac{dq}{dt} = -\frac{q_0}{RC} e^{-t/RC}$$

Curves for $q(t)$ and $i(t)$:





Energy Balance: The Kirchhoff eq. for a charging capacitor is

$$\frac{q}{C} + iR = V$$

Multiplying this eq. by $i = dq/dt$, we find

$$\frac{q}{C}\frac{dq}{dt} + i^2R = Vi$$

or

$$d\left(\frac{q^2}{2C}\right) + i^2R\,dt = Vi\,dt$$

This eq. is interpreted as follows. During a time interval dt an amount of energy $Vi\,dt$ supplied by the

source of emf is expended partly in Joule heating, $i^2R\,dt$, and partly in increasing the stored energy in the electric field of the capacitance, $d(q^2/2C)$. Thus this eq. represents expresses energy conservation for the system in which chemical energy is converted partly into heat and partly into field energy.

The Kirchhoff eq. for a discharging capacitor, $\frac{q}{C} + iR = 0$, becomes

$$-d\left(\frac{q^2}{2C}\right) = i^2 R\,dt$$

which states that the energy removed from the electric field of the capacitor appears in the form of Joule heat in the resistor.

Examples.

1. In the circuit of Sec. 10-1 take $C = 0.1\,\mu F$, $V = 3\,V$ and $R = 10^3\,\Omega$. (The internal resistance of the battery is small compared with R.) The capacitor is uncharged when the switch is closed to position a.

a) Determine the initial current

b) " " charge q_∞ on the capacitor when it is fully charged.

c) Find the time required for the charge on the capacitor to reach 99% of q_∞.

d) Find the time required for the field energy in the capacitor to reach 99% percent of its final value

Solution

a. The initial current is

$$i_0 = \frac{V}{R} = \frac{3\,V}{10^3\,\Omega} = 3\,mA$$

b. The final charge is

$$q_\infty = VC = (3\,V)(0.1\times10^{-6}\,F) = 3\times10^{-7}\,C$$

c. For $q_L = 0.99\,q_\infty$, we have

$$\frac{q}{q_\infty} = \frac{q}{VC} = 1 - e^{-t/RC} = 0.99$$

$$\Rightarrow \quad e^{-t/RC} = 0.01$$

So that

$$t = -RC\ln(0.01) = -(10^3\,\Omega)(10^{-7}\,F)\ln(0.01)$$

or

$$t = 4.61 \times 10^{-9}\ s.$$

d) The energy stored is $U = \frac{1}{2}\frac{q^2}{C}$. Thus

$$\frac{U}{U_\infty} = \frac{q^2}{q_\infty^2} = 0.99$$

so that

$$\frac{q}{q_\infty} = 0.995 = 1 - e^{-t/RC}$$

from which

$$e^{-t/RC} = 0.005$$

or

$$t = -RC\ \ln(0.005) = -(10^3\Omega)(10^{-7}F)\ln(0.005)$$
$$= 5.30 \times 10^{-9}\ s.$$

2. A capacitor C is initially charged to a value $q_0 = VC$ with the polarity indicated in the diagram. Obtain an expression for the charge $q(t)$ as a function of time t after the switch S is closed. Determine the instant at which $q(t) = 0$.

solution

From Sec (10.1) we have

$$q(t) = VC + C_1 e^{-t/RC}$$

Since the charge at time $t=0$ on the capacitor tries to build up a potential that opposes the charging battery we have

$$q(0) = VC + C_1 = -VC \quad \text{or} \quad C_1 = -2VC$$

$$q(t) = VC\left(1 - 2e^{-t/RC}\right)$$

If $q(t) = 0$, then

$$1 - 2e^{-t/RC} = 0$$

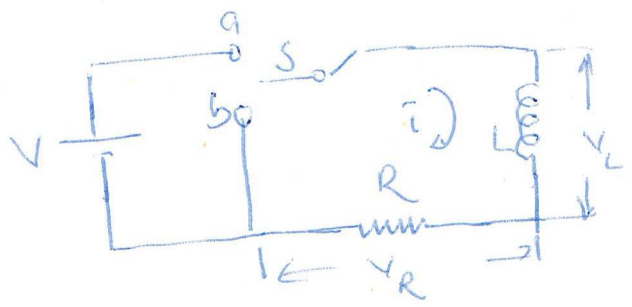
$$e^{-t/RC} = 1/2$$

or

$$t = -RC \ln(1/2) = 0.693RC.$$

10.2 The RL Circuit

Consider, from the process of applying current to an inductor in the circuit shown in the figure below. At



time $t=0$ the switch S is closed to position a and a time dependent (increasing) current $i(t)$ begins to flow. Because of this changing current in the inductor, there is produced a back emf, $L\,di/dt$, that opposes the driving emf V. In the application of Kirchhoff's rule, this back emf appears as a voltage drop V_L. Thus, for a clockwise path in the above circuit, we have

$$V - V_L - V_R = 0$$

or

$$V = L\frac{di}{dt} + iR$$

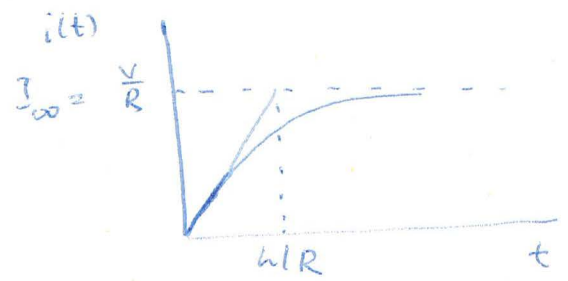
This is similar to the eq. for a charging circuit so that the solution is

$$i(t) = \frac{V}{R} + C_1 e^{-\frac{R}{L}t}$$

where C_1 is the arbitrary constant of integration that is determined by the initial condition, namely, $i=0$ at $t=0$. We find $C_1 = -V/R$, so that

$$i(t) = \frac{V}{R}\left(1 - e^{-\frac{R}{L}t}\right)$$

Curve of $i(t)$:



The quantity L/R is the inductive time constant τ_L of the circuit. When $t \gg L/R$, the current approaches the asymptotic value, $I_\infty = V/R$. For all practical purposes we may consider this current to have been reached after a time of approximately $5\tau_L$, for then $i = 0.993\, V/R$.

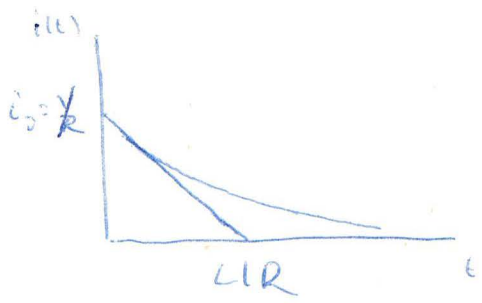
The above process is called energizing the inductor. Next, let us move the switch s to position 2. The current decreases from its initial value $i_0 = V/R$ in accordance with the eq.

$$L\frac{di}{dt} + iR = 0$$

Then, $c_1 = i_0$ so that we obtain

$$i(t) = i_0 e^{-R/L\,t}$$

Curve of $i(t)$:



This process is called de-energizing the inductor.

Energy Balance: Consider first the process of energizing the inductor. We have

$$L\frac{di}{dt} + iR = V$$

$$Li\frac{di}{dt} + i^2R = Vi$$

Which may be rewritten in the form

$$d\left(\frac{Li^2}{2}\right) + i^2 R\,dt = V i\,dt$$

This is interpreted as that during a time interval dt an amount of energy $Vi\,dt$ supplied by the battery is expended partly in Joule heating, $i^2 R\,dt$, and partly in increasing the stored energy in the magnetic field of the inductor, $d(\frac{1}{2}Li^2)$.

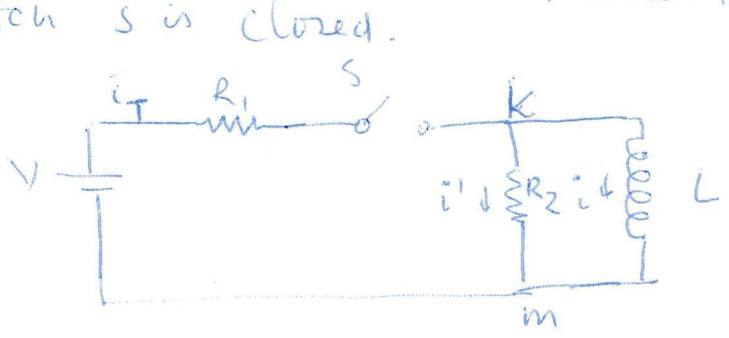
In the case of the de-energizing process one can write the loop eq. as

$$-d(\tfrac{1}{2}Li^2) = i^2 R\,dt$$

which states that the energy removed from the magnetic field of the inductor appears in the form of Joule heat in the resistor.

Examples.

1. Obtain an expression for the current $i(t)$ through the inductor L in the circuit shown in the diagram when the switch S is closed.



Kirchhoff's first rule for junction K gives us that the total battery current is $i_T = i' + i$. Applying the 2nd rule for the two loops, we find

$$(i + i')R_1 + L\frac{di}{dt} = V$$

$$(i + i')R_1 + i'R_2 = V$$

By eliminating i' between these two eqs there result,

$$i\,\frac{R_1 R_2}{R_1 + R_2} + L\frac{di}{dt} = \frac{R_2}{R_1 + R_2}V$$

or if we define,

$$R' = \frac{R_1 R_2}{R_1 + R_2}\,, \quad V' = \frac{R_2}{R_1 + R_2}V$$

we will find

$$i R' + L \frac{di}{dt} = \mathcal{E}'$$

Which can be solved just similar to the case of energizing the inductor in Sec. (9-2). or

$$i(t) = \frac{\mathcal{E}'}{R'}(1 - e^{-\frac{R'}{L}t}) = \frac{V}{R_1}(1 - e^{-\frac{R'}{L}t})$$

2. The switch S in the circuit of this section is thrown from b to a. After one inductive time constant a, the total energy transformed to Joule heat in the resistor is $0.168\, V^2 \tau_L / R$. And that b, the energy stored in the magnetic field is $0.200\, V^2 \tau_L / R$. c, Show that the equil. energy stored in the magnetic field is $0.500\, V^2 \tau_L / R$.

Solution

a) The current in this circuit is given as a function of time by

$$i(t) = \frac{V}{R}(1 - e^{-R/L t})$$

After one inductive time constant, i.e., $t = L/R$; we have

$$i = \frac{V}{R}(1 - e^{-1})$$

$$i^2 = \frac{V^2}{R^2}(1 - e^{-1})^2$$

Hence

$$i^2 R \cdot \tau_L \equiv U_R = \frac{V^2}{R} \tau_L (1 + e^{-2} - 2e^{-1})$$

or

$$U_R = 0.40 \frac{V^2}{R} \tau_L$$

b) $i(L/R) = V/R\,(1 - e^{-1})$

$$U_L = \tfrac{1}{2} L i^2 = \frac{L}{2} \frac{V^2}{R^2}(1 - e^{-1})^2 = 0.5 \frac{V^2}{R} \tau_L (1 + e^{-2} - 2e^{-1})$$

or

$$U_L = 0.20 \frac{V^2}{R} \tau_L$$

c) At equil. $t \to \infty$ and $i(t) = \frac{V}{R} \equiv$ constant.

a, The current in the circuit is given as a function of time by

$$i(t) = \frac{V}{R}\left(1 - e^{-\frac{R}{L}t}\right)$$

The rate of energy dissipation across the resistor is

$$\frac{dU_R}{dt} = i^2R = \frac{V^2}{R^2}\left(1 - e^{-\frac{R}{L}t}\right)^2 R = \frac{V^2}{R}\left(1 + e^{-\frac{2R}{L}t} - 2e^{-\frac{R}{L}t}\right)$$

$$dU_R = \frac{V^2}{R}(\quad)\,dt$$

or,

$$U_R(\tau_L) = \frac{V^2}{R}\int_0^{\frac{L}{R}} \left(1 + e^{-\frac{2R}{L}t} - 2e^{-\frac{R}{L}t}\right) dt$$

$$= \frac{V^2}{R}\left(\frac{L}{R} - \frac{L}{2R}e^{-2} + \frac{2L}{R}e^{-1} + \frac{L}{2R} - \frac{2L}{R}\right)$$

$$= \frac{V^2}{R}\frac{L}{R}\left(\frac{3}{2e} - \frac{1}{2e^2} - \frac{1}{2}\right) = 0.168\, V^2/R\, \tau_L.$$

b, $dU_L = Li\,di = Li\frac{di}{dt}dt$

$$i = V/R\left(1 - e^{-\frac{R}{L}t}\right),\quad di = \frac{V}{L}e^{-\frac{R}{L}t}dt$$

$$dU_L = L\frac{V^2}{RL}\left(e^{-\frac{R}{L}t} - e^{-\frac{2R}{L}t}\right)dt$$

$$U_L(\tau_L) = \frac{V^2}{R}\int_0^{L/R}\left(e^{-\frac{R}{L}t} - e^{-\frac{2R}{L}t}\right)dt$$

$$= \frac{V^2}{R}\left(-\frac{L}{Re} + \frac{L}{2Re^2} + \frac{L}{R} - \frac{L}{2R}\right) = \frac{V^2}{R}\tau_L\left(\frac{1}{2e^2} + \frac{1}{2} - \frac{1}{e}\right)$$

$$= 0.200\, V^2/R\, \tau_L$$

c, $dU_L = Li\,di = \frac{V^2}{R}\left(e^{-\frac{R}{L}t} - e^{-\frac{2R}{L}t}\right)dt$

$$U_L(\infty) = \frac{V^2}{R}\int_0^{\infty}\left(e^{-\frac{R}{L}t} - e^{-\frac{2R}{L}t}\right)dt$$

$$= \frac{V^2}{R}\left(-\frac{L}{R}e^{-\frac{R}{L}t} + \frac{L}{2R}e^{-\frac{2R}{L}t}\right)_0^{\infty}$$

$$= \frac{V^2}{R}\left(\frac{L}{R} - \frac{L}{2R}\right) = \frac{1}{2}\frac{V^2}{R}\frac{L}{R}$$

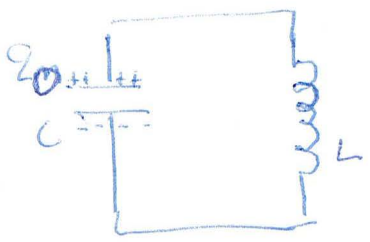
or, $U_L(\infty) = 0.5\,\frac{V^2}{R}\tau_L$.

Sol.

$$U_{L_1 L_2} = \tfrac{1}{2} L i^2 = \tfrac{1}{2} L \frac{N^2}{R^2} I \; 0.8 \; \frac{N^2}{R^2} I^2 L$$

10.3 The LC circuit

In the fig. below we have a circuit consisting of a capacitor of capacitance C and an inductor of self-inductance L.



The LC system here makes oscillations. Let us to see this. Let us assume that initially the capacitor carries a charge q_0 and the current i in the inductor is zero. At this instant the energy stored in the capacitor is given by

$$U_E = \frac{q_0^2}{2C}$$

whereas that in the inductor is

$$U_B = \tfrac{1}{2} L i^2 = 0.$$

The capacitor now starts to discharge through the inductor, +ve charge carriers moving clockwise. This means that a current $i \; (= dq/dt)$ is established in the inductor. As q decreases, the energy stored in the electric field also decreases. This energy is transferred to the magnetic field that appears around the inductor because of the current i that is building up there. Thus the electric field decreases, the magnetic field builds up, and energy is transferred from the former to the latter. In this way all the charge on the capacitor will have disappeared. The electric field in the capacitor will be zero, the energy stored having been transferred entirely to the magnetic field of the inductor. Then we will have a max. current in the inductor.

Next, the large current in the inductor continues

to transport positive charge from the upper plate of the capacitor to the lower plate; energy now flows from the inductor back to the capacitor as the electric field builds up again. Eventually, the energy will have been transferred completely back to the capacitor. But now the capacitor is charged in the opposite direction.

The capacitor will start to discharge again, the current now being counter clockwise. Reasoning as before, we see that the circuit finally returns to its initial situation and that the process continues at a definite frequency f to which corresponds a definite angular frequency $\omega\ (=2\pi f)$. Once started, such LC oscillations (in the ideal case described, in which the circuit contains no resistance) continues indefinitely, energy being shuttled back and forth between the electric field in the capacitor and the magnetic field in the inductor.

One can in deed see that the LC oscillation is analogous to the SHM of a mass m attached to a spring. For this consider the table below.

Mechanical		Electromagnetic	
Spring	$U_p = \frac{1}{2}kx^2$	capacitor	$U_E = \frac{1}{2}\frac{q^2}{C}$
mass	$U_k = \frac{1}{2}mv^2$	inductor	$U_B = \frac{1}{2}Li^2$
	$v = \frac{dx}{dt}$		$i = \frac{dq}{dt}$

The table suggests that a capacitor is in some formal way like a spring and an inductor is like a mass and that certain electromagnetic quantities "correspond" to certain mechanical ones, namely,

q	(corresponds	to	x
i	"	"	v
C	"	"	$1/k$
L	"	"	m

In the end let us obtain the functional dependence of the charge on time t.

The total energy U present at any instant in an oscillating LC circuit which either equals the max. electric p.E. stored within the capacitor or the max. magnetic p.E. stored within the inductor is given by

$$U = U_B + U_E = \frac{q^2}{2C} + \tfrac{1}{2} L i^2$$

which gives

$$\frac{dU}{dt} = \frac{q}{C}\frac{dq}{dt} + Li\frac{di}{dt} = 0$$

or

$$\frac{i}{C}\frac{dq}{dt} + L\frac{dq}{dt}\frac{d^2q}{dt^2} = 0$$

or

$$L\frac{d^2q}{dt^2} + \frac{q}{C} = 0$$

which is an eq. similar to that of the harmonic oscillator, i.e.,

$$m\frac{d^2x}{dt^2} + Kx = 0$$

So the former eq. has solution

$$q(t) = q_0 \cos(\omega t + \delta)$$

Now we have

$$i(t) = \frac{dq}{dt} = -\omega q_0 \sin(\omega t + \delta)$$

$$\frac{di}{dt} = \frac{d^2q(t)}{dt^2} = -\omega^2 q_0 \cos(\omega t + \delta)$$

Substituting the expressions for q and d^2q/dt^2 in the DE, we find

$$-L\omega^2 q_0 \cos(\omega t + \delta) + \frac{1}{C} q_0 \cos(\omega t + \delta) = 0$$

which yields

$$\omega = \frac{1}{\sqrt{LC}}$$

So if ω is given this value. Then the solution of the DE describing the time development of the charge q in the circuit is in deed

$$q(t) = q_0 \cos(\omega t + \delta).$$

δ is determined by the initial condition that prevail at time $t=0$. If, according to our previous discussion, we assume $q_m = q_0$ at $t=0$, we require $\delta = 0$.

Example. A $1.0\,\mu f$ capacitor is charged to 50 volts. The charging battery is then disconnected and a 10mH coil is connected across the capacitor, so that LC oscillations occur. What is the max. ~~Stored energy in the capacitor~~ current in the coil. Assume that the circuit contains no resistance.

The principle of conservation of energy imply that the max. ~~current~~ stored energy in the capacitor must equal the max. stored energy in the inductor. or

$$\frac{1}{2}\frac{q_0^2}{C} = \frac{1}{2} L i_0^2$$

where q_0 is the max. charge and i_0 is the max. current. Note that the max. current and the max. charge do not occur at the same time but one fourth a cycle apart. So.

$$i_0 = q_0\sqrt{\frac{1}{LC}} = C V_0 \sqrt{\frac{1}{LC}} = V_0\sqrt{\frac{C}{L}}$$

or

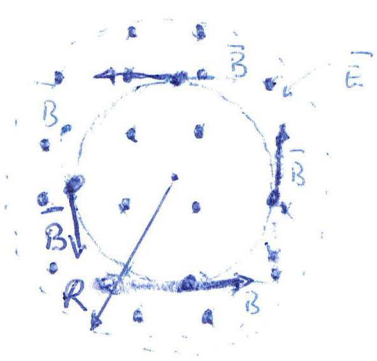
$$i_0 = 50\,V \sqrt{\frac{10^{-6}\,F}{10^{-2}\,H}} = 0.5\,A.$$

Chapter 11 Maxwell's Equations and Electromagnetic Waves

11.1 Induced Magnetic Fields

In chapter 9 we discussed Faraday's law of induction which states a changing magnetic field produces an electric field. Now let us introduce a new concept, namely, a changing electric field produces a magnetic field. This concept, which is the symmetrical counter part of Faraday's law of induction, is of fundamental significance.

To see this, consider a uniform electric field $\vec{E}$ filling a cylindrical region of space, as shown in the figure below.



(a) (b)

The electric field in (a) might be produced by a circular parallel plate capacitor, as suggested in fig (b). We assume E is increasing at a steady rate dE/dt, which means that the charge must be supplied to the capacitor plates at a steady rate; to supply this charge requires a steady current i into the positive plate and an equal steady current out of the negative plate. In this process, experiments show that a magnetic field is set up by this changing electric field. Fig (a) above shows B for four arbitrary points and, further, it indicates an example of the symmetry of nature: a changing magnetic field includes an electric field (Faraday's law) and a changing electric field, on its turn, includes a magnetic field.

To write a quantitative expression, we think of analogy with Faraday's law of induction.

$$\oint_c \vec{E}\cdot d\vec{l} = -\frac{d\Phi_B}{dt}$$

which asserts that an electric field is produced by a changing magnetic field. For the symmetrical counterpart we write

$$\oint_c \vec{B}\cdot d\vec{l} = \mu_0\varepsilon_0\frac{d\Phi_E}{dt}.$$

This eqn asserts that a magnetic field can be produced by a changing electric field. The difference in sign is due to the fact that the induced $\vec{B}$ is opposite to the direction of the induced electric field $\vec{E}$ for a similar case of increasing $\vec{B}$

If there exists a conduction (or free) current i in addition with a changing electric field, we write

$$\oint \vec{B}\cdot d\vec{l} = \mu_0\varepsilon_0\frac{d\Phi_E}{dt} + \mu_0 i$$

Examples 1. A parallel plate capacitor with circular plates is being charged as in the figure of this section. a) Derive an expression for the induced magnetic field at various radii r. Consider both $r<R$ and $r>R$.

Solution

For a changing electric field we have

$$\oint_c \vec{B}\cdot d\vec{l} = \mu_0\varepsilon_0\frac{d}{dt}\int_S \vec{E}\cdot d\vec{S}$$

For $r \leq R$,

$$B\cdot 2\pi r = \mu_0\varepsilon_0\frac{d}{dt}[E\,\pi r^2] = \mu_0\varepsilon_0\pi r^2\frac{dE}{dt}$$

Solving for B we find

$$B = \frac{1}{2}\mu_0\varepsilon_0 r\frac{dE}{dt} \quad (r \leq R)$$

For $r \geq R$,

$$B\cdot 2\pi r = \mu_0\varepsilon_0\frac{d}{dt}[E\cdot\pi R^2] = \mu_0\varepsilon_0\pi R^2\frac{dE}{dt}$$

Or,

$$B = \frac{\mu_0\varepsilon_0 R^2}{2r}\frac{dE}{dt}.$$

2. The last eq. in this sec. shows that the term $\epsilon_0 \frac{d\Phi_E}{dt}$ has the dimensions of a current; We call it the displacement current (designated, i_d). In example L how does the displacement current through a concentric circular loop of radius r vary with r? consider both $r<R$ and $r>R$.

Solution

We have

$$\oint_c \vec{B}\cdot d\vec{l} = \mu_0 i_d$$

(or)

$$i_d = \frac{1}{\mu_0}\oint_c \vec{B}\cdot d\vec{l}$$

From example L.

i. for $r<R$, $\oint_c \vec{B}\cdot d\vec{l} = \mu_0\epsilon_0\pi r^2 \frac{dE}{dt}$

$$\Rightarrow i_d = \frac{1}{\mu_0}\mu_0\epsilon_0\pi r^2\frac{dE}{dt} = \epsilon_0\pi r^2\frac{dE}{dt}$$

ii. for $r>R$, $\oint \vec{B}\cdot d\vec{l} = \mu_0\epsilon_0\pi R^2 \frac{dE}{dt}$

$$\Rightarrow i_d = \epsilon_0\pi R^2 \frac{dE}{dt}$$

CHAPTER 11.2 The Basic Equations of Electromagnetism (Maxwell's Equations.

At this point let us summarize our discussion of the electromagnetic field. The entire theory of the electromagnetic field ~~is~~ boils down to the discussion of four laws. They are called Maxwells equations since it was Maxwell who, in addition to formulating the fourth law, recognized the theory, together with the equation

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}),$$

for the electromagnetic force acting on a charge q, constitute the basic framework of the theory of electromagnetic interactions. The electric charge q and the current I are called the sources of the electromagnetic field since given q and i, Maxwell's equations (table below) allow the calculation of $\vec{E}$ and $\vec{B}$.

Maxwells Equations for the Electromagnetic Field.	Law	Form
1.	Gauss's law for the electric field	$\oint_S \vec{E} \cdot d\vec{S} = q/\epsilon_0$
2.	Gauss's Law for the magnetic field	$\oint_S \vec{B} \cdot d\vec{S} = 0$
3.	Faradays law of Induction	$\oint_C \vec{E} \cdot d\vec{\ell} = -\frac{d}{dt} \int_S \vec{B} \cdot d\vec{S}$
4.	Ampere-Maxwell Law	$\oint_C \vec{B} \cdot d\vec{\ell} = \mu_0 i + \mu_0 \epsilon_0 \frac{d}{dt} \int_S \vec{E}\, d\vec{S}$ $= \mu_0 \int_S \left(\vec{J} + \epsilon_0 \frac{d\vec{E}}{dt} \right) \cdot d\vec{S}$

11.3 Electromagnetic plane Waves

So far our discussion has been limited to fields that

are produced by static charges or steady currents. But here we consider situations in which a system produces time-varying electric and magnetic fields that propagate through space. This combined field constitutes electromagnetic radiation and carries energy away from the source. We now examine the simple case of a monochromatic (single frequency) plane electromagnetic wave that travels through ~~empty~~ medium space in which there are no charges or currents.

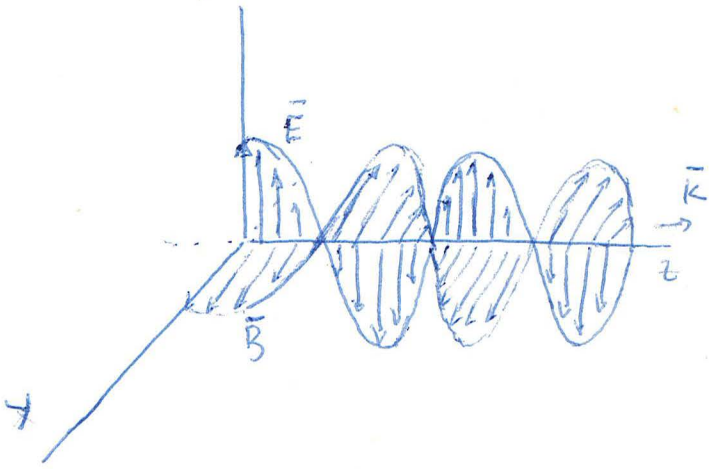
Assume that we have a plane electromagnetic wave (whose wave fronts are planes at a considerable distance from its source) that propagates along the z-axis. The lines of $\bar{E}$ will be parallel to the x-axis whereas those of $\bar{B}$ will be parallel to the y-axis. The values of $\bar{E}$ and $\bar{B}$ for this wave depend, thus, only on z and t (not on x or y). We postulate that they are given by

$$\bar{E} = E_0 \cos(kz - \omega t)\,\bar{i}$$

and

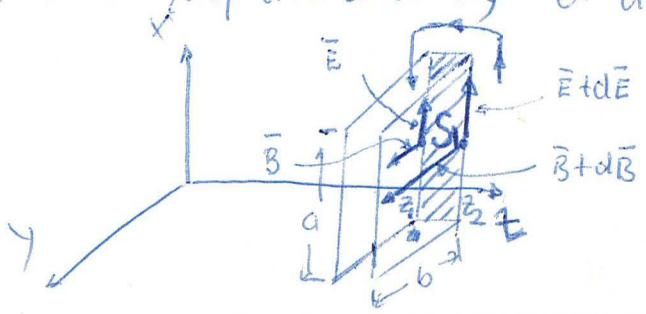
$$\bar{B} = B_0 \cos(kz - \omega t)\,\bar{j}$$

Pictorially we represent the propagation of such an electromagnetic wave as in the figure below.



That means the $\bar{E}$-field oscillates ~~along~~ parallel to the x-axis and the $\bar{B}$-field oscillates ~~along~~ parallel to the y-axis with the wave as a whole moving in the z-direction.

Next, consider two fixed rectangular plane surface with the same sides a and b, parallel to the x-y plane, and separated by a dist. dz as shown below.



$dz = z_2 - z_1$

As the wave passes across the shaded surface S_1, the magnetic ϕ_B over it will change and this flux, according to Faraday's law, will give rise to induced electric fields around this surface. These induced electric fields are, in fact, simply the electric comp. of the traveling electromagnetic wave.

Let us traverse the surface S_1 counter clockwise so as to apply Faraday's law of induction:

$$\oint_{C_1} \vec{E}' \cdot d\vec{\ell} = -\frac{d}{dt} \int_{S_1} \vec{B} \cdot d\vec{S}, \quad C_1 \text{ encloses } S_1.$$

Then,

$$\oint_{C_1} \vec{E} \cdot d\vec{\ell} = \int_{z_1}^{z_2} \vec{E}' \cdot d\vec{\ell} + \int_{x_1}^{x_2} \vec{E} + d\vec{E} \cdot d\vec{\ell} + \int_{z_2}^{z_1} \vec{E}' \cdot d\vec{\ell} + \int_{x_2}^{x_1} \vec{E} \cdot d\vec{\ell}$$

$\vec{E}' \perp d\vec{\ell}$ $\qquad$ $\vec{E}' \perp d\vec{\ell}$

$$\Rightarrow \oint_{C_1} \vec{E} \cdot d\vec{\ell} = (E + dE)a - E \cdot a = dE \cdot a$$

The flux ϕ_B for the rectangle S_1 is

$$\phi_B = B \cdot (dz \cdot a)$$

where we have assumed that for a very small thickness dz the variation of B over S_1 is negligible. Differentiating we obtain

$$\frac{d\phi_B}{dt} = a\, dz\, \frac{dB}{dt}$$

i.e.,

$$dE \cdot a = -a\, dz\, \frac{dB}{dt}$$

or

$$\frac{dE}{dz} = -\frac{dB}{dt}$$

In evaluating dE/dz, it is assumed that t is const. Also, in evaluating dB/dt it is assumed that z is const. Since what we need is the time rate of change of B at a particular position on the surface S_1. We then wrote

$$\boxed{\frac{\partial E}{\partial z} = -\partial B/\partial t}$$

The minus sign in this eq. is appropriate and necessary,

for, although E is increasing with ~~x~~ at the site of the shaded rectangle S_1 in the above fig. B is decreasing with t. Substituting into this eq. the expressions for $\bar{E}$ and B we find

$$-kE_0 \sin(kz-\omega t) = -\omega B_0 \sin(kz-\omega t) \quad \cdots (*)$$

or

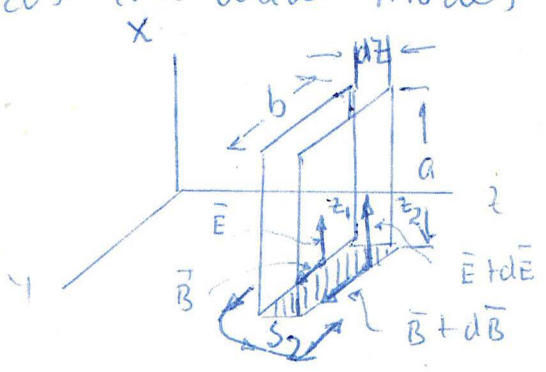
$$\frac{\omega}{k} = \frac{E_0}{B_0}$$

originally we wrote the same quantity, $kz-\omega t$, for the $\bar{E}$ and $\bar{B}$ assuming that they are in phase. Suppose that they differ by a phase difference δ is introduced by writing $\bar{B} = B_0 \cos(kz-\omega t+\delta)\hat{j}$. Then, we obtain an eq. corresponding to eq. (*) that contains $\sin(kz-\omega t+\delta)$ on the right-hand side and $\sin(kz-\omega t)$ on the left-hand side. But we require this eq. to be valid for any time t. For this to be true, δ must be zero and the electric and magnetic field strengths are in deed in phase.

Now since ω/k represents the phase velocity v of the plane electromagnetic wave we write

$$v = E_0/B_0$$

we now turn our attention to the fig. below in which the flux Φ_E for the shaded rectangle S_2 is decreasing with time as the wave moves through it.



$$z_2 - z_1 = dz$$

According to Maxwells 3rd eq. (with $i=0$, because there are no conduction currents in a traveling electromagnetic wave),

$$\oint_{C_2} \vec{B}\cdot d\vec{s} = \mu_0 \varepsilon_0 \frac{d}{dt}\int_{S_2} \vec{E}\cdot d\vec{s}$$

this changing flux will induce a magnetic field at points around the periphery of the rectangle S_2. This induced magnetic field is simply the magnetic comp. of the

electromagnetic wave.

Here again let us traverse the surface S_2 counter-clockwise.

$$\oint_{C_2} \vec{B}\cdot d\vec{l} = \int_{y_1}^{y_2} \vec{B}\cdot d\vec{l} + \int_{z_1}^{z_2} (\vec{B}+d\vec{B})\cdot d\vec{l} + \int_{y_2}^{y_1} \vec{B}\cdot d\vec{l} + \int_{z_2}^{z_1} \vec{B}\cdot d\vec{l}$$

$$\Rightarrow \oint_{C} \vec{B}\cdot d\vec{l} = -(\vec{B}+d\vec{B})\,b + B\,b = -dB\cdot b$$

And the electric field flux ϕ_E over S_2 is

$$\phi_E = E\,dz\cdot b$$

Assuming $\vec{E}$ varies slowly over S_2. Differentiating we obtain

$$\frac{d\phi_E}{dt} = b\,dz\,\frac{dE}{dt}$$

i.e.,

$$-dB\,b = \mu_0\epsilon_0\, b\,dz\, dE/dt$$

or

$$-\frac{dB}{dz} = +\mu_0\epsilon_0\frac{dE}{dt}$$

As before we write $\boxed{-\frac{\partial B}{\partial z} = \mu_0\epsilon_0\,\partial E/\partial t}$ $\equiv$ $\boxed{\frac{\partial B}{\partial z} = -\mu_0\epsilon_0\frac{\partial E}{\partial t}}$

If we use the expressions of E and B in this last eq we will find

$$+ k B_0 \sin(kz-\omega t) = +\epsilon_0\mu_0\omega E_0 \sin(kz-\omega t)$$

i.e.,

$$\frac{E_0}{B_0} = \frac{k}{\epsilon_0\mu_0\omega}$$

But

$$\frac{E_0}{B_0} = \frac{\omega}{k} = v$$

so that

$$v = \frac{1}{\epsilon_0\mu_0 v}$$

and

$$v = \frac{1}{\sqrt{\epsilon_0\mu_0}} = \frac{1}{\sqrt{(8.85419\times10^{-12}\ F/m)(4\pi\times10^{-7}\ T\cdot m/A)}}$$

$$= 2.99792\times10^{8}\ m/s$$

which is the speed of light in free space! Thus the propagation velocity of an electromagnetic wave in vacuum

is just c, the velocity of visible light in vacuum. No restriction has been placed on the value of ω, so the result applies to all forms of electromagnetic radiation. Experiments have verified that, in vacuum, radio waves, microwaves, infrared radiation, visible light, and X-rays all travel with the velocity c.

$$c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$$

Example:

1. A monochromatic plane electromagnetic wave in vacuum has a propagation vector $\vec{K} = K\hat{k}$. At time t, we have $\vec{E}(\vec{r},t) = (200\ V/m)\cos(Kz-\omega t)\hat{i}$. Determine the magnetic vector $\vec{B}(\vec{r},t)$.

Solution

Since $\vec{B}$ has to be $\perp$ to both $\vec{E}$ and $\vec{k}$ it will be directed along the y-axis and from Faraday's law of induction we have

$$\frac{\partial \vec{E}}{\partial z} = -\frac{\partial B}{\partial t}$$

(i.e.)

$$\frac{\partial B}{\partial t} = -\frac{\partial \vec{E}}{\partial z} = E_0 K \sin(Kz-\omega t), \text{ where } E_0 = 200\frac{V}{m}.$$

$$\Rightarrow \quad \vec{B}(\vec{r},t) = \frac{E_0 K}{\omega}\cos(Kz-\omega t) + C(z)$$

Since $\vec{E}$ and $\vec{B}$ have to be in phase at any instant of time t, we take $C(z)=0$, i.e.)

$$\vec{B}(\vec{r},t) = \frac{E_0}{c}\cos(Kz-\omega t)\hat{j}, \quad c = \frac{\omega}{K}.$$

But

$$\frac{E_0}{c} = \frac{200\ V/m}{3\times 10^8\ m/s} = 6.67\times 10^{-7}\ \frac{Wb}{m^2}$$

or

$$\vec{B}(\vec{r},t) = (6.67\times 10^{-7}\ \frac{Wb}{m^2})\cos(Kz-\omega t)\hat{j}$$

2. The magnetic field vector for a monochromatic plane electromagnetic wave in vacuum has only an x-comp., $B_x = B_0 \cos(ky - \omega t)$, with $B_0 = 1.6 \times 10^{-6}$ T. What is the direction of the propagation vector $\bar{k}$? Determine the electric field vector $\bar{E}(\bar{r}, t)$.

solution

Since the field vector $\bar{B}$ depends on the space variable y, the propagation vector $\bar{k}$ is directed along the y-axis. From Amperes law (that generalized by Maxwell) we have

$$-\frac{\partial B}{\partial y} = \mu_0 \epsilon_0 \frac{\partial E}{\partial t}$$

(i.e.)

$$\frac{\partial E}{\partial t} = -\frac{1}{\mu_0 \epsilon_0} \frac{\partial B}{\partial y} = \frac{k B_0}{\mu_0 \epsilon_0} \sin(ky - \omega t)$$

$$\Rightarrow \quad E(\bar{r}, t) = \frac{k B_0}{\omega \mu_0 \epsilon_0} \cos(ky - \omega t) = c B_0 \cos(ky - \omega t)$$

The const. of integration $c(y)$ being ignored due to the fact that $\bar{E}$ and $\bar{B}$ have to be in phase.

But

$$c B_0 = 3 \times 10^8 \frac{m}{s} \times 1.6 \times 10^{-6} \frac{V \cdot s}{m^2} = 4.8 \times 10^2 \, V/m$$

or since $\bar{E} \perp \bar{B}, \bar{k}$ we write

$$\bar{E}(\bar{r}, t) = (480 \, V/m) \cos(ky - \omega t) \hat{k}$$

11.4 The Poynting Vector

One of the imp. characteristics of an electromagnetic wave is that it can transport energy from point to point. The rate of energy flow per unit area in a plane electromagnetic wave can be described by a vector $\bar{S}$, called the poynting vector. We define $\bar{S}$ from

$$\bar{S} = \frac{1}{\mu_0} \bar{E} \times \bar{B}$$

Unit

$$[S] = \left[\frac{1}{\mu_0} E B\right] = \frac{A \cdot A}{V \cdot s} \cdot \frac{V}{m} \cdot \frac{V \cdot s}{m^2} = \frac{V \cdot A}{m^2} = \frac{(J/C)(C/s)}{m^2}$$

or

$$[\bar{S}] = W/m^2$$

The direction of $\bar{S}$ gives the direction in which the energy moves, whereas its magnitude gives the rate of energy flow across a unit area. The vectors $\bar{E}$ and $\bar{B}$ refer to their instantaneous values at the point in question. If the above expression for $\bar{S}$ is applied to a traveling plane electromagnetic wave, $\bar{E} \times \bar{B}$, and hence $\bar{S}$, point in the direction of propagation.

Example:

An observer is a great dist. r from a point light source whose power output is P_0. Calculate the magnitude of the electric and magnetic fields. Assume that the source is monochromatic, that it radiates uniformly in all directions, and that at distant points it behaves like a traveling plane wave.

Solution

The power that passes through a sphere of radius r is $\bar{S}\, 4\pi r^2$, where $\bar{S}$ is the average value of the poyinting vector at the surface of the sphere. This power must equal P_0, or

$$P_0 = \bar{S}\, 4\pi r^2$$

From the def of $\bar{S}$, we have

$$\bar{S} = \left(\frac{1}{\mu_0} EB\right), \quad \bar{E} \perp \bar{B} \text{ for a plane wave}$$

But $E = cB$ and thus

$$\bar{S} = \frac{1}{\mu_0 c} \bar{E}^2$$

Assume E is given by

$$E = E_0 \cos(kz - \omega t)$$

$$E^2 = E_0^2 \cos^2(kz - \omega t)$$

and

$$\bar{E}^2 = \frac{E_0^2}{T} \int_0^T \cos^2(kz - \omega t)\, dt = \frac{E_0^2}{2T} \int_0^T [1 + \cos 2(kz - \omega t)]\, dt$$

$$\frac{E_0^2}{2T} \left(t + \frac{1}{2\omega} \sin 2(kz - \omega t) \right)_0^T$$

$$\bar{E}^2 = \frac{E_0^2}{2T}\left\{T - \frac{1}{2\omega}\sin 2(Kz - 2\pi) + \frac{1}{2\omega}\sin 2Kz\right\} - \frac{E_0^2}{2}$$

which gives

$$P_0 = \frac{E_0^2}{2\mu_0 c}\cdot 4\pi r^2$$

or

$$E_0 = \frac{1}{r}\sqrt{\frac{P_0\mu_0 c}{2\pi}} \quad \text{and} \quad B_0 = \frac{E_0}{c} = \frac{1}{r}\sqrt{\frac{P_0\mu_0}{2\pi c}}$$

11.5 Electromagnetic momentum and Radiation pressure

Since an electromagnetic wave transports energy it must carry mom. as well and there is a simple relation between the energy density u and the poynting vector $\bar{S}$ and the mom. density vector $\bar{P}$ as we may seen the following discussions.

Consider a plane electromagnetic wave with field vectors $\bar{E}$ and $\bar{B}$ (vacuum propagation is required).

$$u_E = \frac{1}{2}\epsilon_0 E^2, \quad u_B = \frac{B^2}{2\mu_0} = \frac{E^2}{2\mu_0 c^2} = \frac{1}{2}\epsilon_0 E^2 = u_E$$

So the total energy density is

$$u = u_E + u_B = \epsilon_0 \bar{E}^2$$

For the poynting vector $\bar{S}$ the magnitude is given by

$$|\vec{S}| = \frac{1}{\mu_0}|\vec{E}\times\vec{B}| = \frac{1}{\mu_0}EB = \frac{E^2}{\mu_0 c} = c\,\epsilon_0 E^2$$

That is

$$|\vec{S}| = cu.$$

But according to the Special theory of relativity energy U and mom. P are related as

$$U = Pc$$

and hence the energy density u is related to the mom. density $\bar{P}$. or

$$u = \bar{P}c$$

whence

$$|\vec{S}| = c\,Pc = c^2 P$$

or

$$p = \frac{1}{c^2}|\vec{S}|$$

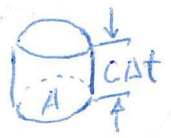
Since both the mom. and the energy flow rate per unit area $\vec{S}$ are along the direction of propagation we write

$$\vec{p} = \frac{\vec{S}}{c^2} = \frac{1}{\mu_0 c^2}\vec{E}\times\vec{B}$$

Since $\frac{1}{c^2} = \epsilon_0 \mu_0$, this relation becomes

$$\vec{p} = \epsilon_0 \vec{E}\times\vec{B}$$

If a wave traveling in vacuum is incident on the surface of a substance that totally absorbs the wave, a pressure will be exerted on the surface. As indicated in the fig. below, the total wave mom. contained



within a volume $Ac\Delta t$ will be absorbed by the surface (which is $\perp$ to the direction of $\vec{p}$) during a time interval Δt. Thus, a force F is exerted by the wave on an area A of the surface, where

$$\text{Impulse} = (\text{Mom. density})\times(\text{volume})$$

$$F\Delta t = \frac{1}{c} u\, Ac\Delta t$$

The force per unit area is the radiation pressure P_{rad}:

$$P_{rad} = \frac{F}{A} = u$$

As usual pressure is measured in N/m^2 or Pa.

Example: A pulsed source of microwaves produces bursts of 20 GHz radiation, with each burst having a duration of 1 ns. A parabolic reflector ($R = 6$ cm) is used to produce a parallel beam of radiation, as indicated in the diagram. The average output power of the source during each pulse is 25 KW.

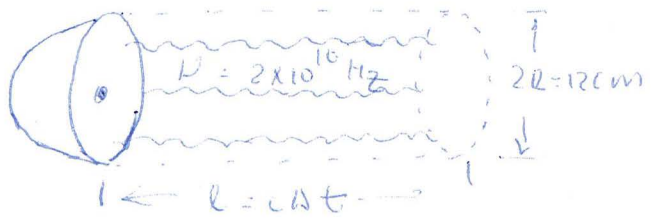
a) What is the wavelength of the radiation?

b, What is the total energy U in each pulse?

c, Determine the average energy density $\bar{u}$ and the average magnitude of the poynting vector, $\bar{S}$, during each pulse.

d, Determine the average mom. density $\bar{p}$ and the average radiation pressure P_{rad} on a totally absorptive surface that is $\perp$ to the direction of propagation of the pulse.

e, Calculate the values of E_0 and B_0 in the wave.



a) $\lambda = c/\nu = \frac{3\times10^8 m/s}{20\times10^9 Hz} = 0.015m$ Solution

b, The total energy in the pulse is $U = \bar{P}\Delta t = 2.5\times10^4 J/s \times 10^{-9} s = 2.5\times10^{-5} J$

c, The length of the pulse is $l = c\Delta t = 3\times10^8 m/s \times 10^{-9} s = 0.30m$

so that the average energy density is

$$\bar{u} = \frac{U}{\pi R^2 l} = \frac{2.5\times10^{-5} J}{\pi (0.06m)^2 (0.30m)} = 7.37\times10^{-3} J/m^3$$

Then we have

$$\bar{S} = \bar{u}c = 7.33\times10^{-3} J/m^3 \times 3.00\times10^8 m/s = 2.22\times10^6 W/m^2$$

and the average radiation pressure is

$$\bar{P}_{rad} = \bar{u} = 7.37\times10^{-3} Pa.$$

d, The average mom. density is

$$\bar{p} = \frac{1}{c^2}\bar{S} = \frac{1}{c}\bar{u} = 7.37\times10^{-3} J/m^3 / 3\times10^8 m/s = 2.46\times10^{-11} \frac{J\,s}{m^4}$$

and the average radiation pressure is

$$\bar{P}_{rad} = \bar{u} = 7.37\times10^{-3} Pa.$$

e, We have

$$u_{E,0} = \tfrac{1}{2}\epsilon_0 E_0^2 = \frac{1}{2}\bar{u} \Rightarrow E_0 = \sqrt{2\bar{u}/\epsilon_0} = \sqrt{\frac{2(7.37\times10^{-3} J/m^3)}{8.85\times10^{-12} C/V\cdot m}}$$

$$= 4.08\times10^4 V/m.$$

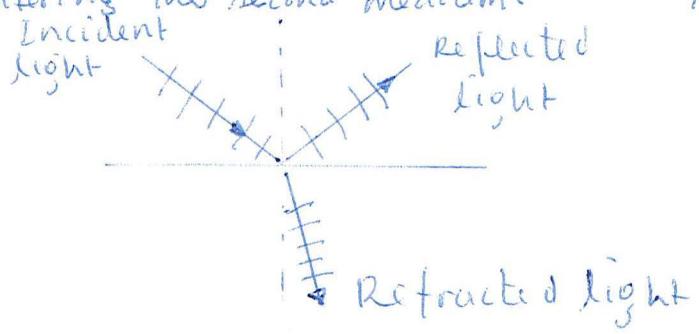
and

$$E_0/B_0 = c \Rightarrow B_0 = \frac{E_0}{c} = \frac{4.08\times10^4 V/m}{3\times10^8 m/s} = 1.33\times10^{-4} Wb/m^2.$$

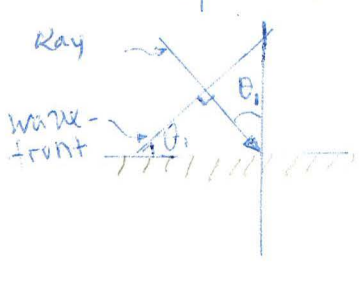
Chapter 12 Reflection and Refraction

12.1 The Laws of Reflection and Refraction

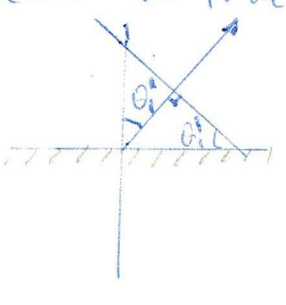
The figure below shows a plane light wave incident on the interface of two diff. media such as air and glass. In this process part of the light is reflected back into the first medium and the remaining part is transmitted into the second medium. The transmitted light is called the refracted light since it bends upon entering the second medium.



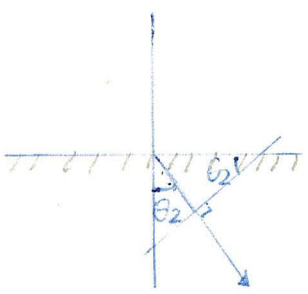
The angle of incidence θ may be defined either as the angle that an incident wave front makes with the interface or the angle that an incident ray makes with the normal to the interface. A similar diff. holds for the angles of reflection and refraction.



θ_1 is the angle of incidence



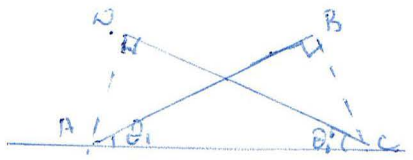
θ_1' is the angle of reflection



θ_2 is the angle of refraction

Reflection:

To obtain the relation between θ_1 and θ_1' consider the figure below.



$\overline{AB}$ is incident wave front and $\overline{CD}$ is reflected wave-front.

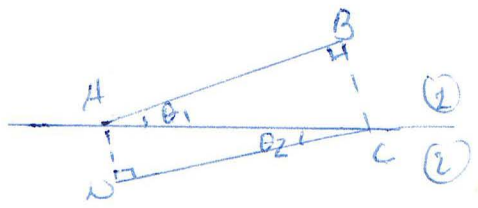
In time Δt the point B on the incident wave front $\overline{AB}$ moves to point C where as point A goes to point D in the same time interval Δt. Thus, $\overline{DC}$ corresponds to the reflected wave front. That is $\overline{BC}$ and $\overline{AD}$ are incident and reflected ray segments having the same length $v\Delta t$, where v is the wave velocity in the medium above the interface.

The angles $\angle ADC$ and $\angle ABC$ are both right angles. Therefore, the triangles ΔABC and ΔADC, which have a common side $\bar{AC}$, are congruent. Thus we conclude that the angle of incidence θ_1 is equal to the angle of reflection θ_1'; i.e.,

$$\theta_1' = \theta_1$$

Refraction:

Next we find the relation between θ_1 and θ_2 by considering the following figure.



$\bar{AB}$ is an incident wavefront and $\bar{CD}$ is a refracted wavefront.

In the same time interval, say Δt, point A and point B (where the incident wavefront $\bar{AB}$ have moved respectively to point D and point C to give us the new (refracted) wavefront, $\bar{DC}$. $\bar{BC}$ and $\bar{AD}$ represent incident and refracted rays, respectively. If the velocity of light in the first medium is v_1 and the velocity in the second medium is v_2, we have

$$BC = v_1 \Delta t, \quad \bar{AD} = v_2 \Delta t$$

so that

$$\sin\theta_1 = \frac{\bar{BC}}{\bar{AC}} = \frac{v_1 \Delta t}{\bar{AC}} \quad \text{and} \quad \sin\theta_2 = \frac{\bar{AD}}{\bar{AC}} = \frac{v_2 \Delta t}{\bar{AC}}$$

which give

$$\sin\theta_1 / \sin\theta_2 = \frac{v_1}{v_2} \quad \cdots (*)$$

Define,

$$n = \frac{c}{v}.$$

ratio

This is called the index of refraction of the medium. Then with this def. we obtain

$$\sin\theta_1 / \sin\theta_2 = n_2 / n_1 \equiv n_{21}$$ relative index of refraction of medium (2) wrt medium (1).

or

$$n_1 \sin\theta_1 = n_2 \sin\theta_2 \quad \cdots (**)$$

Which is the law of refraction. But the frequency, being the no. of waves moving past a given point per unit

time, is the same in both media and we write

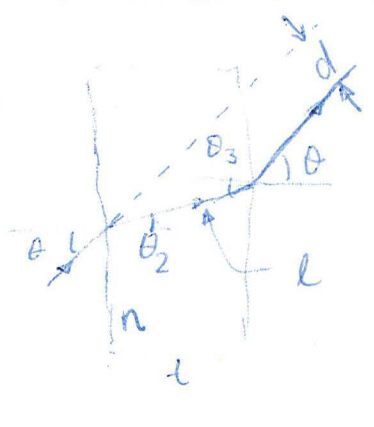
$$\nu = \frac{v_1}{\lambda_1} = v_2/\lambda_2 \quad \text{or} \quad v_1 = \nu\lambda_1 \text{ and } v_2 = \nu\lambda_2$$

whence,

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2} = \frac{\lambda_1}{\lambda_2} = \frac{n_2}{n_1}$$

This is once again the law of refraction. This law is called Snell's law.

Example: A narrow beam of light passes through a plate of glass with thickness t and refractive index n. The beam enters from air at an angle θ. Derive an expression for the deviation d of the ray in terms of t, θ and n. Reduce the expression for the limiting value of small θ.



$$\sin\theta = n\sin\theta_2 = n\sin\theta_3$$

$$\frac{d}{t} = \sin(\theta - \theta_2), \quad \frac{t}{l} = \cos\theta_2 \Rightarrow \quad \frac{d}{t} = \frac{\sin(\theta - \theta_2)}{\cos\theta_2}$$

or

$$d = t\,\frac{\sin(\theta - \theta_2)}{\cos\theta_2} = t\sin\theta - t\cos\theta\tan\theta_2$$

using Snell's law and simplifying we find

$$d = t\sin\theta\left(1 - \frac{\cos\theta}{\sqrt{n^2 - \sin^2\theta}}\right)$$

If θ is small, $\cos\theta \approx 1$, $\sin\theta \sim \theta$ and neglecting $\sin^2\theta = \theta^2$ in the radical we obtain

$$d = t\theta\left(1 - \frac{1}{n}\right)$$

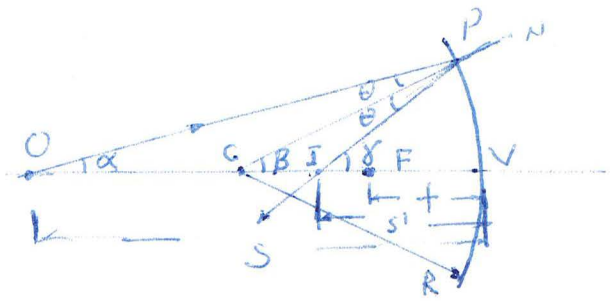
or

$$d = t\theta\,\frac{n-1}{n}$$

12.2 Spherical Mirrors

Concave Spherical Mirrors:

The figure below shows a concave spherical mirror with radius of curvature R. The point C represents the center of curvature and whereas F is a point midway between C and the vertex V of the mirror. A point O there is a point source which is a dist. s away from V. The ray OP in the mirror at an angle θ with the normal N is reflected at P making the same angle θ with this normal. All other rays from O for which the angle α is small ($\alpha \ll 1$) also intersect the axis of the mirror at I (or very close to I). Thus, I is the image point of O. It is a dist. s' from the vertex.



Angles β and γ are exterior angles of triangles ΔOPC and ΔCPI, respectively. Thus, $\beta = \alpha + \theta$ and $\gamma = \beta + \theta$; hence,

$$2\beta = \alpha + \gamma \quad \cdots (*)$$

Let the length of arc $\overset{\frown}{PV}$ be l; then, because $\overline{CV} = R$, we have $\beta = l/R$. If in addition $\alpha \ll 1$ and $\gamma \ll 1$, we have $\alpha \approx l/s$ and $\gamma = l/s'$. With these expressions used in (*) we find:

$$\frac{1}{s} + \frac{1}{s'} = \frac{2}{R}$$

The point at which an optical system causes incident parallel rays to converge is called the focal point of the system. When the source O in the above figure is moved far away from the mirror ($s \to \infty$), the incident rays become parallel and the image dist. s' approaches the limiting value R/2. Thus, the focal point F of the spherical mirror is located a dist. R/2 from the vertex, V. So the last eq. above may now be expressed

as

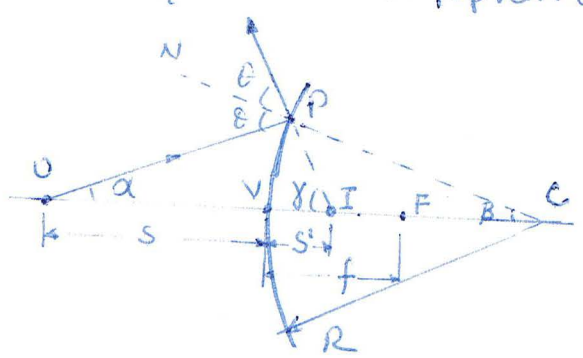
$$\frac{1}{s} + \frac{1}{s'} = \frac{1}{f} \quad \text{with} \quad f = \frac{R}{2}$$

one should note that the object and image points can be interchanged (i.e.) $s \to s'$ and $s' \to s$ without affecting the above eq. This is a property of any focusing system and is referred to as the principle of reversibility.

Sign convention: If the image is formed on the other side of the mirror opposite to the object, this image is called virtual as opposed to real image that is formed on the same side of the mirror as the object. This is because no actual rights intersect at the image (for virtual images) s' is negative.

- Convex Spherical Mirrors:

Consider the convex spherical mirror with radius of curvature R as shown in the figure below. The ray OP is reflected from the mirror surface at P with equal angles of incidence and reflection θ. Regardless of the position of the object O, the reflected rays appear to diverge from an image located behind the mirror. Thus, for any s, we have $s' < 0$, and the image formed by a convex spherical mirror is always virtual.



In this figure θ is an external angle of triangle ΔOPC and 2θ is an external angle of triangle ΔIOP. Thus $\theta = \alpha + \beta$ and

$$2\beta = \gamma - \alpha \quad \text{or} \quad -2\beta = \alpha - \gamma \qquad (**)$$

Let us now extend the sign convention we have introduced. The points C, and F and I are all located behind the mirror, so the distances to these points as measured from the vertex V are taken to be negative. Thus, $f = \overline{VF} < 0$, and $s' = \overline{VI} < 0$, and $r = \overline{VC} = -R$. (For a concave mirror, $r = +R$). Further, the angles β and γ are both reckoned to be negative. As before, with $\overline{OV} = l$, we have $\beta = -l/r$; then, with $\alpha \ll 1$ and $\gamma \ll 1$, we have $\alpha \approx l/s$, $\gamma \approx -l/s'$. With these expressions in (**) we find

$$\frac{1}{s} + \frac{1}{s'} = \frac{2}{r} = -\frac{2}{R}$$

or

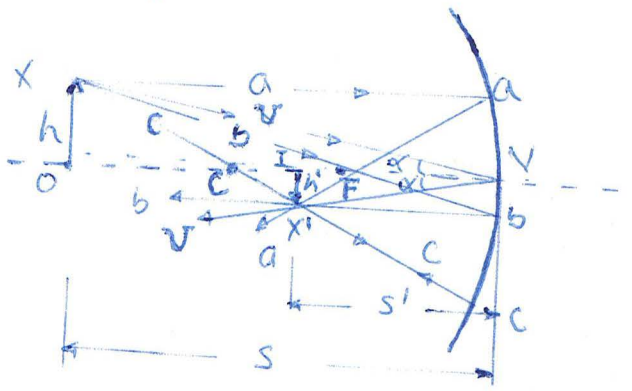
$$\frac{1}{s} + \frac{1}{s'} = \frac{1}{f}, \text{ convex mirror with } f = -\tfrac{1}{2}R$$

So for both concave and convex mirrors we have identical equations and we can write for all spherical mirrors

$$\frac{1}{s} + \frac{1}{s'} = \frac{1}{f} \quad \text{with } f = \frac{1}{2}r = \begin{cases} \tfrac{1}{2}R, \text{ concave mirror} \\ -\tfrac{1}{2}R, \text{ convex } \text{"} \end{cases}$$

The distances s, s' and f are positive when measured in front of the mirror and are negative when measured behind the mirror. Images formed in front of the mirror ($s'>0$) are real, and images formed behind the mirror ($s'<0$) are virtual. A convex mirror can form only virtual images of real objects.

Objects with Finite Size: The image of an extended object formed by a mirror is determined by constructing some rays called principal rays. The figure below shows these rays emanating from the tip X of an object in the form of luminous arrow.



1. Ray a leaves X parallel to the axis and upon reflection passes through the focal point F
2. Ray V is directed toward the vertex V and is reflected at an angle α with the axis equal to the angle of incidence
3. Ray b passes through the focal point F and upon reflection emerges parallel to the axis.
4. Ray C passes through the mirror center C and is reflected back along its original path.

However, any two of these principal rays will suffice to locate the image point X' of the arrow tip.

The height of the object arrow measured from the axis is h and the height of the image arrow is h'. Then, using the triangles formed by the incident and reflected portions of the ver tex ray V, we have

$$\tan\alpha = \frac{h}{s} = -\frac{h'}{s'}$$

where we assume that positive dist. is measured above the axis and negative dist. is measured below the axis so that h' is negative. We define the transverse or lateral magnification m to be

$$m = \frac{h'}{h} = -\frac{s'}{s}$$

When $m < 0$, the image is inverted wrt the object.

For a small object we can also define an axial or longitudinal magnification μ. Consider an object that has a length Δs along the mirror axis and is a mean dist. s from the ver tex; the image has a mean dist. s' and a corresponding axial length $\Delta s'$. Now we have

$$\frac{1}{s} + \frac{1}{s'} = \frac{1}{f}$$

Taking differentials, with f fixed, we find

$$-\frac{ds}{s^2} - \frac{ds'}{s'^2} = 0$$

We approximate ds by Δs and ds' by $\Delta s'$; then the axial magnification is defined by the relationship

$$\mu = \frac{\Delta s'}{\Delta s} = -\left(\frac{s'}{s}\right)^2 = -m^2$$

Note that when the object is located outside the center C of a concave mirror (so that $s > 2f$), the image formed is real, inverted, and diminished in size. But from the principle of reversibility we know that when the object is moved inside the focal point (so that $s < f$), the image becomes virtual, erect (i.e., oriented the same as the object), and magnified. A concave mirror is therefore a magnifying mirror for objects located inside the focal point.

For a convex mirror, however, regardless of the position of the object, the image is always virtual, erect, and diminished in size.

Example: a. A point source of light is located 20 cm in front of a concave spherical mirror with

R = 30 cm. Determine the position and character of the image point. b, If the concave mirror is replaced by a convex mirror of the same radius, determine the position and character of the new image point.

Solution

a, $f = \frac{1}{2}R = \frac{1}{2}(30\,cm) = 15\,cm$, $s = +20\,cm$. Then,

$$\frac{1}{s'} = \frac{1}{f} - \frac{1}{s} = \frac{1}{15cm} - \frac{1}{20cm} = \frac{1}{60cm}$$

(i.e.)

$$s' = 60\,cm$$

The image point is located infront of the mirror and is real.

b, For the convex mirror, $f = -\frac{1}{2}R = -\frac{1}{2}(30\,cm) = -15\,cm$. $s = +20\,cm$. Then,

$$\frac{1}{s'} = \frac{1}{f} - \frac{1}{s} = -\frac{1}{15cm} - \frac{1}{20cm} = -\frac{7}{60cm}$$

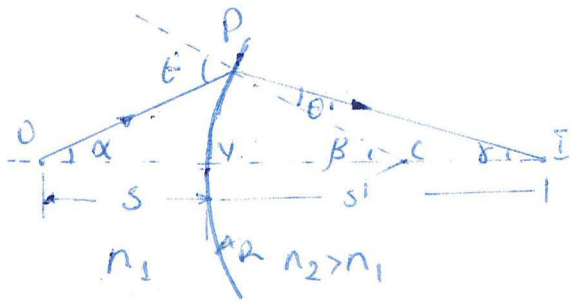
(i.e.)

$$s' = -60/7\ cm.$$

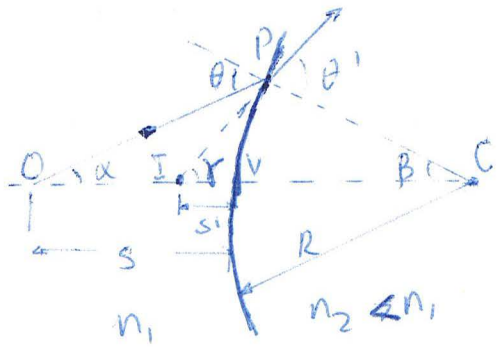
The image point is located behind the mirror and is virtual.

12-3 Refraction at Spherical Surfaces

The figure below shows a convex spherical interface with radius of curvature R that separates two media, one with index of refraction n_1 and one with index n_2. A point source of light is located at O. The line that connects O and the center C of the surface defines the axis of the system and the vertex V. The object dist is $\bar{OV} = s$ and the image dist is $\bar{VI} = s'$.



(a) $s' > 0$



b) $s' < 0$

For $n_2 > n_1$, fig. (a), an incident ray OP may be sufficiently refracted toward the normal at P that the ray will intersect the axis at the image point I. Thus, a real image is formed behind the refracting surface. On the other hand, if $n_2 < n_1$, fig. (b), an incident ray will be refracted away from the normal and will always appear to diverge from the point I. Thus, a virtual image is formed in front of the refracting surface. The situation is therefore the opposite of that of image formation by mirrors, in which real images appear in front of mirrors and virtual images appear behind. For refracting systems, the "front" surface is the surface on which the light is first incident.

The sign convention here is that the image dist. s' is positive for real images and negative for virtual images. Moreover, the surface radius $r = +R$ as in the above figures, when the center C is located behind the vertex, and $r = -R$ if the surface were (or care.)

With these conventions let us derive the formulas for a refracting surface. As before, we have $\theta = \alpha + \beta$ and $\beta = \theta' + \delta$. For small angles θ and θ' we can write $\sin\theta \approx \theta$ and $\sin\theta' \approx \theta'$; then Snell's law gives

$$n_1 \theta = n_2 \theta'$$

Which using the expressions of θ and θ' gives

$$n_1(\alpha+\beta) = n_2(\beta-\gamma) \text{ or } n_1\alpha + n_2\gamma = (n_2-n_1)\beta$$

We also have $\overline{PV} = \ell$, $\alpha \approx \ell/s$, $\gamma \approx \ell/s'$, and $\beta = \ell/r = \ell/R$ because $r = +R$ and all the angles are positive. Then the above eq. can be expressed as

$$\frac{n_1}{s} + \frac{n_2}{s'} = \frac{n_2-n_1}{R}, \text{ convex surface}$$

For a concave surface, the derivation proceeds in a similar manner, except that $r = -R$ and the distances s' and R is with negative. The resulting eq. is

$$\frac{n_1}{s} + \frac{n_2}{s'} = -\frac{n_2-n_1}{R} \text{ concave surface.}$$

We may combine the last two equations as

$$\frac{n_1}{s} + \frac{n_2}{s'} = \frac{n_2-n_1}{r}$$

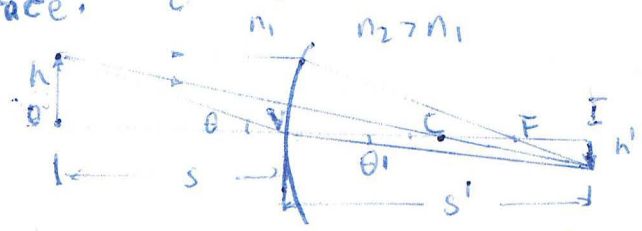
with

$r = +R$, convex surface

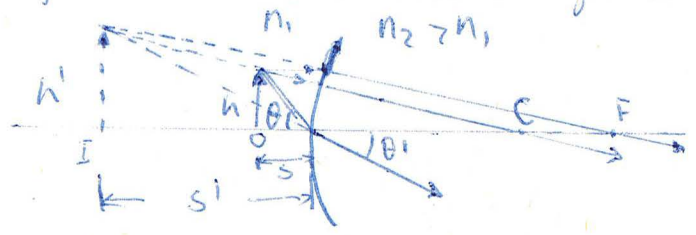
$r = -R$, concave "

For either a concave or convex surface, $s' > 0$ corresponds to a real image located behind the surface and $s' < 0$ corresponds to a virtual image located in front of the surface.

Objects with Finite Size: As is the case of mirrors, the image formed by a refracting system of surfaces of an extended object can be determined by considering the geometry of principal rays. On the fig. below, three principal rays emanating from the tip of an object of height h are shown. Here a real image of height h' is formed behind the surface.



On the other hand, when the object is moved close to the the refracting surface, a virtual image is formed in front of the surface. (Fig. below).



For rays close to the axis of the refracting surface (paraxial rays) we have

$$\tan\theta \approx \theta = h/s \quad \text{and} \quad \tan\theta' \approx \theta' = -h'/s'$$

These give

$$h = s\theta \quad \text{and} \quad h' = -s'\theta' \qquad \cdots (*)$$

But using Snells law, we also write

$$n_1\theta = n_2\theta' \qquad \cdots (**)$$

So, in view of these two last equations (*) and (**) the lateral magnification m can be expressed as

$$m = h'/h = -\frac{s'\theta'}{s\theta} = -\frac{n_1 s'}{n_2 s}$$

where the negative sign again means that the image is inverted when $s > 0$ and $s' > 0$.

The axial magnification μ is obtained by taking differentials of the eq.

$$\frac{n_1}{s} + \frac{n_2}{s'} = \frac{n_2 - n_1}{r}$$

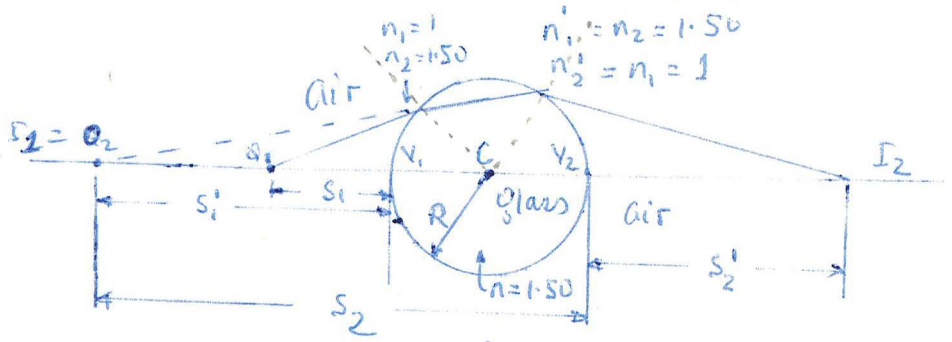
for which

$$-n\frac{ds}{s^2} - n'\frac{ds'}{s'^2} = 0$$

Thus, we find

$$\mu = \frac{\Delta s'}{\Delta s} = \frac{ds'}{ds} = -\frac{n_2}{n_1}m^2$$

Example: A point source of light (in air) is placed at O_1, a dist. $2R$ from the center of a glass sphere with radius of curvature R and refractive index $n = 1.50$. Find the image point of the source formed by paraxial rays.



Solution

The rays from the source at O_1 first strike the convex surface face of the sphere. Thus, in using the eq. for refracting surfaces, we have $r = +R$. Also, $n_1 = 1$, $n_2 = 1.50$, and $s_1 = R$;

We have

$$\frac{1}{s_1'} = \frac{1}{1.5}\left(\frac{1.5-1}{R} - \frac{1}{R}\right) = -\frac{1}{3R}$$

or

$$s_1' = -3R$$

Hence, the image formed by the first refraction is virtual and is located at I_1, a dist. 3R in front of the vertex V_1.

The refracted ~~light~~ ~~sphere~~ within the sphere appears to have originated at the point I_1; thus this point serves as the virtual object for the second refraction. The ray strikes the concave surface of the interface with V_2. So in using the e.q. of the refracting surface, we have $r = -R$ for the surface radius. Also, $n_1' = 1.50$, $n_2' = 1$ and $s_2 = 3R + 2R = 5R$. Then,

$$\frac{1}{s_2'} = \frac{1-1.50}{-R} - \frac{1.50}{5R} = \frac{1}{5R}$$

or

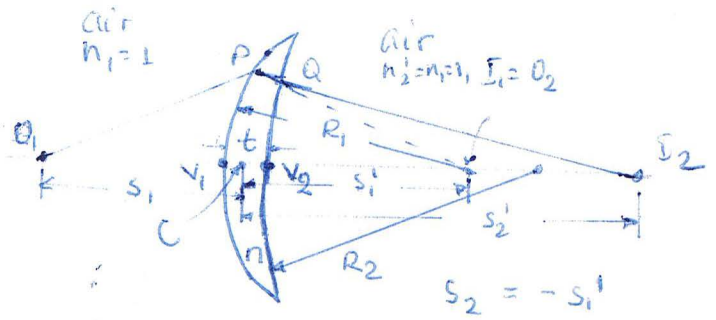
$$s_2' = 5R$$

Thus, the final image is located a dist. 5R behind the rear surface of the sphere; and the image is real.

12.4 Thin Lenses

A thin lens is one whose thickness along any line parallel to the axis is small compared with the radii of curvature of the lens surfaces.

Consider the figure below in which we have a lens of small thickness t. R_1 and R_2 are the radii of curvature of the convex



surface and the concave face of the lens. We see that ray QP is refracted by the interface at P. The object dist. is $s_1 = \overline{QV_1} \approx \overline{QC}$, and the surface radius is $r_2 = +R_1$. The image that would be formed by this surface alone, if the medium were extended, is real and its position is given by

$$\frac{1}{s_1} = \frac{n-1}{R_1} - \frac{n}{s_1'} \quad \cdots (*)$$

A second refraction of the ray occurs at C_2. The (virtual) object O_2 for this refraction is the image I_1 that results from the first refraction. The interface is again convex, so the surface radius is $r_2 = +R_2$. So for this refraction we write

$$\frac{n}{s_2} = \frac{1-n}{R_2} - 1/s_2' \qquad \ldots (**)$$

Now, $|s_1'| = \bar{V_1 I_1} \approx \bar{CI_1}$ and $|s_2| = \bar{V_2 O_2} \approx \bar{CI_1}$. According to the sign convention, $s_1' > 0$ and $s_2 < 0$; therefore $s_2 = -s_1'$. (This cond. is always true; for example, if the image point I_1 is infront of V_1, we would have $s_1' < 0$; but, then, O_2 would be in front of V_2 so that $s_2 > 0$.) Substituting $s_2 = -s_1'$ in (**), we find

$$-\frac{n}{s_1'} = \frac{1-n}{R_2} - 1/s_2'$$

and substituting this result into eq. (*) we find

$$\frac{1}{s_1} + 1/s_2' = (n-1)\left(1/R_1 - 1/R_2\right)$$

This eq. can be rewritten for the lens as a whole by dropping the subscripts on s_1 and s_2'. Moreover, the eq. applies for either convex or concave surfaces if we replace R_1 and R_2 by r_1 and r_2. Thus,

$$\frac{1}{s} + 1/s' = (n-1)\left(1/r_1 - 1/r_2\right)$$

where s and s' are measured from the center of the lens with r_1 and r_2, following the sign convention previously established. If either refracting surface of the lens is concave, the surface radius r for that surface is negative.

A lens which focuses parallel light rays to a single point (called focal point) is called converging lens and a lens for which parallel light rays striking its surface seem to diverge from a single point (focal point) is called diverging lens. This definition implies that a lens can have two focal points corresponding to its two faces. As with mirrors, the focal length f of a lens is defined from as the dist s' of the image when $s \to \infty$ (or vice versa). Thus we write

$$1/f = (n-1)\left(1/r_1 - 1/r_2\right)$$

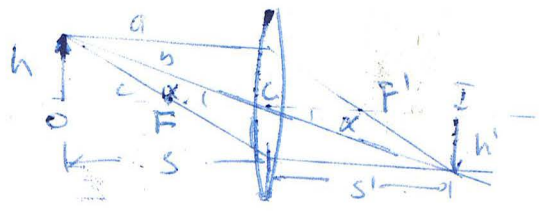
which is known as the lens makers formula.

The thin lens eq. now becomes

$$1/s + 1/s' = 1/f$$

This is called the Gaussian form of the lens eq. and it is identical to the mirror eq.

Objects with Finite size: The image-forming properties of a thin lens for an extended object are determined by considering the behaviour of the principal rays just as for the case of a single interface. The fig. below shows how three principal rays are used to determine the position of the image.



Ray b passes undeviated through the centre of the lens; this is true if only the lens is thin.

We have

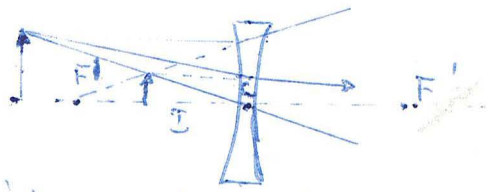
$$\tan\alpha = \frac{h}{s} = -\frac{h'}{s'}$$

which gives for the lateral magnification m the relation

$$m = \frac{h'}{h} = -\frac{s'}{s}$$

where the negative sign again indicates an inverted image when $s > 0$ and $s' > 0$.

The fig. below shows image formation of a diverging lens. The image is always erect, virtual and on

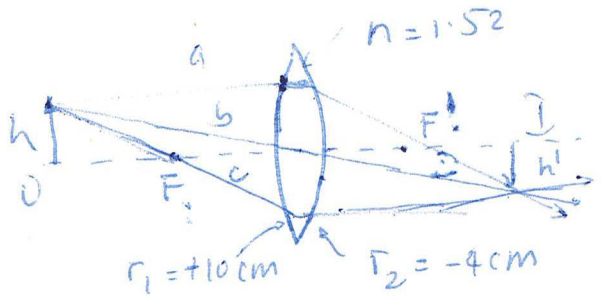


the same side of the lens as the object whatever position of the object is considered.

Examples.

1. The thin lens shown in the diagram is made from glass ($n = 1.520$) and has radii of curvature $R_1 = 10\,cm$ and $R_2 = 4\,cm$.
   a) Determine the focal length f of the lens.
   b) A luminous object with height $h = 2\,cm$ is located

30 cm in front of the lens. Determine the position, character, and size of the image.



a, Using the lens maker's eq. we find

$$1/f = (n-1)(1/r_1 - 1/r_2) = (1.52-1)\left(\frac{1}{10\,cm} - \frac{1}{-4\,cm}\right)$$

from which

$$f = 5.59\,cm$$

b, $s = +30\,cm$, then

$$1/s' = \frac{1}{f} - 1/s = \frac{1}{5.59\,cm} - \frac{1}{30\,cm}$$

from which

$$s' = 6.72\,cm$$

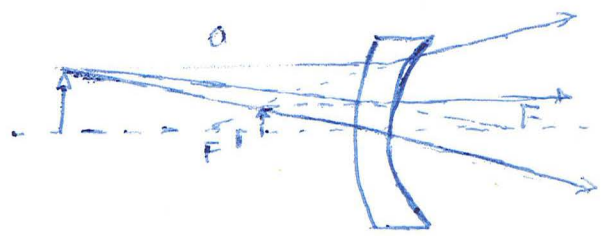
The magnification is

$$m = -s'/s = -\frac{6.72\,cm}{30\,cm} = -0.224$$

So that the image is real & inverted, with a height

$$h' = mh = -0.244 \times 2\,cm = -0.448\,cm$$

2. Rework Example 1 in the case that the second surface of the lens is convex; that is, $r_2 = 4\,cm$ instead of $-4\,cm$



a, The focal length is now

$$1/f = (1.52-1)\left(\frac{1}{10\,cm} - \frac{1}{4\,cm}\right)$$

from which

$$f = -12.82\,cm$$

i.e., F' is 12.82 cm infront of the lens.

b, $s = +30\,cm$, then

$$1/s' = \frac{1}{-12.82\,cm} - \frac{1}{30\,cm}$$

from which

$$s' = -8.98\,cm$$

The magnification is

$$m = -s'/s = -\frac{-8.98\,cm}{30\,cm} = 0.299$$

DO. that the image is virtual and erect with height

$$h' = mh = 0.299(2\,cm) = 0.598\,cm.$$

Chapter 13 Interference ~~and Diffraction~~

13.1 Conditions of Interference

In order for the phenomenon of interference to occur the following conditions should be fulfilled:

1. two or more waves must occupy the same region of space
2. the actual or effective sources must have the same frequency and a constant phase difference - this is the condition of coherence.
3. To observe constructive interference (intensity maxima) and destructive interference (intensity minima) the vector sum of the overlapping waves should have a max. amplitude for the former case and a minimum amplitude for the latter case. The max. amplitude is observed when the comp. waves vibrate in the same direction at a given point and the min. amplitude is observed when these comp waves vibrate in opposite directions at the point in question. There it is

two waves

However, from Maxwell's theory we have reached the conclusion that light is electromagnetic in nature and hence a light wave is nothing but the vibration of transverse electric and magnetic fields which propagate through space. The physical effects of an incident light depend on the energy flux delivered to a surface. This is measured by the poynting vector $\vec{S}$ and the time averaged energy flux represents a quantity called the wave intensity I. For example, for a single wave with $E = E_0 \cos(kx - \omega t)$, the wave intensity at a particular plane, $x = x_0$ say, is

since light is electromagnetic in nature

$$I = \bar{S} = \frac{1}{\mu_0}\overline{|E \times B|} = \frac{1}{\mu_0}\overline{EB}, \quad \text{plane el. mag. wave.}$$

Now, $B = B_0 \cos(kx - \omega t)$ and $B_0 = E_0/c$ for a wave propagating in vacuum (or air). Thus, $B = E/c$; then we find

$$I = \overline{E^2}/c\mu_0$$

so that

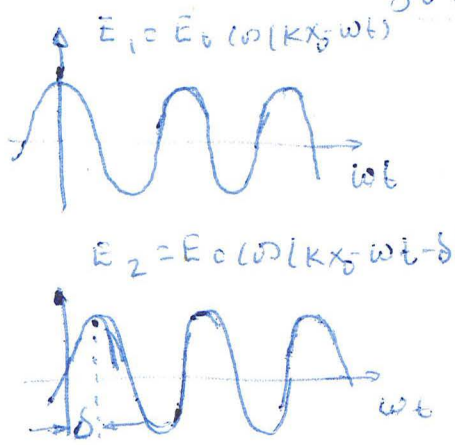
$$I = \frac{E_0^2}{c\mu_0}\cos^2(kx - \omega t) = \frac{E_0^2}{2c\mu_0}$$

where the units of I are W/m^2, in SI.

Example: Two coherent, equal amplitude electromagnetic plane waves are propagating in the x-direction with parallel polarizations. The waves arrive at a plane, $x = x_0$, with

their wavefronts essentially parallel. The phase difference of the two waves at that plane is δ. What is the wave intensity on the plane $x = x_0$? (For convenience, consider the waves to be infinite in extent, i.e., purely monochromatic).

Solution



The fig. shows the time variation of the electric vectors, E_1 and E_2, for the two waves on the plane $x = x_0$. The superposition of the two waves gives

$$\bar{E} = E_1 + E_2 = E_0 [\cos(kx_0 - \omega t) + \cos(kx_0 - \omega t - \delta)]$$

or

$$\bar{E} = 2E_0 \cos(kx_0 - \omega t - \delta/2) \cos \delta/2$$

The (time-averaged) wave intensity is given by

$$I = \overline{E^2} / \mu_0 c$$

so that in this case

$$\overline{E_R^2} = 4E_0^2 \cos^2 \delta/2 \; \overline{\cos^2(kx_0 - \omega t - \delta/2)}$$

$$= 2E_0^2 \cos^2 \delta/2$$

and

$$I = \frac{2E_0^2}{\mu_0 c} \cos^2 \delta/2$$

One can see from this expression that the wave intensity is proportional to the square of the wave amplitude E_0. Now define,

$$I_0 = \frac{1}{2}(E_0^2 / \mu_0 c), \text{ intensity of one wave.}$$

whence

$$I = 4I_0 \cos^2 \delta/2$$

N.B. The magnetic field strength H is related to $\bar{B}$ (in vacuum) as

$$\bar{H} = \frac{\bar{B}}{\mu_0} \text{ or } H = B/\mu_0$$

For a plane el. mag. wave, $B = E/c$ in which

$$H = \frac{\bar{E}}{\mu_0 c} = \sqrt{\frac{\epsilon_0}{\mu_0}} E$$

and

$$\frac{\bar{E}}{H} = \frac{\bar{E}_0}{H_0} = \sqrt{\frac{\mu_0}{\epsilon_0}}$$

$\frac{E_0}{H_0}$ has dimensions of V/A or Ohms.

We write

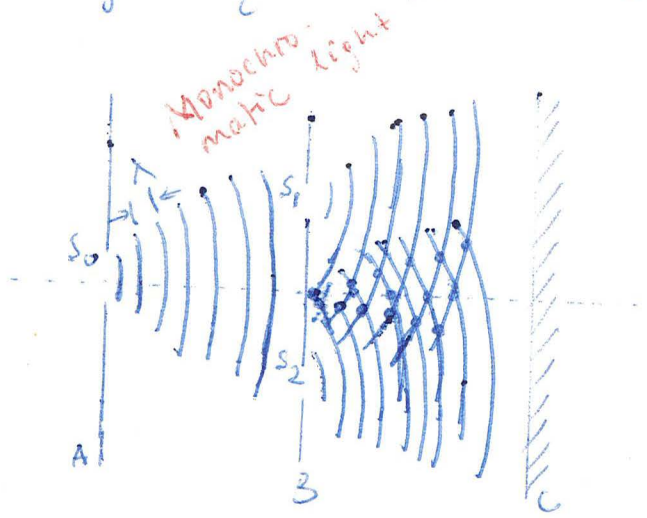
$$\frac{E_0}{H_0} = \sqrt{\frac{\mu_0}{\epsilon_0}} \equiv Z_0 , \quad Z_0 = 377\,\Omega .$$

The quantity Z_0 is called the impedence of free space.

13.2 Young's Double-Slit Experiment

This experiment demonstrates that light produces interference effects and thus can be treated as a wave phenomenon. This was the first of several crucial experiments that eventualy established the wave theory of light and it was performed by Thomas Young in 1801.

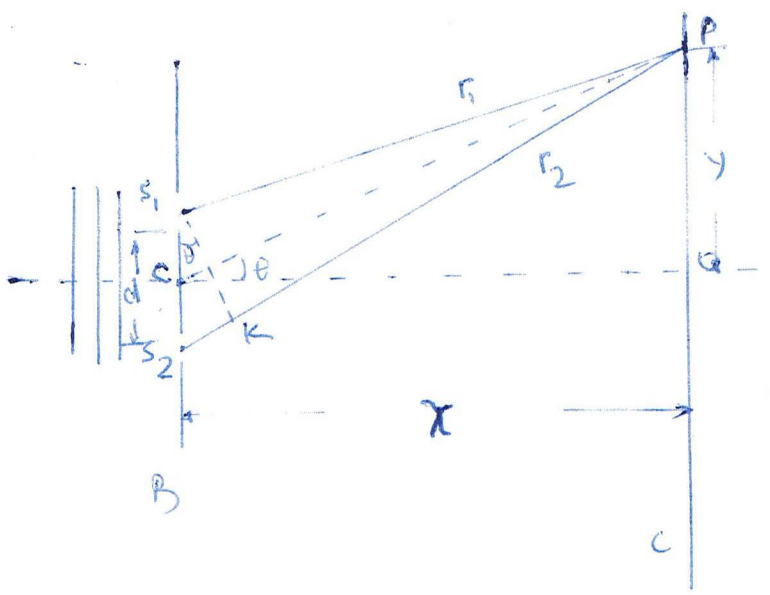
Young allowed sunlight to fall on a pinhole (slit) S_0 punched in a screen A. As represented in the figure below, diverging waves radiate from this slit and impinge on the pair of narrow, parallel slits, S_1 and S_2, that are separated by a dist. d and are equidistant from S_0. The result is that S_1 and S_2 become sources of coherent secondary waves of equal amplitude, which radiate outward and interfere in the region to the right of the slits, as indicated in the figure.



S_1 & S_2 are equidistant from S_0. Behind the screen B constructive & destructive interference effects are observed as a result of the two coherent secondary waves of the slits S_1 and S_2.

Let us analyze young's experiment quantitatively. Assume that the incident light consists of a single wavelength only. In the figure below, P is an arbitrary point on the screen C, a dist. r_1 and r_2 from the narrow slits S_1 and S_2, resp. Draw a line from S_1 to K in such a way that the lines PS_1 and PK are equal. If d, the slit spacing is much smaller than the dist. between the two screens, S_1K is then almost $\perp$ to both r_1 and r_2. This means that the angle $\angle K S_1 S_2$ is almost equal to angle $\angle PCO$,

both angles being marked θ in the figure. This is equivalent to saying that the lines r_1 and r_2 may be taken as parallel.



The light waves r_1 and r_2 may be represented at P as:

r_1: $E_1 = E_0 \cos(kr_1 - \omega t)$

r_2: $E_2 = E_0 \cos(kr_2 - \omega t)$

If $x \gg d$, then

$S_2K = r_2 - r_1 = d \sin\theta$

The two rays arriving at P, r_1 and r_2, from S_1 and S_2 are in phase at the source slits, both being derived from the same wavefront in the incident plane wave. Because the rays contain diff. numbers of waves, they arrive at P with a phase difference. The no. of wavelengths contained in S_2K, which is the path difference, determines the nature of interference at P. This is so because the effect of the optical path difference is to bring a difference in the phase of the waves.

To have a max. (constructive interference) interference at P, $S_2K (= d\sin\theta)$ must contain an integral no. of wavelengths, or

$$S_2K = \pm m\lambda, \quad m = 0, 1, 2, \ldots$$

which can be written as

$$d\sin\theta = \pm m\lambda, \quad m = 0, 1, 2, \ldots \text{ (maxima)}$$

Note that each maximum above 0 has a symmetrically located maximum below 0. There is a central maximum, described by $m = 0$.

For a minimum (destructive interference) at P, $S_2K (= d\sin\theta)$ must contain a half-integral no. of wavelengths, or

$$d\sin\theta = \pm(m' + \tfrac{1}{2})\lambda, \quad m' = 0, 1, 2, \ldots \text{ (minima)},$$

Intensity: Because the waves are in phase when they leave the slits S_1 and S_2, the wave from S_2 arrives at the point P(y) with some phase difference δ relative to the wave from S_1. To convert S_2K into a phase difference, first divide by λ to measure the path-length difference in wavelengths and then multiply by 2π to convert to radians. Thus,

$$\delta = \frac{2\pi d}{\lambda}\sin\theta$$

Also.
$E_1 = E_0\cos(kr_1 - \omega t)$
$E_2 = E_0\cos(kr_2 - \omega t)$
$\Rightarrow \delta = k(r_2 - r_1) = \frac{2\pi}{\lambda} d\sin\theta$
$\Big\}\ E_2 = E_0\cos(kr_1 - \omega t + \delta)$

Then, we have the conditions of the example in sec. 13-6, so that the intensity at the point P(y) = P(θ) is

$$I(\theta) = 4I_0\cos^2\left(\frac{\pi d}{\lambda}\sin\theta\right)$$

For $\theta = 0$, there is an intensity maximum. The next intensity maxima, on either side of the central maximum, have, $\sin\theta = \pm\lambda/d$, and so on. Or intensity maxima are in general given by

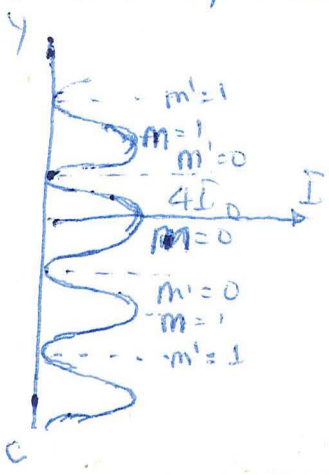
$$d\sin\theta = \pm m\lambda, \quad m = 0, 1, 2, \ldots$$

Destructive interference and, hence, intensity minima (zero intensity) occur when

$$d\sin\theta = \pm(m' + \tfrac{1}{2})\lambda, \quad m' = 0, 1, 2, \ldots$$

The result of this interference is to produce on the screen a pattern that consists of a series of alternating bright and dark bands called interference fringes. The central bright fringe ($m = 0$) is called the zero-order fringe; the next pair make the first-order fringes, and so forth.

The intensity at each maximum is $4I_0$. Because the intensity curve has a cosine squared shape, the average intensity over the screen is $2I_0$.



Example: What is the linear dist. on screen C of the 2nd figure in this section between adjacent maxima? The wavelength λ is 5460 A; the slit separation d is 0.10mm, and the slit-screen dist. x is 20 cm.

Solution

If θ is small enough, we can use the approximation $\sin\theta \approx \tan\theta \approx \theta$, and from the figure

$$\tan\theta = y/x$$

now,

$$d\sin\theta \approx d\frac{y}{x} = m\lambda, \quad m = 0, 1, 2, \ldots \text{ maxima.}$$

or

$$y_m = m\frac{\lambda x}{d}, \quad m = 0, 1, 2, \ldots$$

So for any two adjacent maxima of orders m and m+1 we have

$$y_m = m\frac{\lambda x}{d}$$

$$y_{m+1} = (m+1)\frac{\lambda x}{d}$$

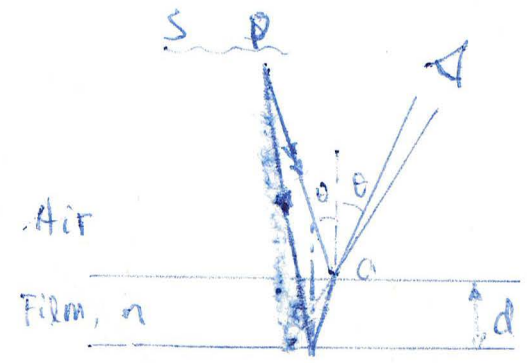
Thus, the separation Δy is given as

$$\Delta y = y_{m+1} - y_m = \lambda x/d = \frac{5.5\times10^{-7}\,m\,(20\times10^{-2}\,m)}{0.1\times10^{-3}\,m} = 1.1\,mm.$$

That means as long as θ is small, the separation of the interference fringes is independent of m- the fringes are evenly spaced. If the incident light contains more than one wavelength the separate interference patterns, which will have diff. fringe spacings, will be superimposed.

13.3 Interference produced by Thin Films

The figure below shows a film of uniform thickness d and index of refraction n. The film is illuminated by a broad source of monochromatic light S. There exists on this



source a point P s.t. two rays, identified by the single

and double arrows, can leave p and reach an observer, as shown after passing through point a. These two rays follow diff. paths in going from p to the observer, one being reflected from the upper surface of the film, the other from the lower surface. So at point a there will be formed either a bright or a dark interference pattern. These waves are coherent because they both originate from the same point p on the light source.

The interference pattern at a depends on two factors:

- the wavelength must refer to the wavelength of light in the film λ_n and not to its wavelength λ in air (λ_a); i.e., we are concerned with optical path lengths rather than geometrical path lengths.

$$\lambda_n = \lambda/n$$

- If light is incident on an optically denser medium, the incident and reflected light waves undergoes a phase difference of π change of π.

For near normal incidence an angle of incidence ($\theta \approx 0$) the geometrical phath difference is approximately equal to $2d$, see now using these two facts the interference pattern can be obtained at which results in a phase difference of $(2\pi/\lambda_n)(2d)$. Then, the total phase difference is

$$\frac{2\pi}{\lambda_n}(2d) + \pi \equiv \delta, \quad m = 0, \pm 1, \pm 2, \ldots$$

or for maximum intensity at a we must have

$$\frac{2\pi}{\lambda_n}(2d) - \pi = 2m\pi \qquad m = 0, \pm 1, \pm 2, \ldots$$

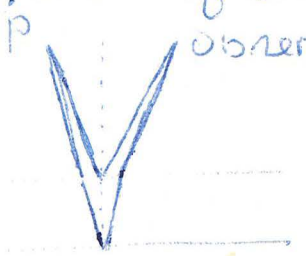
(i.e.,)

$$2d = (m + \tfrac{1}{2})\lambda_n \Rightarrow 2nd = (m + \tfrac{1}{2})\lambda_a, \; m = 0, \pm 1, \pm 2, \ldots$$

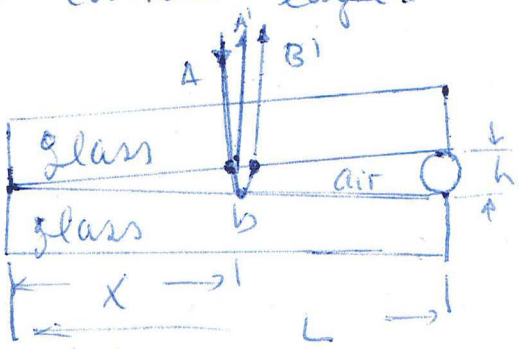
The condition for a minimum intensity is

$$2nd = m'\lambda_a \qquad m' = 0, \pm 1, \pm 2, \ldots$$

These equations hold when the index of refraction of the film is either greater or less than the indices of the media on each side of the film.



Example: Two flat glass plates are separated by a thin spacer with thickness h. For the case of near-normal incidence, locate the interference maxima wrt the contact edge.



The thickness of the glass plates is large compared with the wavelength of the incident light, so the observable interference effects are due entirely to the wedge-shaped air gap. The ray AB' undergoes a phase change of π upon reflection at point b. The thickness of the air gap at this point is $t = (\frac{h}{L})x$, introduces an additional phase difference $\delta = 4\pi h x / L\lambda_a$. Hence, interference maxima occur at positions for which

$$\frac{4\pi h}{L\lambda_a}x + \pi = 2m\pi \quad , \quad m = 1, 2, 3$$

At the contact edge, the medium is continuous, so there is no reflection and this edge is dark.

The spacing Δx between successive bright fringes is given by

$$\frac{4\pi h(x+\Delta x)}{L\lambda_a} - \frac{4\pi h x}{L\lambda_a} = 2(m+1)\pi - 2m\pi$$

from which

$$\Delta x = \frac{L}{2h}\lambda_a$$

For $L = 20\,cm$, $h = 0.1\,mm$, and $\lambda_a = 589\,nm$ (sodium yellow light), we find

$$\Delta x = \frac{20\,cm}{2(0.01\,cm)}(5.89\times10^{-7}\,m) = 5.89\times10^{-4}\,m = 0.589\,mm.$$

Chapter 14 Diffraction by a Single Slit

Introduction: Diffraction is the bending of light around an obstacle such as the edge of a slit. This phenomenon is also one of the conditions that lead to the conclusion that light is wavelike in nature. One can see the diffraction of light, for example, by putting a small object some dist. away from a point light source. Here part of the region directly behind the object will be illuminated and this is opposite to the case that would be expected from the rectilinear propagation of light. In the latter case the region behind the object is supposed to be completely dark representing the shadow of the object.

14-1 Diffraction by a Single Slit

There are two cases to consider for the diffraction patterns that are produced by a single slit:

1. Fresnel diffraction: This is a general case where the light source and/or the screen on which diffraction pattern is displayed are a finite dist. from the diffracting aperture; the wavefronts that fall on the diffracting aperture in this case and that leave it to illuminate any point of the diffusing screen are not planes; the corresponding rays are not parallel.

2. Fraunhofer diffraction: This is a simplified version of the case in (1) and it results if the source and screen are moved to a large dist. from the diffracting aperture. The wavefronts arriving at the diffracting aperture from the distant source are planes, and the rays associated with these wavefronts are parallel to each other. Similarly, the wavefronts arriving at any point of the distant screen are planes, the corresponding rays also being parallel.

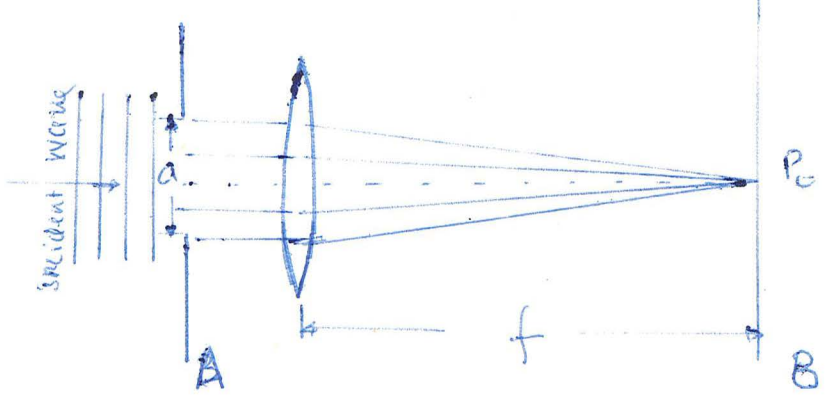
For source and screen at a finite distances from the aperture, Fraunhofer conditions are met by using two converging lenses, one in between the source and the aperture and the other in between the aperture and the screen so that plane wavies are obtained from in front of and beyond the aperture. assumed Fraunhofer conditions for young's double-slit experiment in section 13.1.

Although Fraunhofer diffraction is a limiting case of the more general Fresnel diffraction, it is an imp. limiting case and is easier to handle mathematically. So we

deal only with the Fraunhofer diffraction.

Consider the figure below in which a plane light wave is incident on an aperture A that consists of a single slit and moves, after diffraction, to a screen B.

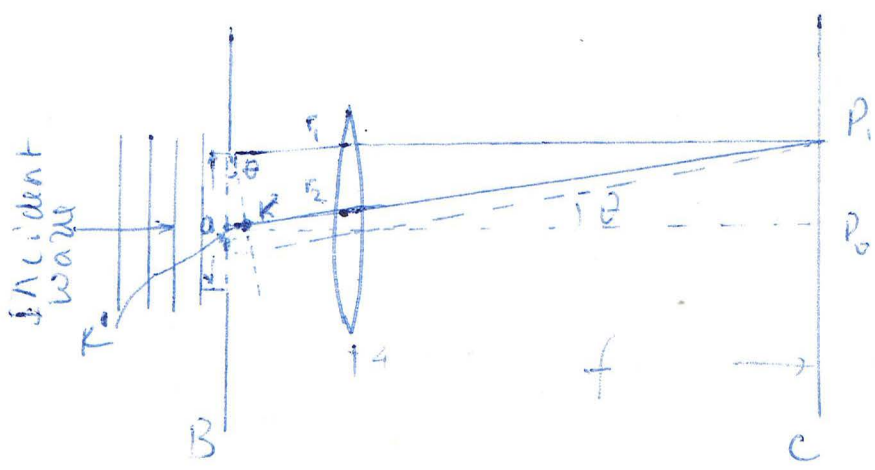
The oncoming plane wave falls normally on a



The slit width is a and the screen B is within the focal plane of the lens. plane waves fall on B.

long narrow slit of width a. P_0 is a central point on the screen. The rays extending from the slit to P_0 all have the same optical path lengths. since they are in phase at the plane of the slit, they will still be in phase at P_0, and the central point of the diffraction pattern that appears on B has a max. intensity.

We now consider another point P_1 on the screen (fig. below). Light rays which reach P_1 leave the slit at an angle θ as shown. Ray r_1 originates at the top



$\overline{KK'} = \lambda/2$

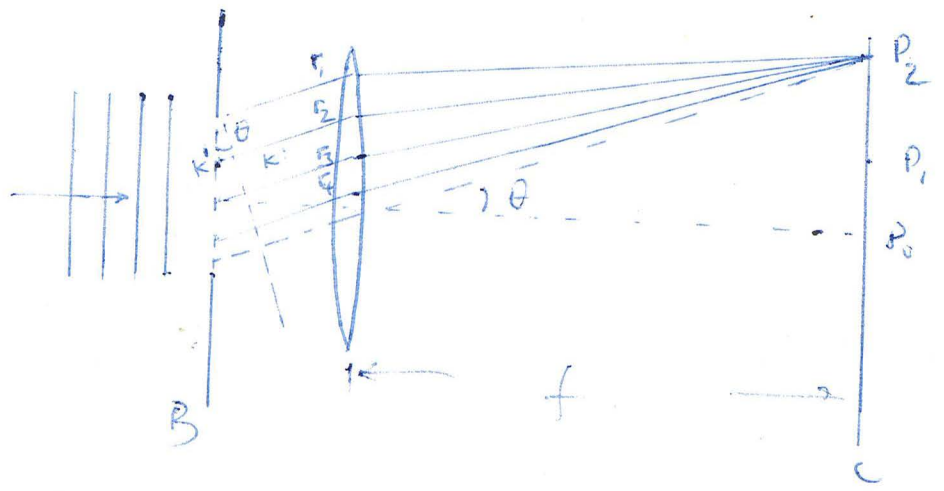
of the slit and ray r_2 at its center. If θ is chosen so that the dist. $\overline{KK'}$ in the figure is half a wavelength, r_1 and r_2 will be out of phase and will produce no effect at P_1. In fact, every ray from the upper half of the slit will be canceled by a ray from the lower half, originating a dist. $a/2$ below the first ray. The point P_1, the first minimum of the diffraction pattern, will have zero intensity. The condition shown in the above

figure is

$$\frac{a}{2}\sin\theta = \lambda/2 \quad \text{or} \quad a\sin\theta = \lambda.$$

The central maximum becomes wider as the slit is made narrower. For example, if the slit width is as small as one wavelength ($a = \lambda$), the first min. occurs at $\theta = 90°$, which implies that the central maximum fills the entire forward hemisphere.

In the fig. below the slit is divided into four equal zones, with a ray leaving the top of each zone.



$K'K = \frac{\lambda}{2}$

Let θ be chosen so that the dist $\overline{K'K}$ is half a wavelength. Rays r_1 and r_2 will then cancel at P_2. Rays r_3 and r_4 will also be half a wavelength out of phase and will cancel at P_2. Consider four other rays emerging from the slit a given dist. below the four rays above. The two rays below r_1 and r_2 will cancel uniquely, as will the two rays below r_3 and r_4. We can proceed across the entire slit and conclude again that no light reaches P_2; we have located a second point of zero intensity. The condition described requires that

$$\frac{a}{4}\sin\theta = \lambda/2 \quad \text{or} \quad a\sin\theta = 2\lambda$$

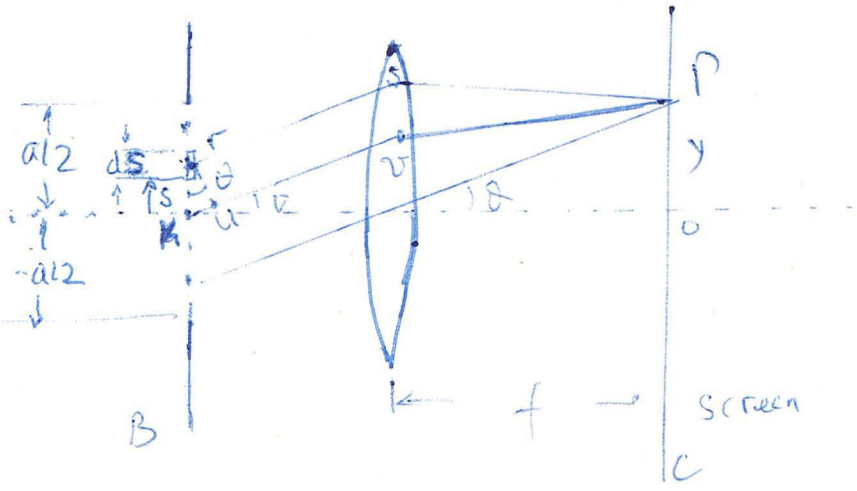
By extension, the general formula for the minima in the diffraction pattern on screen C (including those below the center of the screen also):

$$a\sin\theta = \pm m\lambda \qquad m = 1, 2, 3 \ldots$$

There is a maximum approximately half-way between each adjacent pair of minima.

Intensity distribution in single slit diffraction.

Let us now divide (as shown below) the slit into parallel strips of width dS. Each strip acts as a source of secondary waves that travel outward toward the viewing screen.



since the strip, of width dS shown in the above figure, is an infinitesimal width ds, all the rays originating from it are focused at a single point P on the screen C. Additional rays from other similar secondary sources are also focused at P, and the light intensity there is obtained by superimposing all such rays.

Choose the central ray KO as a reference ray for determining the phase differences of all other rays that arrive at P. The optical path length $\overline{FSP}$ is the same as $\overline{OP}$. ~~There are similar strips between the bounds of rays that originate at source S and P.~~ Thus, the rays FS from the strip source at ds arrive at P with a phase lead, compared with the reference ray, corresponding to the optical path difference $\overline{KU} = S \sin\theta$.

The slit as a whole is illuminated uniformly with the incident plane light wave. Consequently, the electric field amplitude dE_0 at point P due to the secondary line source ds may be expressed as

$$dE_0 = \eta\, ds$$

where η is a const. with appropriate dimensions. The reference ray is considered to have zero phase angle. If the angle θ is very small, all contributions to the electric field intensity on the screen may be considered to be parallel and to lie in the plane of the screen. Thus, the time dependent electric field comp. at P due to the segment ds is

$$dE_s = n\,dS\,\cos(\omega t + \frac{2\pi}{\lambda} s \sin\theta)$$

In terms of a new parameter α which represents the phase (angle. ~~was known~~

$$\alpha = \frac{\pi a}{\lambda}\sin\theta$$

the superposition of all contributions due to the secondary sources from $s = -a/2$ to $s = a/2$ is

$$E = n\int_{-a/2}^{a/2} \cos(\omega t + \frac{2\alpha}{a} s)\,ds$$

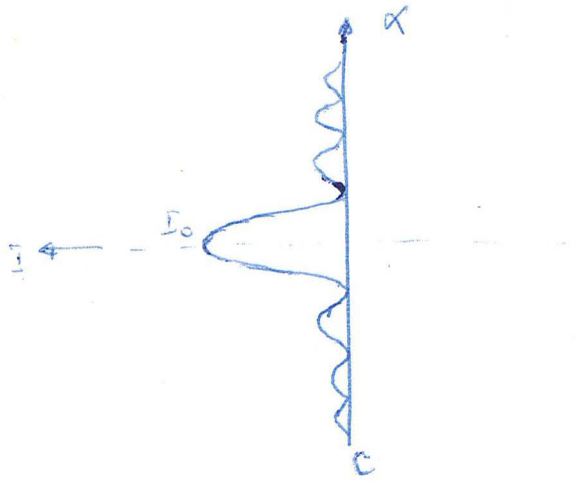
$$= \frac{na}{2\alpha}[\sin(\omega t + \alpha) - \sin(\omega t - \alpha)] = na\left(\frac{\sin\alpha}{\alpha}\right) \cdot \cos\omega t$$

The intensity is equal to $\overline{E^2}/2\mu_0 c$, so we can write

$$I = I_0 \left(\frac{\sin\alpha}{\alpha}\right)^2$$

where $\frac{1}{2} n^2 a^2/\mu_0 c$ has been set equal to I_0.

The fig. below shows the distribution of the intensity on the viewing screen C.



The central position ($\alpha = 0$) corresponds to a max. because

$$\lim_{\alpha \to 0} \frac{\sin\alpha}{\alpha} = 1.$$

The secondary maxima are given closely (but not exactly) by the cond. $\sin\alpha = \pm 1$, (i.e.) as before

$$(*) \qquad \alpha = \frac{\pi a}{\lambda}\sin\theta \simeq \pm \tfrac{1}{2}(2m+1)\pi \quad , \quad m = 1,2,3,\ldots \text{ approx. maxima.}$$

The intensity minima occur when $\sin\alpha = 0$ (excluding $\alpha = 0$); i.e.)

$$(**) \qquad \alpha = \frac{\pi a}{\lambda}\sin\theta = \pm m\pi \quad , \quad m = 1,2,3,\ldots \text{ minima.}$$

which is the result previously derived.

The "half-width" $\Delta\alpha$ of the central max. is defined to be twice the value of α for which $I = \frac{1}{2} I_0$. Thus

$\Delta\alpha = 2.783$ rad, can be solved graphically.

The condition of the infinite slit we have been considering becomes a rectangular aperture, the diffraction pattern will contain contributions from the width a and the length, say b. In the direction ⊥ to the length of the rectangle, the intensity maxima are given by (*). To find the maxima in the direction ⊥ to the width, the width a in (*) must be replaced by the length b > a, which means that the spacing between the maxima in this direction, determined by the values of θ, will be less than in the other direction.

Example: A slit with a width of 0.03 mm is illuminated with plane waves of monochromatic light ($\lambda = 589$ nm) in the geometry of the first figure of this subsection. The focal length of the lens is 60 cm.

a) Determine the locations on the screen of the first minima.

b) Determine the approximate locations on the screen of the first secondary maxima.

c) Find the approximate ratio of the intensity of the first secondary maxima to the intensity of the central maximum.

d) What is the width of the central maximum at half maximum intensity?

Solution

a) Eq. (**) with $m = 1$ gives

$$\frac{a}{\lambda}\sin\theta = \pm 1$$

or

$$\theta_1^{min} = \pm \sin^{-1}\left(\frac{5.89\times10^{-7}\,m}{3\times10^{-5}\,m}\right) = \pm 0.0196 \text{ rad}$$

The position on the screen, measured from the central maximum, is obtained by referring to the ray that passes undeviated through the center of the lens:

$$y_1^{min} = \pm f \tan\theta \approx \pm 60\text{cm}\,(0.0196) = \pm 1.18\text{cm}$$

b) For the approximate location of the first secondary maxima, we use $m = 1$ in (*) so that

$$\theta_1^{max.} \approx \pm\sin^{-1}\frac{3\lambda}{2a} = \pm\sin^{-1}\frac{3(5.89\times10^{-7}\,m)}{2(3\times10^{-5}\,m)} = \pm 0.0295 \text{ rad}$$

from which

$$y_1^{max.} = \pm f\tan\theta_1^{max.} \approx \pm 1.77\text{cm}.$$

c. The approximate intensity ratio is found as

$$I_1^{max}/I_0 = \left(\frac{\sin \alpha_1^{max}}{\alpha_1^{max}}\right)^2$$

$$\alpha_1^{max} = \frac{\pi a}{\lambda} \sin \theta_1^{max} = 3/2\,\pi$$

Thus,

$$I_1^{max}/I_0 = \left[\frac{2}{3\pi} \sin(3\pi/2)\right]^2 = \frac{4}{9\pi^2} = 0.043$$

d, All relevant values of θ are small ($\theta \ll 1$), we can write

$$\alpha = \frac{\pi a}{\lambda} \sin\theta \approx \frac{\pi a}{\lambda}\theta$$

and

$$\Delta\alpha = \frac{\pi a}{\lambda}\Delta\theta$$

Also,

$$\Delta y \approx f\Delta\theta = \frac{f\lambda}{\pi a}\Delta\alpha$$

Then, using $\Delta\alpha = 2.783$, we find

$$\Delta y \approx \frac{60cm\,(5.89\times10^{-7}m)}{\pi\,(3\times10^{-5}m)}(2.783) = 1.04\,cm.$$

14.2 Diffraction by a Double slit

In young's double-slit experiment we assumed that the slits were arbitrarily narrow (i.e., $a \ll \lambda$), which means that the central part of the diffusing screen was uniformly illuminated by the diffracted waves from each slit. When such waves interfere, they produce fringes of uniform intensity. This idealized situation cannot occur with actual slits because the condition $a \ll \lambda$ cannot usually be met. Waves from the two actual slits combining at diff. points of the screen will have intensities that are not uniform but are governed by the diffraction pattern of a single slit. The effect of relaxing the assumption that $a \ll \lambda$ in young's experiment is to leave the fringes relatively unchanged in location but to alter their intensities.

The interference pattern for two infinitesimally narrow slits is given by

$$I(\theta) = 4 I_0 \cos^2\left(\frac{\pi d}{\lambda}\sin\theta\right)$$

But to indicate that this is due to interference we may write

$$I(\theta)_{int} = 4 I_{0,int} \cos^2\left(\frac{\pi d}{\lambda}\sin\theta\right) \quad \ldots(*)$$

The intensity for the diffracted wave from either slit is given by (with a small change of notation) by

$$I(\theta)_{,diff.} = I_{0,diff.}\left(\frac{\sin\alpha}{\alpha}\right)^2 \quad \ldots(**)$$

where

$$\alpha = \frac{\pi a}{\lambda}\sin\theta, \quad a \text{ is the slit width.}$$

The combined effect is found by regarding $I_{0,int}$ as a variable amplitude, given in fact by $I(\theta, diff.)$. This assumption, for the combined pattern, leads to

$$I(\theta) = I_m\left[\cos\left(\frac{\pi d}{\lambda}\sin\theta\right)\right]^2\left[\sin\left(\frac{\pi a}{\lambda}\sin\theta\right)\right]^2$$

(I_m = max. intensity)

in which all the subscripts referring separately to interference and diffraction are dropped. With

$$\beta = \frac{\delta}{2} = \frac{\pi d}{\lambda}\sin\theta \quad \text{and} \quad \alpha = \frac{\pi a}{\lambda}\sin\theta$$

we have

$$I(\theta) = I_m(\cos\beta)^2\left(\frac{\sin\alpha}{\alpha}\right)^2 \quad \ldots(***)$$

At any point on the screen the available light intensity from each slit, considered separately, is given by the diffraction pattern of that slit (eg. **). The diffraction patterns for the two slits, again considered separately, coincide because parallel light rays in Fraunhofer diffraction are focused at the same spot (See the last fig. in sec. (14.2)). Because the two diffracted waves are coherent, they will

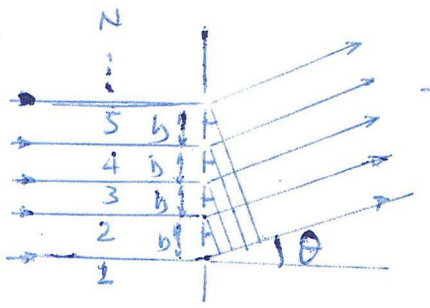
interfere.

The effect of interference is to redistribute the available energy over the screen, producing a set of fringes. In sec. 13.1, where we assumed $a \ll \lambda$, the available energy was virtually the same at all points on the screen, so that the interference fringes had virtually the same intensities (see the last fig. in sec 13.1). If we relax the assumption $a \ll \lambda$, the available energy is not uniform over the screen but is given by the diffraction pattern of a slit of width a. In this case the interference fringes will have intensities that are determined by the ~~coincidence~~ intensity of the diffraction pattern at the location of a particular fringe. Eq. (***) is the mathematical expression of this argument.

14.3 The Diffraction Grating

This is a device used to measure the wavelength of light with high precision. Such gratings are usually prepared by ruling a large no. of narrow, closely spaced parallel grooves on a polished glass plate. Diffraction occurs for both reflected and transmitted light.

Consider the Fraunhofer diffraction of light with wavelength λ by a transmission grating that consists of N equally spaced narrow slits (fig. below). The



grating spacing (the dist. between successive slits) is b, and the diffraction to the point P of interest on the distant viewing screen is specified by the angle θ. For convenience, the ray that passes

through the first slit, labeled 1, is taken to be the reference ray with zero phase angle. The ray labeled 2 has a path length that is shorter by an amount $b \sin\theta$ and therefore has a phase lead, compared with the reference ray, of $\frac{2\pi b}{\lambda} \sin\theta$. The ray labeled 3 has a phase lead of twice this amount, and so forth. If the contribution to the electric field $\bar{E}$ at point P due to the first slit is $E_0 \cos\omega t$, the superposition of the contributions from all N slits gives

$$\bar{E} = \bar{E}_0 \sum_{\ell=0}^{N-1} \cos\left(\omega t + \frac{2\pi}{\lambda} \ell b \sin\theta\right)$$

In complex notation we write

$$\bar{E} = \bar{E}_0 \, \mathrm{Re} \sum_{\ell=0}^{N-1} e^{i(\omega t + 2\ell\beta)}$$

where

$$\beta = \frac{\pi b}{\lambda} \sin\theta$$

This summation is a simple geometric progression, the sum of which is

$$a + ar + ar^2 + ar^3 + \cdots + ar^{N-1} = a\left(\frac{r^N - 1}{r - 1}\right)$$

where, in the present case, $a = e^{i\omega t}$ and $r = e^{i2\beta}$. Thus

$$\bar{E} = \bar{E}_0 \, \mathrm{Re}\, e^{i\omega t} \frac{e^{i2N\beta} - 1}{e^{i2\beta} - 1}$$

Let

$$X = e^{i\omega t}\left(e^{i2N\beta} - 1\right) / \left(e^{i2\beta} - 1\right)$$

Then,

$$X = \frac{e^{iN\beta}}{e^{i\beta}} \frac{e^{iN\beta} - e^{-iN\beta}}{e^{i\beta} - e^{-i\beta}} = e^{i(N-1)\beta} \frac{\left(e^{iN\beta} - e^{-iN\beta}\right)/2i}{\left(e^{i\beta} - e^{-i\beta}\right)/2i}$$

or

$$X = e^{i(N-1)\beta} \sin N\beta / \sin\beta$$

which gives

$$\bar{E} = \bar{E}_0 \, \mathrm{Re}\, e^{i[\omega t + (N-1)\beta]} \frac{\sin N\beta}{\sin\beta}$$

thus or

$$\bar{E} = \bar{E}_0 \frac{\sin N\beta}{\sin\beta} \cos[\omega t + (N-1)\beta]$$

The intensity at P is proportional to the time average of E^2; thus,

$$I = \frac{I_0}{N^2} \left(\frac{\sin N\beta}{\sin\beta} \right)^2$$

where I_0 is the intensity for $\beta = 0$ (i.e., $\theta = 0$).

The principal maxima of the intensity I occur when $\sin\beta = 0$; thus

$$\beta = \frac{\pi b}{\lambda} \sin\theta = \pm m\pi, \quad m = 0, 1, 2, \ldots \quad \square.$$

TO calculate the value of I at the principal maxima, we use the following procedure.

$$\lim_{\beta \to \pm m\pi} \left(\frac{\sin N\beta}{\sin\beta} \right)^2 = \lim_{\beta \to \pm m\pi} \left(\frac{\sin N\beta}{N\beta} \cdot N\beta \right)^2$$

$$= \lim_{\beta \to \pm m\pi} N^2 \left(\frac{\sin N\beta / N\beta}{\sin\beta / \beta} \right)^2$$

$$= N^2 \lim_{\beta \to \pm m\pi} \left(\frac{\sin N\beta}{N\beta} \right)^2 \cdot \lim_{\beta \to \pm m\pi} \left(\frac{\beta}{\sin\beta} \right)^2$$

$$= N^2 \cdot 1 \cdot 1 = N^2$$

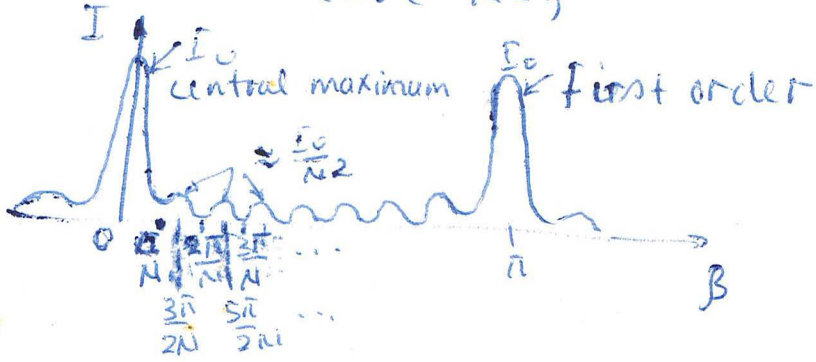
Then, we find

$$I = I_0 \quad \text{principal maxima.}$$

In the fig. above we can see how these maxima arise. For constructive superposition of all of the rays, the difference in path length between successive rays must be an integer no. of wavelengths; i.e., the condition for the principal maxima is $b\sin\theta = \pm m\lambda$, in agreement with the result π.

The value $m = 0$ corresponds to the central maximum. For $m = 1$, the two resulting maxima are referred to as

the first order maxima; for $m = 2$, the maxima are second order; and so forth. Between each pair of principal maxima are $N-2$ additional maxima that correspond to $\sin N\beta \approx 1$; because $\sin\beta$ is also approximately equal to unity, the intensity of these secondary maxima, which does have a regular variation, is nevertheless of the order of I_0/N^2 (see the result for $\bar{I}$). Between each pair of secondary maxima is a minimum for which $\sin N\beta = 0$ (but $\beta \neq \pm m\pi$). The fig. below shows the case for $N = 8$, principal secondary maxima for the case $N = 9$



As N is increased, the intensity at the principal maximum increases as N^2. This effect is due to two factors. First, the total power that reaches the viewing screen is proportional to N, the no. of slits that are open to the incident plane wave. Second, the width of each principal maximum, which can be defined as the dist. between the minima on either side, is proportional to $1/N$ (see the above fig.). Thus, the power density (the intensity) at a principal maximum increases as N^2. Because the intensity of the secondary maxima decreases as $1/N^2$, increasing the no. of slits in the grating both narrows the principal maxima and heightens the contrast with the secondary maxima. This is a point of the greatest importance for the applications of diffraction gratings.

Resolving power:

1. Young Double Slit Expt.

$I = 4I_0 \cos^2\left(\frac{\delta}{2}\right)$; δ - phase difference between rays from the two slits.

$\delta = \frac{2\pi}{\lambda} d \sin\theta$, where d is the slit separation & θ is the angular position of the interference pattern.

$\Rightarrow I = 4I_0 \cos^2\left(\frac{\pi}{\lambda} d \sin\theta\right)$

- Maximum intensity

$\frac{\pi d}{\lambda} \sin\theta = \pm m\pi$, $m = 0, 1, 2, \ldots$

or, $\sin\theta = \pm m\frac{\lambda}{d}$, $m = 0, 1, 2, \ldots$

- Minimum intensity

$\frac{\pi d}{\lambda} \sin\theta = \pm (m' + \frac{1}{2})\pi$, $m' = 0, 1, 2, \ldots$

or, $\sin\theta = \pm (m' + \frac{1}{2})\frac{\lambda}{d}$, $m' = 0, 1, 2, \ldots$

$\frac{E_0}{H_0} = \frac{\omega}{k} = \frac{1}{\epsilon_0 k \omega \mu_0 k}$

$= \mu_0 k_m v \omega$

$= 1$

or, $\frac{E_0}{H_0} = \frac{\omega}{k} = \frac{1}{\epsilon} = \mu v$

$u = \frac{1}{2}\vec{E}\cdot\vec{D} + \frac{1}{2}\vec{H}\cdot\vec{B}$

$u = \frac{1}{2}\epsilon_0 k E_0^2$

$\vec{S} = \vec{E} \times \vec{H}$

$\vec{S} = uc\hat{k}$

$\vec{v} = \frac{\vec{S}}{c^2} = \frac{v}{c}\hat{k}$

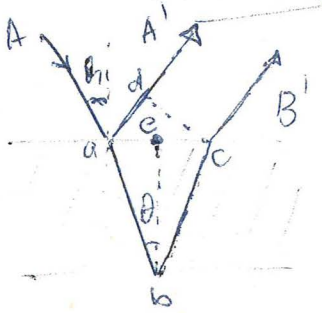
$P_{rad} = \frac{F}{A} = u$.

2.

thickness t

phase difference between AA' & AB'

$\delta = \frac{2\pi}{\lambda_a}$



path difference between AA' & AB'

$= 2nt\cos\theta$

$\Rightarrow \delta = \frac{2\pi}{\lambda_a}(2nt\cos\theta) + \pi$

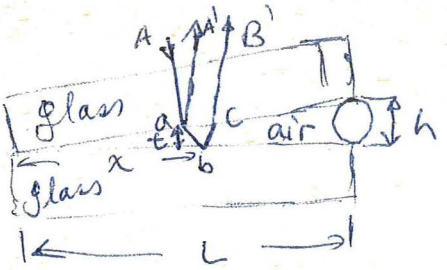
3 Newtons Rings

Radius, $r_m = \sqrt{(m+\frac{1}{2})R\lambda_a}$; $m = 0, 1, 2, \ldots$

R = radius of the lens

λ_a = wavelength of light in air.

4.



Near normal incidence

path difference

$= 2t = \frac{2hx}{L}$

$\Rightarrow \delta = \frac{2\pi}{\lambda_a}\left(\frac{2hx}{L}\right) + \pi$ ← reflection at a dense medium.

Constructive interference

$$\delta = \frac{2\pi}{\lambda_a}\left(\frac{2hx}{L}\right) + \pi = 2m\pi, \quad m = 1, 2, 3, \ldots$$

or,

$$x_m = \frac{1}{4}(2m-1)\frac{\lambda_a L}{h}, \quad m = 1, 2, 3, \ldots$$

5. Fraunhofer diffraction from a single slit

$$S = S_0\left(\frac{\sin\alpha}{\alpha}\right)^2, \quad \alpha = \frac{\pi a}{\lambda}\sin\theta$$

θ – angular position of int. pattern.
a – slit width.

– Central maximum at $\alpha = 0$ with $S = S_0 \sin\alpha$

$$\lim_{\alpha\to 0} \frac{\sin\alpha}{\alpha} = 1.$$

– Secondary maxima

Approximately, these are given where $\sin\alpha = \pm 1$, or

$$\alpha = \frac{\pi a}{\lambda}\sin\theta \simeq \pm(m+\tfrac{1}{2})\pi, \quad m = 1, 2, 3, \ldots$$

or,

$$\sin\theta = \pm(m+\tfrac{1}{2})\frac{\lambda}{a}, \quad m = 1, 2, 3, \ldots$$

– Minima

$\sin\alpha = 0$, $\alpha \neq 0$

That is,

$$\alpha = \frac{\pi a}{\lambda}\sin\theta = \pm m\pi, \quad m = 1, 2, 3, \ldots$$

or,

$$\sin\theta = \pm m\frac{\lambda}{a}, \quad m = 1, 2, 3, \ldots$$

- Angular resolution

$$\delta\theta_{min} \approx \frac{\lambda}{a}, \text{ single slit}$$

$$\delta\theta_{min} \approx \frac{1.22\lambda}{d}, \text{ circular aperture (like lense, eye etc.)}$$

where a is the width of the slit and d is the diameter of the circular aperture.

6. Fraunhofer Diffraction of the Diffraction Grating

$$I = \frac{I_0}{N^2}\left(\frac{\sin N\beta}{\sin\beta}\right)^2, \quad I_0 \text{ intensity at } \beta = 0 \text{ or } \theta = 0.$$

where N is the no. of slits (or ruled lines)

$$\beta = \frac{\pi b}{\lambda}\sin\theta$$

and b is the slit separation (between consecutive slits)

- principal intensity maxima ($\sin\beta = 0$)

or,

$$\beta = \frac{\pi b}{\lambda}\sin\theta = \pm m\pi, \quad m = 0, 1, 2, \ldots$$

$$\sin\theta = \pm m\frac{\lambda}{b}, \quad m = 0, 1, 2, \ldots$$

$\beta = 0$, principal maximum.

- Resolving power: The angular separation between a principal maximum and the adjacent minimum is given by $\Delta\beta = \pi/N$. Or,

$$\Delta\beta = \frac{\pi b}{\lambda}\cos\theta\,\Delta\theta = \pi/N$$

from which the angular ~~separation~~ change $\Delta\theta$ to descend from a principal maximum to the adjacent zero ~~maximum~~ intensity may be calculated.

- Resolving power of a diff. grating

$$R_g = \frac{\lambda}{\Delta\lambda} = mN, \quad m \text{ is the order of diffraction.}$$

- If the width of the slits in the grating is taken into account we have

$$I = \frac{I_0}{N^2}\left(\frac{\sin N\beta}{\sin\beta}\right)^2\left(\frac{\sin\alpha}{\alpha}\right)^2 \quad \alpha = \frac{\pi a}{\lambda}\sin\theta.$$

For the case of a double slit $N=2$ and we will find

$$I = I_0 \cos^2\beta \left(\frac{\sin\alpha}{\alpha}\right)^2.$$

Ring along axis

$$E = \frac{Qz}{2\pi\varepsilon_0(z^2+a^2)^{3/2}}, \quad V = \frac{Q}{4\pi\varepsilon_0}\frac{1}{\sqrt{z^2+a^2}}$$

disk along axis

$$E = \frac{\sigma}{2\varepsilon_0}\left[1 - \frac{z}{(z^2+a^2)}\right], \quad V = \frac{Q}{2\pi\varepsilon_0 R^2}\left[\sqrt{R^2+z^2} - z\right]$$

---

on the axis of a ring

$$B = \frac{\mu_0 I a^2}{2(z^2+a^2)^{3/2}}, \quad B(z \gg a) = \frac{\mu_0 \vec{m}}{2\pi z^3}$$

$$\mathcal{E} = \tfrac{1}{2}\omega B R^2, \quad \Phi_m = -\pi R^2 B\frac{\theta}{2\pi} = \tfrac{1}{2}R^2 B\omega t$$
$$\rightarrow \mathcal{E} = \tfrac{1}{2}\omega B R^2$$

Maxwell disp current : $\vec{j} = \varepsilon_0\frac{d\vec{E}}{dt}$, vacuum.

Magnetism

$\mu_0\vec{M} = \vec{B} - \vec{B}_0$, $\frac{B}{B_0} = \frac{\mu}{\mu_0} = k_m = \chi_m + 1$, $\chi_m < 0$, diam $\chi_m > 0$, para, ferro.

Magnetization current $I_m = 2\pi R M$, $i_m = M$ - linear current den.

- $H = \frac{B_0}{\mu_0} = \frac{B}{\mu_0} - \vec{M}$, $\int \vec{H}\cdot d\vec{r} = NI$, $\vec{B} = \mu_0 k_m \vec{M} = \mu\vec{H}$

- Boundary $B_{1\perp} = B_{2\perp}$, $H_{1\parallel} - H_{2\parallel} = j$ or $H_{1\parallel} = H_{2\parallel}$, $j = 0$.

Heat and Thermodynamics (phys 203)

1. Thermodynamic Systems

When we make a physical discussion of a certain portion of matter what we really do is separate this portion from the environment external to it. We call this portion a system - the next step is to examine how the system interacts with the surrounding and this enables us to describe the system. But here we need to choose suitable observable quantities to describe the behaviour of the system. If these quantities specify the bulk properties of the system, we call them macroscopic quantities. If in a process in which heat is involved, we talk about relations that involve macroscopic quantities such as pressure, volume, temp., internal energy and entropy (among others), then we are dealing with the so called thermodynamics.

On the other hand, we can have microscopic studies of the system by considering the microparticles that make up the system. Now we are giving the so called statistical description of our system.

From the foregoing discussion then we see that there are two ways of describing one and the same thing - the system. We then expect that the macroscopic and microscopic quantities are to be related in some way. In particular we can express the former in terms of the latter. For example, consider a gas in a container as our system. Macroscopically we can measure its pressure using a manometer. But from the microscopic point of view, the pressure is related to the average rate of mom. per unit area which the molecules of the gas deliver when they strike the walls of the container.

1.1 The Zeroth Law of Thermodynamics

Let us consider two identical bodies A & B that are at diff. temperatures which are placed in contact with one another. We can measure their temperatures by a thermometer. After a certain time interval we will find (using our thermometer) that they are both at the same final temp. We then say they are in thermal equil. and we state the Zeroth law of thermodynamics as:

> If two bodies A & B are each in thermal equil. with a third body C (the thermometer), then A & B are in thermal equil. with each other.

1·2 Heat and Energy

Any theory about heat should satisfactorily explain the following facts:

(i) thermal expansion (ii) conservation of heat, i.e. when there is transfer of heat between two bodies the heat lost by one body is equal to the heat gained by the other.

(iii) Specific heat of bodies

(iv) thermal capacity of bodies

(v) Changes of State (vi) propagation of heat radiation in space with the speed of light.

Two theories have been proposed to explain the properties and the effects of heat. These are

(i) the caloric theory (ii) the dynamic theory or the kinetic theory.

The first assumes heat as some type of fluid and in the course of heat transfer from one body to another, this fluid is transferred. However, experiments due to Joule, Rumford and Davy have shown that heat is a form of energy associated with the mechanical motion of material particles. Thus the kinetic theory favoured the dynamical theory and the failure of the caloric theory.

According to Joule, heat and work are equivalent to a scale factor J i.e.)

$$W = JQ$$

where W is the work, Q is the quantity of heat and J (= 4·18 J/cal) is the mechanical equivalent of heat, the calorie being the unit of heat. The other unit is the British Thermal unit (BTU), 1 BTU = 252 cal.

1·2·1 Specific Heat capacity

Substances differ from one another in the quantity of heat needed to produce a given rise of temp. in a given mass. The ratio of the amount of heat energy ΔQ supplied to a body to its corresponding temperature rise ΔT is called the heat capacity C of the body i.e.)

$$C = \Delta Q / \Delta T \qquad [C] = \left[\frac{Q}{T}\right] = \frac{cal}{^\circ C} \; \frac{J}{K} \; \frac{BTU}{^\circ F}$$

The heat capacity per unit mass of a body is called the specific heat, denoted c.

$$c = \Delta Q / m \Delta T \qquad [c] = \left[\frac{Q}{MT}\right] = \frac{cal}{g \cdot ^\circ C} \; \frac{J}{kg^\circ C}$$

It is the characteristic of the material of which the body is composed. But the specific heat depends in general on temperature. At room temp. and over it is a constant taking a fixed value.

The last eq. above does not define the specific heat uniquely. One has to specify the conditions under which the heat ΔQ is added to the specimen. There are many possibilities each leading, in general, to a diff. value of c. To obtain a unique value of c we must specify the conditions, such as specific heat at constant volume c_v, at constant pressure c_p, and so on.

The above discussion tells us that energy must be added to raise the temp. of a body. Energy changes also occur when a material changes from one state to another. Melting (solid to liquid), vaporization (liquid to gas) or sublimation (changing directly from solid to gas) all require the addition of energy. The reverse processes involve the loss of energy from the body. The amount of heat energy per unit mass that must be transferred when any of these processes take place is called the specific heat of transformation or the specific latent heat. It is designated by l. The amount of heat required for a specific change is sometimes given a special name. For example, the heat required to vaporize one unit mass of water at atmospheric pressure is called the heat of vaporization of water. Its value is 539 cal/g. The heat of fusion of water is 80 cal/g and represents the amount of heat that must leave water when it freezes at atmospheric pressure. As with specific heats, heats of transformation vary with the external conditions under which the phase changes take place.

1.2.2 Change of State

Any substance exists in three states viz., solid, liquid and gas. The state of a sub. depends on its temp.

On the other hand we know that subs. are made up of micro-particles - molecules which are in constant motion. But the molecules of a solid sub. are not so free as those of a liquid and in turn these ones are not so free as those of a gas. It is thus clear that the difference between the three states of matter is due to a difference in the motion of the molecules. And to bring a change of state we have to either supply or withdraw heat from the substance. Ice is the solid state of water. By supplying heat to ice it can be changed into water. Similarly by supplying heat to water it is converted into steam which is the gaseous state. The reverse process occur when heat is withdrawn. This is true for all substances. For example permanent gases like oxygen, nitrogen and hydrogen can be liquified at low temperatures.

For a given sub., the change of state occurs at a fixed temp. and at a given pressure and the vice versa. When the

Sub. changes from the solid to the liquid state, the heat supplied at const. temp. (melting point) is used in overcoming the forces of intermolecular attraction. The mean molecular dist. increases and the molecules are more free to move. similarly, when the sub. changes from the liquid to the gaseous state at a fixed temp. (called the boiling point) at standard atmospheric pressure, the heat supplied is used in increasing the mean molecular dist. They are then free to move in the whole space available to them.

Latent Heat of Fusion

Take small pieces of ice in a beaker. Fix a thermometer to note the temp. of ice in the beaker. Heat the beaker slowly. Now the ice melts without the thermometer showing any rise in temp. When the whole ice melts the temp. of water rises and it is indicated by the thermometer. The heat supplied is thus used to change the solid state of ice into the liquid state. This heat is called the latent heat.

Ice takes 80 calories of heat for each gram that melts into water i.e., the latent heat of fusion is 80 cal/g. This same amount of heat is given off when water freezes to ice at 0°C.

Laws of Fusion

1. Every sub. changes its state from solid to liquid at a particular temp. (under normal pressure) called the melting point.
2. As long as the change of state takes place, there is no change in temp.
3. One gram of every sub. requires a definite quantity of heat for change of state from solid to liquid and it is called the latent heat of fusion of that sub. It is diff. for diff. subs.
4. Some subs. show increase in volume on melting e.g. wax while some other subs. show decrease in volume on melting, e.g. ice.
5. The melting point of those subs. which decrease in volume on melting is lowered with increase in pressure.
6. The melting point of those subs which increase in volume on melting is increased with increase in pressure.

Vaporization and Condensation

The conversion of a sub. from liquid to vapour state is called vaporization. Boiling is the phenomenon of vaporization accompanied by ebullition of bubbles. However, vaporiz-

ation can take place without boiling. For example, the water kept on an open plate in summer evaporates and there is a change of state. But there is no boiling. The conversion of water to steam is vaporization. The conversion of a sub. from the vapor state to the liquid state is called Condensation. So conversion of steam to water is condensation.

Laws of Vaporization

1. In every liquid changes its state from liquid to vapour at a particular temp. (under normal pressure) called the boiling point.
2. As long as change of state takes place, there is no change in temp.
3. One gram of every liquid requires a definite quantity of heat for change of state from liquid to vapour and it is called the latent heat of vaporization. It is diff. for diff. subs.
4. All liquids show increase in volume on vaporization
5. The boiling point of a liquid increases with increase in pressure
6. The liquid can boil at a lower temp. under reduced pressure.

Def. The latent heat of vaporization of a liquid is the amount of heat required to convert one gram of liquid into vapour without any change in temp.

For water the latent heat of vaporization is 587 cal/g at a pressure of 76 cm of Hg.

Examples:

1. 1 Kg of ice at -10°C is heated untill the whole of it evaporates. How much heat is required? Latent heat of fusion of ice = 80 cal/g and latent heat of vaporization of water is 540 cal/g.

Solution

- Heat required to raise the temp. of ice from -10°C to 0°C (C_w = 0.5 cal/g·K)

  = 1000 g × ½ cal/g·K × 10 K = 5000 cal

- Heat required to melt ice at 0°C

  = 1000 g × 80 cal/g = 80,000 cal

- Heat required to raise the temp. of water from 0°c to 100°c.
  = 1000g × 1 cal/g·K × 100 K = 100,000 cal
- Heat required to convert water into Steam at 100°c
  = 1000g × 540 cal/g = 540,000 cal

∴ The total quantity of heat required is

Q = (5000 + 80,000 + 100,000 + 540,000) cal
= 725,000 cal

2. 1g of Steam is passed into a calorimeter containing 100 g of ice. Calculate the amount of ice melted.

Solution

Heat given out by 1g of Steam when converted into water at 100°c

= 1g × 537 cal/g = 537 cal

Heat given out by 1g of water when cooled from 100°c to 0°c

= 1g × 1 cal/g·K × 100K = 100 cal

Total quantity of heat given out by 1g of Steam
= (537 + 100) cal = 637 cal

Let the amount of ice melted be m grams.
Heat absorbed by ice when it melts to water at 0°c is (80 cal/g) m.

Heat lost by Steam = Heat gained by ice

637 cal = (80 cal/g) m

m = (637/80) g

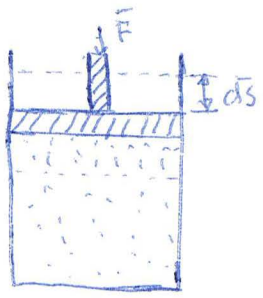
or m = 7.96g, which is the amount of ice that has melted to water at 0°c.

1-3. Heat and Work

Heat, as we have discussed, is the energy that flows from one body to another because of a temp. difference between them (ie) heat is energy in transit. Thus heat is not a property of a body in the sense that we can assign a value to the amount of heat "contained" in a body.

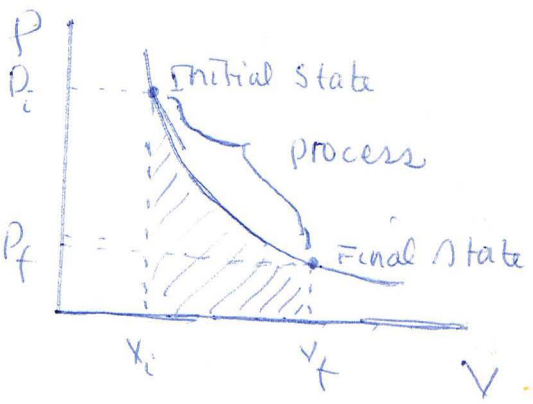
Work is similar to heat in that it is a measure of energy being transmitted from one body to another. Thus work is not a specific property of a body which we can assign a value to work "contained" in the body. But work is a measure of energy transfer by mechanical means, such as by gravitation, electric or magnetic forces. ~~Heat~~ On the other hand, heat is a measure of energy transfer by means of temp. difference.

Suppose now we wish to discuss a gas contained in a cylinder that is fitted with a movable piston (below).



The environment with which the system of molecules ~~interacts~~ of the gas interacts would be the cylinder walls and the piston. The quantities that describe the behaviour of the gas would be the pressure, temperature, volume, and chemical state of the gas. A thermodynamic process in such a system might involve a transference of heat energy Q to the gas as a result of a temp. difference between ~~between~~ the system (gas) and its environment (the cylinder walls). The gas might ~~cause~~ do work W on its environment (the piston) by causing the piston to move because of the force the molecules exert while striking the piston. During this process the gas might change from a configuration described by initial parameters P_i, V_i and T_i to a final configuration with parameters P_f, V_f and T_f. (If changes of state and chemical reactions are involved, they will be described by other appropriate parameters.)

If the changes in the system occur slowly enough that all parts of the gas are always at the same temp. (i.e., the gas is in thermal equil.), the sample of gas is characterized by a specific temp., pressure, and volume at each point in the process. Such a process is called quasi-static. The state of the gas at ~~each~~ any time and the process itself can then be represented in a diagram of pressure versus volume (p-v diagram) as shown for a special case ~~below~~ in the fig. below.



A p-v curve representing the slow expansion of a gas. The work done by the gas is equal to the area under the curve.

The laws of thermodynamics allow us to calculate certain relationships between the initial and final parameters of the system on the one hand and the energy transfers Q and W on the other.

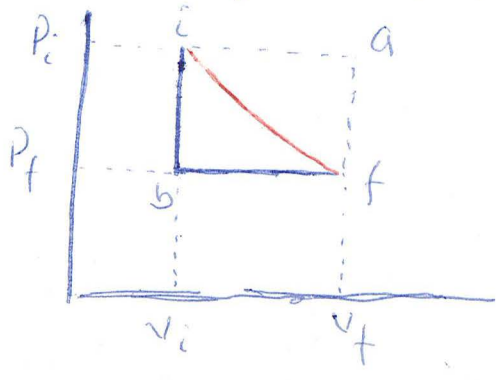
Let us calculate W for the gas expanding in the cylinder (first figure). The work done by the gas in displacing the piston through an infinitesimal dist. ds is

$$dW = \vec{F} \cdot d\vec{s} = pA\,ds = p\,dV$$

where dV is the increase in the volume of the gas as it expands. In general the pressure will not be constant during the process. The total work W done on the piston is obtained by integration (i.e.,

$$W = \int_{V_i}^{V_f} p\,dV$$

There are many diff. ways in which the system can be taken from the initial state i to the final state f. For example (the fig. below), the pressure may be kept constant



from i to a and then the volume kept constant from a to f. Then the work done by the expanding gas is equal to the area under the line ia. Another possibility is the ~~line~~ path ibf, in which case the work done by the gas is the area under the line bf. The continuous curve from i to f is another possible path, in which the work done by the gas is still diff. from the previous ones. We can see, therefore, the work done by a system depends not only on the initial and final states but also on the intermediate states, i.e., on the details of the process.

A similar result follows if we compute the flow of heat during the process.

1.3.1 Isobaric process

If the system (working substance) is taken in an expanding chamber kept at a constant pressure, the process is called an isobaric process. In this process, the temp. and volume of the system change. If an amount ΔQ of heat is given to the system, it is partly used in increasing the temp. of the system by ΔT and partly used in doing external work. We have seen that when a gas expands the work it does on its environment is

$$W = \int p\,dV$$

where p is the pressure exerted by the gas and dV is the increase in volume of the gas. So in this process if the pressure remains constant and the volume changes by a finite amount, say, from V_i to V_f, then the work done by the gas will be

$$W = \int_{V_i}^{V_f} p\,dV = p \int_{V_i}^{V_f} dV = p(V_f - V_i)$$

1.3.2 Adiabatic process
Boyle - Mariottes Law (Gas Thermometer)

A process that takes place in such a way that no heat flows into or out of the system is called an adiabatic process. Experimentally such processes are achieved either by sealing the system off from its surroundings with heat-insulating material or by performing the ~~system~~ process quickly. Because the flow of heat takes time, any process can be made practically

adiabatic if it is performed quickly enough.

When work is done on the working sub (system), there is rise in temp. because the external work done on the system increases its internal energy. When work is done by the working ~~sub~~ system, it is done at the cost of its internal energy. If the system is perfectly insulated from the surroundings, there is fall in temp.

Thus during an adiabatic process, the system is perfectly insulated from the surroundings. All along the process there is change in temp.

Examples of Adiabatic process

1. The compression of the mixture of oil ~~and~~ vapour and air during compression stroke of an internal combustion is an adiabatic process and there is rise in temp.
2. The expansion of the combustion products during the working stroke ~~stroke~~ of an engine is an adiabatic process and there is fall in temp.
3. The sudden bursting of a cycle tube is an adiabatic process.

In general the processes that take place suddenly or quickly are adiabatic processes.

1.3.3 Gases

In gases the intermolecular dist. is larger than that of a solid or a liquid and the molecules of a gas are free to move about in the entire space available to them. Hence a gas has no shape. If the space occupied by a gas is increased, the gas will occupy the whole of the new space uniformly. This is due to the fact that the molecules of a gas are always in rapid random motion and the force of inter molecular attraction is negligibly small. The molecules move about independently in straight lines and the only restriction to this movement is collision with another molecule or collision with the walls of the container.

1.3.3.1 The concept of Ideal or perfect Gas and its Equation of State

Let a certain amount of gas be confined in a container of volume V. We can reduce the density of this gas

either by removing some gas from the container or by putting the gas in a larger container. We find from experiment that, at low densities all gases tend to show a certain simple relation among the thermodynamic variables P, V and T and the intermolecular attractions are negligibly small. This is then a simplified description and any gases obeying this situation (or restriction) is called an ideal gas.

The relation between the thermodynamic variables P, V and T of a gas is called the equation of state of the gas. (at low density).
For low enough values of the density experiments show that (1) for a given mass of gas held at a constant temp., the pressure is inversely proportional to the volume (Boyle's law), and (2) for a given mass of gas held at a constant temp-pressure, the volume is directly proportional to the temp. (Charles law). These experimental facts are summarized by a single relation

$$\frac{PV}{T} = \text{constant}$$

The volume occupied by a given gas at a given pressure and temp. is proportional to its mass. Thus the constant in the above eq. must also be proportional to the mass of the gas. This constant is written as nR, where n is the no. of moles of the gas in the sample and R is a const. that must be determined for each gas by expt. At low enough densities R has the same value for all gases, namely,

$$R = 8.314 \text{ J/mol.K} = 1.986 \text{ cal/mol.K}$$

It is called the universal gas constant.
We then write the ideal gas eq. of state

$$PV = nRT.$$

So an ideal gas obeys this eq. under all conditions. Actually there is no such thing as truly ideal gas, but all real gases approach the ideal gas abstraction in their behaviour if their densities are low enough.

1.4 Thermodynamic Functions

We have diff. quantities which are functions of some independent variables and are used to specify

(state)
the behaviour of a physical system. These are.

1. The internal energy E : this is the sum total of energy of all kinds that a body may have and it includes chemical, mechanical, etc. energies of the body.

2. The enthalpy, H: It is given by the relation H=E+PV

3. The Helmholtz free energy F: It is a function given as F = E-TS where S is called the entropy.

4. The Gibbs free energy G. It is defined as G = F+PV

By using these functions then we can see whether a given system is in Thermodynamic equil. or not i.e., whether the system has fixed values of pressure, temp. (gives equil.) or not.

Reversible Processes

Suppose that a system is taken from its initial state to a final state. This can be brought about by a process that takes place within the system. Now if the system can definitely come back from its final state to its initial state by fulfilling or restoring the constraints of this earlier state the process is called reversible.

Or a reversible process is one that can take place in the reverse order and bring the system back to its initial configuration if the constraints of the initial configuration are restored.

As an example consider a cylinder containing a gas at a certain pressure and temp. The cylinder is fitted with a frictionless piston. If the pressure is decreased the gas expands slowly and maintains a constant temp. (isothermal process). The energy required for this epansion is continuously drawn from the surroundings. If the pressure on the piston is increased, the gas contracts slowly and maintains constant temp. (isothermal process). The energy liberated during compression is given to the surroundings. This is also true for an adiabatic process provided the process takes place infinitely slowly.

The process will not be reversible if there is any loss of heat due to friction, radiation or conduction. If the changes take place rapidly, the process will not be reversible. The energy used in overcoming friction cannot be retraced.

The conditions of reversibility for any process can be stated as follows.

1. The pressure and temp. of the system must not differ appreciably from those of the surroundings at any stage of the process.

2. All the processes taking place must be infinitely (infinitesimally) slowly.

3. The ~~working~~ process takes place without the effects of friction.

Note however that a complete reversible process is only an ideal case. In an actual process, there is always loss of heat due to friction, conduction or radiation.

1.4.1 The First Law of Thermodynamics

Let a system change from an initial (equilibrium) state i to a final equilibrium state f in a definite way, the heat absorbed by the system being Q and the work done by the system being W. Then we compute Q-W. Now we start the process again and change the system from the same state i to the same state f, but this time by a diff. path. We do this repeatedly using diff. paths each time. We find that in every case the quantity Q-W is the same (i.e., although Q & W each depend on the path taken, Q-W does not depend at all on how we take the system from state i to state f but only on the initial and final (equil.) states.

In mechanics we know that when an object moves from an initial point to a final point under the effect of gravity, the work done does not depend on the path taken by the object but it rather depends on the positions of the two points. This means that there is a potential energy function of the space coordinates of the object whose final value minus its initial value equals the work done on this object. In thermodynamics, too, experiments tell us that when a system changes from a state i to a state f, the quantity Q-W depends only on these two states and not at all on the path taken between these end points. We conclude that there is a function whose final value minus its initial value equals the change Q-W. We call this function the internal energy.

Let U denote the internal energy of the system. Then

$$\Delta U = U_f - U_i = Q - W.$$

where U_i is the internal energy of the system in state i and U_f in state f.

This eq. is known as the first law of Thermodynamics.

In applying the above eq. Q is considered to be positive when heat enters the system and W is positive when work is done on the system.

If the system undergoes only an infinitesimal change in state, only an infinitesimal amount of ~~work is done~~ heat dQ is absorbed and only an infinitesimal amount of work dW is done so that the internal energy change is also infinitesimal dU. So in differential form the first law becomes

$$dU = dQ - dW$$

The first law applies to every process in nature that starts in one equil. state and ends in another. As mentioned earlier a system is in equil. when it can be described by a set of constant parameters such as pressure, temp., volum, magnetic field etc.

2.4.2 Some Applications of the First Law of Thermodynamics

1. Isobaric process:

As calculated in Sec. ( ) the work done by a gas when expanding at const. pressure is

$$W = p(V_f - V_i)$$

Consider the change of phase of a mass m of liquid to a vapour occurring at const. pressure and temp. Let V_ℓ be the volume of the liquid and V_v the volume of the vapour. The work done by the sub. in expanding from V_ℓ to V_v at const. pressure is

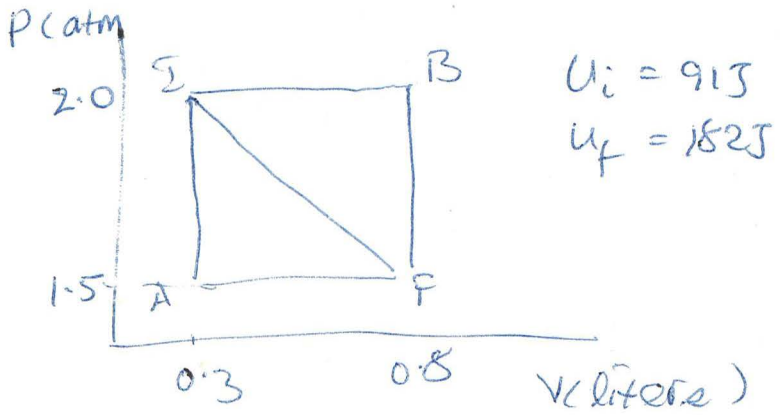
$$W = p(V_v - V_\ell)$$

Let ~~H~~ ℓ be the specific heat of vaporization. Then the heat absorbed by the mass m of liquid during change of phase is

$$Q = m\ell$$

From the first law of Thermodynamics, we have

Ex. One mole of gas initially at a pressure of 2 atm and a volume of 0.3 liters has an internal energy equal to 91 J. In its final state, the pressure is 1.5 atm, the volume is 0.8 liters, and the internal energy equals 182 J. For 3 paths IAF, IBF, and IF in the fig. below, Calculate a, the work done by the gas b, the net heat transferred in the process.



$U_i = 91\,J$

$U_f = 182\,J$

Path IAF

a, $W = \int_{IAF} p\,dV = \int_{IA} p\,dV + \int_{AF} p\,dV = \int_{AF} p\,dV = 1.5\times1.01\times10^5 \int_{0.3\,lit}^{0.8\,lit} dV = 1.5\times1.01\times10^5\,(0.8-0.3)\times10^{-3}\,J$

or, $W = 75.75\,J$

b, $Q = \Delta U + W = (192-91)\,J + 75.75\,J = 176.75\,J$

Path IBF

a, $W = \int_{IBF} p\,dV = \int_{IB} p\,dV = 2\times1.01\times10^5 \int_{0.3}^{0.8} dV = 2\times1.01\times10^5\,(0.8-0.3)\,10^{-3}\,J$

or, $W = 1.01\times10^2\,J = 101\,J$

b, $Q = \Delta u + W = (182-91)\,J + 101\,J = 192\,J$

Path IF

a, $\frac{1.5-2}{0.8-0.3} = \frac{p-2}{v-0.3}$ or $p = 2/(V-0.3)\,M(p\,(-v+2.3)\,1.01\times10^5\,J$

$W = 1.01\times10^5 \int_{0.3}^{0.8} (-v+2.3)\,dV = 1.01\times10^5 \left(-\frac{v^2}{2}+2.3v\right)_{0.3}^{0.8} = 1.01\times10^5\times0.875 = 8.84\times10^4\,J$

b, $Q = \Delta u + W = (192-91)\,J + 8.84\times10^4\,J = 88491\,J.$

$$\Delta U = Q - W$$

i.e.)

$$\Delta U = mL - p(V_0 - V_L)$$

2. Adiabatic process.

For an adiabatic process $Q = 0$ and the first law gives

$$\Delta U = U_f - U_i = -W$$

Hence, the internal energy of a system increases exactly by the amount of work done on the system and the internal energy decreases by exactly the amount of work done by the system on its environment.

1.4.3 Cycles

Suppose that we have a system (a real gas, say) in an equil. state in a cylinder-piston arrangement. By making changes in the environment of the system we can carryout, as we please, a wide variety of processes. We can let the gas expand or we can compress it; we can add or substract heat energy; we can do these things and others reversibly or irreversibly (a process in which a system never attains its initial equil. state (coming back in the reverse order.) again). We can also do a sequence of processes such that the system returns to its original equil. state; we call this a cycle. If the processes involved are all reversible, we call it a reversible ~~order~~ cycle.

1.4.3.1 The Carnot cycle

An imp. concept in the study of cycles is the concept of a heat engine which is used to convert heat into mechanical work. Sadi carnot (French) conceived of an engine which is free from all the defects of practical engines. Its efficiency is max. and it is an ideal heat engine.

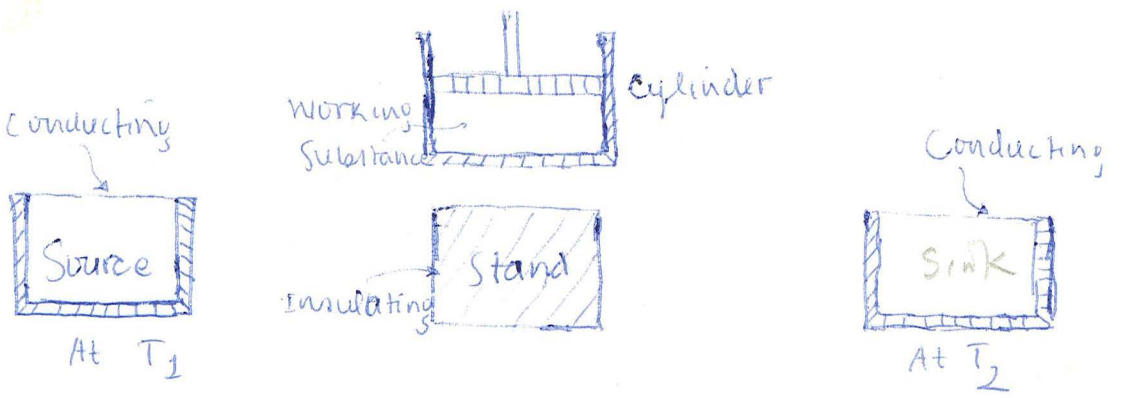
For any engine there are three essential requisites:

(1) Source. It is a device having infinite thermal capacity and which can ~~absorb~~, ~~retain~~ or reject unlimited quantity of heat without any change in its temp. Thus the

source should be at a fixed high temp. T_1, say, from which the heat engine can draw heat.

(2) sink :- This also has infinite thermal capacity and its temp. remains constant at lower value T_2, say. so any amount of heat can be rejected to the sink without its temp. being changed.

3) working substance :- A cylinder with non-conducting sides and conducting bottom contains a gas as the working sub.

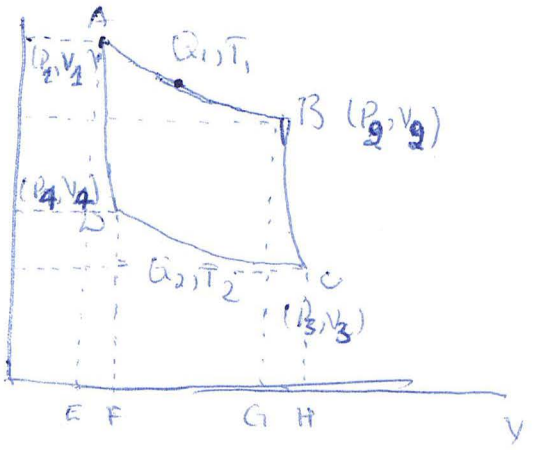


A perfect nonconducting and frictionless piston is fitted into the cylinder. The working sub. undergoes a complete cycle operation (as shown above).

A perfectly nonconducting stand is also provided so that the working sub. can undergo adiabatic operation.

Let us now come to the discussion of the Carnot cycle.

The working sub. (system) is a gas and the cycle consists of two isothermal (const. temp.) and two adiabatic reversible processes. The cycle is shown on the P-V diagram below.



Step 1 Place the engine containing the working sub. over the source at the temp. T_1. The working sub. is also at a temp. T_1. Its pressure is P_1 and volume V_1 as shown by point A in the above fig. Decrease

the pressure. The volume of the working sub. increases. Work is done by the working sub. As the bottom of the cylinder is perfectly conducting to the source, it absorbs heat and maintains the initial temp. T_1 at all stages of the expansion. The process is completely isothermal. Let the amount of heat absorbed by the working sub. be Q_1 at the temp. T_1. Then point B is obtained.

Consider one mole of the working sub. (perfect gas).

Work done from A to B (isothermal process) is, say

$$W_1 = \int_{V_1}^{V_2} P\,dV = RT_1 \log_e \frac{V_2}{V_1} = \text{area of } ABGE$$

Step 2: place the engine on the stand having an insulated top. Decrease the pressure on the working sub. The volume increases. The process is completely adiabatic. Work is done by the working sub at the cost of its internal energy. The temp. falls. The working sub. undergoes adiabatic change from B to C. At C the temp. is T_2.

Work done from B to C (adiabatic process) is, say, W_2

$$W_2 = \int_{V_2}^{V_3} P\,dV$$

$$= k \int_{V_2}^{V_3} \frac{dV}{V^\gamma}, \quad \text{since for an adiabatic process } PV^\gamma = \text{const.} = k,\ \gamma = \frac{c_p}{c_v}.$$

$$= \frac{k V_3^{1-\gamma} - k V_2^{1-\gamma}}{1-\gamma},$$

$$= \frac{P_3 V_3 - P_2 V_2}{1-\gamma}, \quad P_3 V_3^\gamma = P_2 V_2^\gamma = k$$

$$= \frac{R(T_2 - T_1)}{1-\gamma} = \frac{R(T_1 - T_2)}{\gamma - 1}, \quad \text{since } P_2 V_2 = RT_1 \text{ and } P_3 V_3 = RT_2.$$

$$= \text{area of } BCHG$$

Step (3): place the engine on the sink at temp. T_2. Increase the pressure. Work is done on the working sub. As the base is conducting to the sink, the process is isothermal. A quantity of heat Q_2 is rejected to the sink at temp T_2. Finally point D is reached.

Work done from C to D (isothermal process) is say, W_3

$$W_3 = \int_{V_3}^{V_4} P\,dV = RT_2 \log_e \frac{V_4}{V_3} = -RT_2 \log_e \frac{V_3}{V_4} = -\text{area of } CHFD$$

"-" indicates that work is done on the sub.

(4) Place the engine on the insulating stand. Increase the ~~temp~~ pressure. The volume decreases. The process is completly adiabatic. The temp. rises and finally point A is reached.

Work done from D to A (adiabatic process) is say, W_4

$$W_4 = \int_{V_4}^{V_1} P\,dV = -\frac{R(T_1 - T_2)}{\gamma - 1} = -\text{area of DFEA}$$

Now the net work done by the working sub. in one complete cycle is

$$W = W_1 + W_2 + W_3 + W_4$$
$$= W_1 + W_3$$
$$= RT_1 \log_e \frac{V_2}{V_1} - RT_2 \log_e \frac{V_3}{V_4}$$

Points A & D are on the same adiabatic.

$$T_1 V_1^{\gamma-1} = T_2 V_4^{\gamma-1} \Rightarrow \frac{T_2}{T_1} = \left(\frac{V_1}{V_4}\right)^{\gamma-1}$$

Similarly, for points B and C we have

$$\frac{T_2}{T_1} = \left(\frac{V_2}{V_3}\right)^{\gamma-1}$$

i.e., $\left(\frac{V_1}{V_4}\right)^{\gamma-1} = \left(\frac{V_2}{V_3}\right)^{\gamma-1}$ or $\frac{V_1}{V_4} = \frac{V_2}{V_3}$ or $\frac{V_3}{V_1} = \frac{V_3}{V_4}$

and hence

$$W_1 = RT_1 \log_e \frac{V_2}{V_1} - RT_2 \log_e \frac{V_2}{V_1}$$
$$= R \log_e \frac{V_2}{V_1} (T_1 - T_2)$$

But the net amount of heat absorbed by the working sub. in the cycle is $Q_1 - Q_2$. Further, since the system has come back to its initial state, there is no net change in its internal energy. Thus the first law of thermodynamics gives

$$W = Q_1 - Q_2$$

Def. Efficiency $= \eta = \frac{\text{output}}{\text{Input}} = \frac{W}{Q_1}$

for our case $\eta = \frac{Q_1 - Q_2}{Q_1} = 1 - \frac{Q_2}{Q_1}$

Heat is supplied from the source from A to B only.

$$Q_1 = RT_1 \log_e \frac{V_2}{V_1}$$

$$\therefore \eta = \frac{W}{Q_1} = \frac{Q_1 - Q_2}{Q_1}$$

$$= \frac{R(T_1 - T_2)\log_e \frac{V_2}{V_1}}{RT_1 \log_e \frac{V_2}{V_1}}$$

$$\text{or } \eta = 1 - T_2/T_1$$

The Carnot's engine is perfectly reversible. It can be operated in the reverse direction also. Then it works as a refrigerator. That is the heat Q_2 is taken from the sink and external work is done on the working sub. such that it gives heat Q_1 ($> Q_2$) to the source at a higher temp.

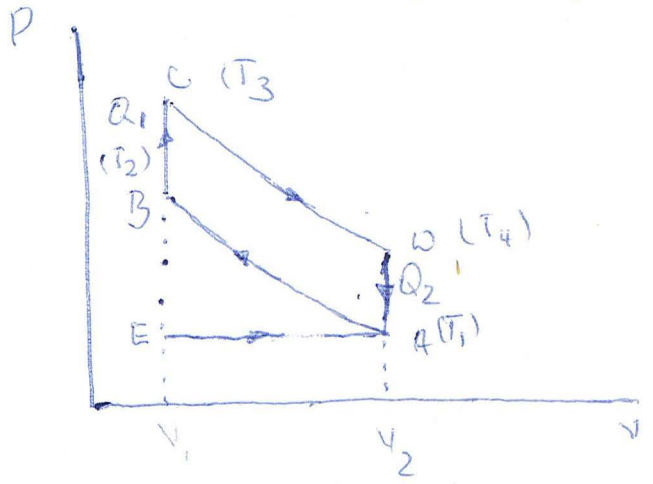
The isothermal process will take place only when the piston moves slowly to give enough time for the heat transfer to take place. The adiabatic process will take place when the piston moves extremely fast ~~and~~ to avoid heat transfer. Any practical engine cannot satisfy these conditions & le practical engines have an efficiency less than the Carnot's engine.

1.4.3.2 The Otto cycle

In the Otto engine (internal combustion engine), air is the working substance and petrol vapour acts as the fuel. The following conditions are assumed in the discussion of the otto cycle:

1. The working sub. is air all the time and it behaves as a perfect gas.
2. There is no friction.

(fig. below) On the basis of these ideal conditions, the otto cycle is divided into the following operations:



1. E to A represents the charging stroke. The mixture of

air and petrol vapour is allowed to enter at atmospheric pressure. The final volume is V_2 at temp. T_1.

2. A to B represents the adiabatic compression. There is no friction and no flow of heat through the walls of the cylinder. The volume changes from V_2 to V_1 and the temp. changes from T_1 to T_2. There is change in pressure also. This represents the compression stroke. The compression ratio is nearly 8 i.e., $V_2/V_1 = 8$. The pressure changes from one atmosphere to about 18 atmospheres.

3. B to C represents the ignition stage. A spark is produced and the mixture of air and petrol vapour is ignited. The pressure increases from 18 to about 80 atmospheres and the temp. changes from T_2 to T_3 (from 500°C to 2000°C).

4. C to D represents the working stroke. The gas expands adiabatically and the engine works. The volume changes from V_1 to V_2. The pressure and temp. decrease.

5. As the point D is reached, the exhaust valve opens. D to A represents the change of pressure to the atmospheric pressure and the temp. changes from T_4 to T_1.

6. A to E represents the exhaust stroke. The exhaust gases are completely discharged from the cylinder. Thus the initial condition of the engine is restored.

Efficiency :- Consider that 1g of the working sub. is used in the process. A quantity of heat Q_1 at a higher temp. is absorbed from B to C

$$Q_1 = 1 \cdot C_v (T_3 - T_2)$$

A quantity of heat Q_2 is rejected at a lower temp. from D to A.

$$Q_2 = 1 \cdot C_v (T_4 - T_1)$$

Thus,

$$\frac{Q_2}{Q_1} = \frac{T_4 - T_1}{T_3 - T_2}$$

$$\Rightarrow \eta = 1 - \frac{Q_2}{Q_1} = 1 - \frac{T_4 - T_1}{T_3 - T_2}$$

The points C and D lie on the same adiabatic

$$\therefore T_4 V_2^{\gamma-1} = T_3 V_1^{\gamma-1}$$

The points A and B lie on the same adiabatic

$$T_1 V_2^{\gamma-1} = T_2 V_1^{\gamma-1}$$

Then using these two last eqs.

$$(T_4 - T_1) V_2^{\gamma-1} = (T_3 - T_2) V_1^{\gamma-1}$$

i.e.)

$$\frac{T_4 - T_1}{T_3 - T_2} = \left(\frac{V_1}{V_2}\right)^{\gamma-1}$$

Substituting this expression for $(T_4 - T_1)/(T_3 - T_2)$ in the eq form one has

$$\eta = 1 - \left(\frac{V_1}{V_2}\right)^{\gamma-1} = 1 - \left(\frac{1}{V_2/V_1}\right)^{\gamma-1} = 1 - \left(\frac{1}{\rho}\right)^{\gamma-1}$$

Where ρ is called the adiabatic compression ratio.

1.4.7. Entropy and the Second Law of Thermodynamics

1.4.1 The Second Law of Thermodynamics

A heat engine is chiefly concerned with the conversion of heat energy into mechanical work. On the other hand, a refrigerator is a device to cool a certain space below the temp. of its surroundings. The first law of Thermodynamics is a qualitative statement which does not preclude the possibility of the existence of either a heat engine or a refrigerator. The first law does not contradict the existence of a 200% efficient heat engine or a self-acting refrigerator.

In practice, these two are not attainable. These phenomena are recognized and this led to the formulation of a law governing these two devices. It is called the second law of Thermodynamics.

Kelvin and Planck stated the second law as follows.

"It is impossible to get a continuous supply of work from a body (or engine) which can transfer heat with a single heat reservoir." According to this statement, a single reservoir at a single temp cannot continuously transfer heat into work. It means there should be two reservoirs for any heat engine one reservoir (called source) is taken at a higher temp. and the other reservoir (called sink) is taken at a lower temp.

Further, the second law implies zero degree absolute temp. is not attainable, because if an engine works between any temp. higher than zero degree kelvin and zero degree kelvin, it means it uses a single reservoir which contradicts the Kelvin-Planck statement of the second law. Similarly no engine can be 100% efficient.

The Second law has two parts:

1. According to Kelvin,

"It is impossible to get a continuous supply of work from a body by cooling it to a temp. lower than that of its surroundings"

In a heat engine the working sub. does some work and rejects heat to the sink. The temp. of the source must be higher than the surroundings and the engine will not work when the temps. of the source and sink are the same.

E.g. Take a steam engine. The steam (working sub.) at high pressure is introduced into the cylinder of the engine. The temp. of the steam increases and hence expands to do external work. The contents remaining behind after doing work are rejected to the surroundings. The temp. of the working sub. rejected to the surroundings is higher than the temp. of the surroundings. If this working sub. rejected by this engine is used in another engine, it can do work and the temp. of the working sub. will fall further. It means the working sub. can do work only if its temp. is higher than that of the surroundings.

2. According to clausius.

"It is impossible to make heat flow from a body at a lower temp. to a body at a higher temp. without doing external work on the working sub"

This statement seems to be quite obvious and there are examples showing it to be true.

In general then the Second law of Thermodynamics gives us the conditions under which heat can be converted into work.

2.4.4.2 Entropy ~~and the Second law of Thermodynamics~~

The Second law of Thermodynamics is related to a thermodynamic variable called entropy. For a Carnot cycle we have seen that

$$\frac{Q_1}{T_1} = \frac{Q_2}{T_2}$$

in which the Qs were taken as positive quantities, i.e., we dealt only with the absolute values of the Qs. Considering them as algebraic quantities, Q being positive when heat enters the system and negative when heat leaves the system, we can write

$$\frac{Q_1}{T_1} + \frac{Q_2}{T_2} = 0$$

This eq. states that the sum of the quantity Q/T is zero for a Carnot cycle.

But any reversible cycle is equivalent to an assembly of Carnot cycles. So, for an arbitrary reversible cycle in which there are isothermal - adiabatic sequence of processes (as in the Carnot cycle) we write

$$\sum \frac{Q}{T} = 0$$

or, in the limit of infinitesimal temp. differences between the isotherms

$$\oint \frac{dQ}{T} = 0.$$

The sign $\oint$ indicates that the integral is evaluated for a complete traversal of the cycle, starting (and ending) at any arbitrary point of the cycle.

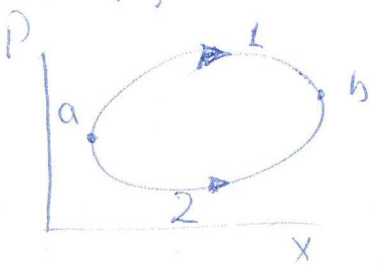
If the integral of a quantity around any closed path is zero, the quantity is called a State variable; i.e., it has a value that which is characteristic only of the state of the system, regardless of how that state was arrived at. We call the variable in this case the entropy S and we have

$$dS = \frac{dQ}{T} \quad \text{and} \quad \oint dS = 0.$$

Unit: $[S] = \left[\frac{Q}{T}\right] =$ Joule/°K, Calorie/°K.

Internal energy U, Helmholtz free energy F, Gibbs free energy G, the enthalpy H, the pressure P and the temp. T are other State variables and eqs. of the form $\oint dX = 0$ hold for each of them. Heat Q and work W are not state variables and in general $\oint dQ \neq 0$ and $\oint dW \neq 0$, as one can show for a special case of the Carnot cycle.

The property of a State variable expressed by $\oint dX = 0$ can also be expressed by saying that $\int dX$ between any two equil. States has the same value for all reversible paths connecting those States. Let us prove this for entropy. Consider the fig. below



$$\oint ds = 0 \Rightarrow {}_1\int_a^b ds + {}_2\int_b^a ds = 0$$

Since the cycle is reversible

$${}_1\int_a^b ds - {}_2\int_a^b ds = 0$$

or,

$${}_1\int_a^b ds = {}_2\int_a^b ds$$

This result tells us that the quantity $\int_a^b ds$ between any two equil. states a and b of the system is independent of the path connecting these states, for 1 and 2 are quite arbitrary paths.

The change in entropy between a and b is then

$$S_b - S_a = \int_a^b ds = \int_a^b \frac{dQ}{T} \quad \text{(reversible process)}$$

Where the integral is evaluated over any reversible path connecting these two states.

1.4.4.3 Entropy and The Second Law

The second law of Thermodynamics is a generalization from experience so that we cannot prove it. This law may also be stated as

Any process that starts in one equil. state and ends in another will go in the direction that causes the entropy of the system plus the environment to increase.

This is true only for what we call irreversible processes because it is only such processes that have a "natural direction". But reversible processes can go equally well in either direction and the entropy of the system plus the environment remains unchanged here. This is so because if heat dQ is transferred from the environment to the system the entropy of the environment decreases by dQ/T, whereas that of the system increases by dQ/T, the net change for the system plus environment being zero. The fact that the process is irreversible means that the environment and the system can differ in temp. only by a differential amount dT when heat transfer takes place whereas this is large for an irreversible process.

Another class of processes of interest are adiabatic processes (reversible or irreversible); they involve no transfer of heat with the environment, so the only entropy change possible is that of the system. From our statement of the second law and facts about reversible processes (above), we conclude that

$$S_f = S_i \quad \text{(reversible adiabatic process)}$$

and

$$S_f > S_i \quad \text{(irreversible adiabatic process)}$$

where S_f and S_i are the final and initial entropies of the system.

~~1.4.4~~ 2 The Kinetic Theory of Gases

Thermodynamics deals only with macroscopic variables, such as pressure, temperature and volume. Its basic laws, expressed in terms of such quantities, say nothing about the fact that matter is made up of atoms (or molecules).

On the other hand, the experiments on the conversion of heat and into work have clearly shown that heat is a form of energy, and this energy is connected with the motion of molecules of which matter is made of. The molecules are the smallest particles of a sub. that can exist in free state. The molecules possess the characteristic properties of the parent sub. The molecules are in a state of continuous motion with all possible velocities and the kinetic theory gives an explanation to the nature of this motion and the nature of heat energy. The kinetic theory of ~~gas~~ matter is based on the following three points:

(i) matter is made up of molecules

(ii) molecules are in rapid motion

(iii) molecules experience forces of attraction between one another.

So now if we have a gas kept in a certain container, the molecules of the gas will collide continuously with the walls of the container and reflect back in their original direction of motion. This will then result in change of momentum of the molecules. According to Newtons second law of motion, the rate of change of momentum per unit area of the wall surface

corresponds to the force exerted per unit area. The force per unit area measures the pressure of the gas.

Postulates of The Kinetic Theory of Gases

(i) The gas is composed of small particles called molecules. Depending on the nature of the gas, each mol. consists of one atom or a group of atoms.

(ii) The dist. between the molecules is large as compared to that of a solid or liquid and hence the forces of intermolecular attraction are negligible.

(iii) The molecules are continuously in motion with varying velocities and the molecules move in straight lines between any two consecutive collisions. The collisions do not alter the molecular density of the gas i.e., on the average the no. of molecules present in a unit volume remains the same. Also, the molecules do not accumulate at any place within the volume of the gas.

(iv) The size of the molecules is infinitesimally small as compared to the average dist. traversed by a mol. between any two consecutive collisions. The dist. between any two consecutive collisions is called free path and the average dist. is called the mean free path. The mean free path is dependent on the pressure of the gas. If the pressure is high the mean free path is less and if the " " low " " " " more.

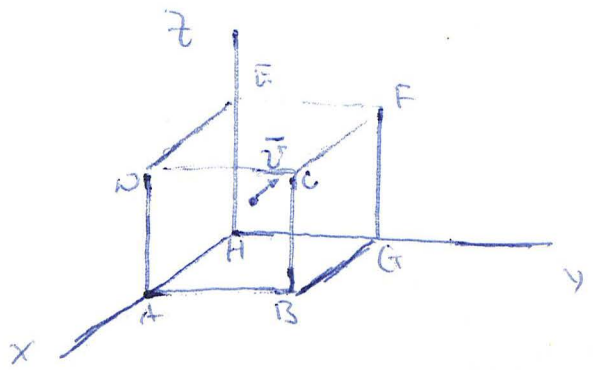
(v) The time of impact is negligible in comparison with the time taken to traverse the free path.

(vi) The molecules are perfectly hard elastic spheres and the whole of their energy is kinetic.

2.1 The process Kinetic Calculation of Pressure

As mentioned earlier, the continuous impact of the molecules on the walls of the containing vessel accounts for the pressure of the gas.

To simplify matters, consider a gas in a cubical vessel of side l whose walls are perfectly elastic (Fig. below)



Some Basic Concepts in Differential and Integral Calculus

1. **Differentiation**

Def. A function f defined on a domain I is said to be differentiable at the point x_0 if x_0 in I iff

$$\lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

exists. When this limit exists it is denoted by $f'(x_0)$ or $df(x_0)/dx$ or $Df(x_0)$ and is called the derivative at x_0. If the function is differentiable at every point of I then f is said to be differentiable on I and its derivative is a function on I denoted by $f'(x)$ or $\frac{df(x)}{dx}$ for all $x \in I$.

Examples.

1) Let $f(x) = x$ for all $x \in R$ (set of real nos.)

To find the derivative of f use the following steps

Step 1: $f(x+h) = x+h$

Step 2: $f(x) = x$

Step 3: $[f(x+h) - f(x)]/h = (x+h-x)/h = 1$

Step 4: $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h \to 0} 1 = 1$, for all $x \in R$, i.e.,

$f'(0) = f(-1/2) = f(3) = f(-1)$, etc. $= 1$.

2) $g(x) = 4x^2+3$, $x \in R = (-\infty, \infty)$

1. $g(x+h) = 4(x+h)^2 + 3$
2. $g(x) = 4x^2+3$
3. $\frac{g(x+h) - g(x)}{h} = \frac{4(x+h)^2+3 - (4x^2+3)}{h} = 8x+4h$
4. $\lim_{h \to 0} [g(x+h) - g(x)]/h = 8x$

Hence $g'(x) = 8x$ for all $x \in (-\infty, \infty)$.

3) $f(x) = \sin x$ for all $x \in (-\infty, \infty)$

1. $f(x+h) = \sin(x+h) = \sin x \cos h + \sin h \cos x$
2. $f(x) = \sin x$
3. $\frac{f(x+h) - f(x)}{h} = \frac{\sin x \cos h + \sin h \cos x - \sin x}{h} = \sin x \left(\frac{\cos h - 1}{h}\right) + \cos x \frac{\sin h}{h}$
4. $\lim \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \left[\sin x \left(\frac{\cos h - 1}{h}\right) + \cos x \frac{\sin h}{h}\right]$

$= \sin x \cdot 0 + \cos x \cdot 1 = \cos x$

4. $f(x) = \cos x$

1. $f(x+h) = \cos(x+h) = \cos x \cos h - \sin x \sin h$
2. $f(x) = \cos x$
3. $\dfrac{f(x+h)-f(x)}{h} = \dfrac{\cos x \cos h - \sin x \sin h - \cos x}{h} = \cos x\left(\dfrac{\cos h - 1}{h}\right) - \sin x \dfrac{\sin h}{h}$.
4. $\lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0}\left[\cos x\left(\dfrac{\cos h - 1}{h}\right) - \sin x \dfrac{\sin h}{h}\right]$

$= \cos x \cdot 0 - \sin x \cdot 1 = -\sin x$

Deep analysis shows that the derivative of a function at a point denotes (or represents) the slope of the tangent line to the graph of the function at that point.

Algebraic properties of the derivative

1. $(f+g)'(x) = f'(x) + g'(x)$
2. $(f \cdot g)'(x) = f'(x)\,g(x) + f(x)\,g'(x)$
3. $\left(\dfrac{f}{g}\right)'(x) = \dfrac{f'(x)\,g(x) - f(x)\,g'(x)}{g^2(x)}$, $g(x) \neq 0$.

One can prove these properties by applying the def. of the derivative.

Applications

1. Derivatives of monomials and polynomials

a) $f(x) = x$ has derivative $f'(x) = 1$

b) one can show that the const. function $h(x) = c$ has derivative $h'(x) = 0$.

Theorem: If $M(x) = x^n$, $n \in \mathbb{N}$ then $M'(x) = n x^{n-1}$.

proof: $(x+h)^n = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} h + \binom{n}{2} x^{n-2} h^2 + \dots + \binom{n}{n-1} x\, h^{n-1} + h^n$

where $\binom{n}{r} = \dfrac{n!}{r!(n-r)!}$.

$$\frac{f(x+h)-f(x)}{h} = \frac{\binom{n}{0}x^n + h\binom{n}{1}x^{n-1} + \dots + \binom{n}{n-1}x\,h^{n-1} + h^n - x^n}{h}$$

$$= n x^{n-1} + \frac{n(n-1)}{2!}h x^{n-2} + \dots + n x h^{n-2} + h^{n-1}, \text{ where } \binom{n}{0}=n, \binom{n}{n-1}=n.$$

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = n x^{n-1}.$$

Examples: $(x^5)' = 5x^4$, $(x^3)' = 3x^2$ etc.

Also, by applying the division property we find that if $N(x) = 1/x^n$, $n \in \mathbb{N}$ (set of natural nos),

$$N'(x) = -n/x^{n+1}.$$

* If $L(x) = c x^n$, where c is a const., $L'(x) = n c x^{n-1}$.

If

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \dots a_1 x + a_0$$

is a polynomial function of degree n, by applying the the additive property of differentiation and the rule for differentiating monomials we find

$$P'(x) = a_n n x^{n-1} + a_{n-1}(n-1) x^{n-2} + \dots + a_1.$$

Examples:

a. If $P(x) = 2x^3 - 3x^2 + 4x + 3$, $P'(x) = 6x^2 - 6x + 4$.

b. If $P(x) = \frac{1}{2}x^4 + \frac{3}{5}x^3 - \frac{1}{3}x$, $P'(x) = 2x^3 + \frac{9}{5}x^2 - \frac{1}{3}$.

2. Derivative of Rational Functions

A rational function is a quotient of two polynomials, which can be expressed as $P(x)/Q(x)$, $P(x)$ & $Q(x)$ are polynomials.

By applying the division property, we have

$$\left(\frac{P(x)}{Q(x)}\right)' = \frac{P'(x)Q(x) - P(x)Q'(x)}{Q^2(x)}, \quad Q(x) \neq 0.$$

Examples:

$$\left(\frac{2x^3-3}{x^2+1}\right)' = \frac{6x^2(x^2+1) - (2x^3-3)2x}{(x^2+1)^2} = \frac{2x^4+6x^2+6x}{(x^2+1)^2}.$$

3. Derivative of other functions

a) $f(x) = \tan x = \sin x / \cos x$

$$\left(\frac{\sin x}{\cos x}\right)' = \frac{(\sin x)'\cos x - (\sin x)(\cos x)'}{\cos^2 x}$$

$$= \frac{\cos x \cdot \cos x - \sin x(-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= 1/\cos^2 x = \sec^2 x .$$

or $(\tan x)' = \sec^2 x$

b) $f(x) = \cot x = \frac{\cos x}{\sin x}$

$$\left(\frac{\cos x}{\sin x}\right)' = \frac{(\cos x)'\sin x - (\sin x)'\cos x}{\sin^2 x} = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x}$$

$$= -1/\sin^2 x = -\operatorname{cosec}^2 x$$

or $$(\cot x)' = -\operatorname{cosec}^2 x .$$

Chain Rule: If f and g are two continuous functions s.t. g is differentiable at x and f is differentiable at $g(x)$, then

$$[f(g(x))]' = f'(g(x)) \cdot g'(x).$$

Derivatives of Logarithmic and Exponential Functions

By def. the no. e is given by

$$e = \lim_{t \to 0} (1+t)^{1/t} \quad *$$

- Now let $a > 0$, then

$(\log_a x)' = 1/x \log_a e$, prove this result by using * and the def. of the derivative.

examples: $(\log_2 x)' = 1/x \log_2 e$.

$(\log_{\sqrt{2}} x^2)' = \frac{1}{x^2} \cdot 2x \log_{\sqrt{2}} e = \frac{2}{x} \log_{\sqrt{2}} e$.

- Let $a > 0$. The exponential function a^x is defined by $y = a^x$ iff $\log_a y = x$. Then

$$(a^x)' = a^x / \log_a e$$

Proof:

$$\log_a a^x = x$$

Taking derivatives on both sides

$$(\log_a a^x)' = (x)' = 1$$

By the chain rule

$$(\log_a a^x)' = \frac{1}{a^x} \log_a e \; (a^x)'$$

When ce

$$(a^x)' = a^x / \log_a e = a^x \ln a.$$

In particular, $(e^x)' = e^x$.

Note that we can also find the 2nd, 3rd, etc. derivatives of a function.

2. Integration; Indefinite Integral

Def. The function F is an antiderivative of the continuous function f iff for all x in the domain of f

$$F'(x) = f(x)$$

1, The function $F(x) = x^2/2$ is an antiderivative of the function $f(x) = x$. This function also has other antiderivatives namely. for example,

$$F_1(x) = x^2/2 + 7, \quad F_2(x) = x^2/2 - 97\tfrac{1}{2},$$

and in general the function

$$f(x) = x^2/2 + c$$

where c is arbitrary constant is an anti-derivative of $f(x) = x$.

2. For $x \neq 0$, the function $F(x) = 1/x^2 + c$ is an anti-derivative of $f(x) = -2/x^3$ for arbitrary const. c.

3. For an arbitrary const. c, $F(x) = -\cos x + c$ is an antiderivative of $f(x) = \sin x$.

4. Since $\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n$, the function $F(x) = \frac{x^{n+1}}{n+1} + c$ is an antiderivative of $f(x) = x^n$ for $n \neq -1$ and an arbitrary const. c.

5. Since $\frac{d}{dx}(\tan x) = \sec^2 x$, the function $F(x) = \tan x + c$ is an antiderivative of $f(x) = \sec x^2$.

Theorem 1.1.

If $F(x)$ is an antiderivative function of $f(x)$, then $F(x)+c$ where c is an arbitrary const. is also an antiderivative of $f(x)$.

proof.

$$\frac{d}{dx}(F(x)+c) = \frac{dF(x)}{dx} + \frac{dc}{dx} = \frac{dF(x)}{dx} + 0 = f(x)$$

Thus $F(x)+c$ is also an antiderivative of $f(x)$.

Theorem 1.2

If $F(x)$ & $G(x)$ are antiderivatives of the function $f(x)$, then

$$F(x) = G(x)+c$$

where c is a const.

Proof:

According to the theorem, $F'(x) = f(x)$ and $G'(x) = f(x)$

Define

$$L(x) = F(x) - G(x)$$

$$\Rightarrow L'(x) = [F(x) - G(x)]' = f(x) - f(x) = 0$$

Thus $L(x)$ is a const. or,

$$L(x) = F(x) - G(x) = C \equiv \text{constant}$$

i.e.)

$$F(x) = G(x) + c.$$

Theorem 1.3

If F & G are antiderivatives of the continuous functions f and g resp., and k is any const.

1. kF is an an antiderivative of kf
2. $F+G$ " " " " $f+g$
3. $F-G$ " " " " $f-g$.

proof of 2.

Since $(F+G)' = F' + G' = f+g$, it follows that $F+G$ is an antiderivative of $f+g$.

Def. The Set of all antiderivatives of a continuous

function f is called the indefinite integral of f and is denoted symbolically by

$$\int f(x)\,dx = F(x) + c$$

the f is called the integrand and x is called the variable of integration.

Thus if $F(x)$ is an antiderivative of the continuous function f and c is any const., then the relation

$$\int f(x)\,dx = F(x) + c$$

holds and the theorem is now expressed as

1. $\int k f(x)\,dx = k\int f(x)\,dx = k F(x) + c$
2. $\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx = F(x) + G(x) + c$
3. $\int [f(x) - g(x)]\,dx = \int f(x)\,dx - \int g(x)\,dx = F(x) - G(x) + c$

The process of finding the set of antiderivatives of the continuous function f is called Integration of the function f wrt the variable of integration. Hence by means of indefinite integration we find the antiderivatives of a given function and thus indefinite integration is the inverse of differentiation.

Examples:

$\int x^2\,dx = x^3/3 + c$; $\int x\,dx = x^2/2 + c$; $\int dx = x + c$;

$\int 2\,dx = 2x + c$; $\int \sin x\,dx = -\cos x + c$; $\int \cos x\,dx = \sin x + c$;

$\int x^n\,dx = x^{n+1}/(n+1) + c$, $n \neq -1$; $\int x^{-1}\,dx = \ln x + c$;

$\int (x^2 + 3x - 4)\,dx = \int x^2\,dx + \int 3x\,dx - \int 4\,dx = \frac{x^3}{3} + \frac{3x^2}{2} - 4x + c$.

1. First order Linear Differential Equation

$$P(x)\frac{dy(x)}{dx} + Q(x)y(x) = R(x)$$

$$y' + \frac{Q(x)}{P(x)}y = \frac{R(x)}{P(x)}$$

Define $A(x) = \frac{Q(x)}{P(x)}$, $B(x) = \frac{R(x)}{P(x)}$

$$\Rightarrow \quad y' + A(x)y = B(x)$$

Consider the homogeneous part of the DE

$$y' + A(x)y(x) = 0$$

$$\Rightarrow \frac{dy}{y} = -A(x)\,dx, \quad \ln y = -\int A(x)\,dx + A_0', \quad y = e^{A_0} e^{-\int A(x)dx}, \quad A_0 \equiv \text{const.}$$

$$y = C\, e^{-\int A(x)dx}, \quad C = e^{A_0}.$$

Let $C = C(x)$.

$$\Rightarrow y = C(x)\, e^{-\int A(x)dx}$$

Use this value of y in the original DE.

$$y' + A(x)y = B(x)$$

$$\Rightarrow C' e^{-\int A(x)dx} - A(x) C e^{-\int A(x)dx} + A(x) C e^{-\int A(x)dx} = B(x)$$

$$C' = B(x) e^{\int A(x)dx}$$

or,

$$C = \int B(x) e^{\int A(x)dx} + C'$$

$$\therefore y = C(x) e^{-\int A(x)dx}$$

$$\Rightarrow \boxed{y = e^{-\int A(x)dx} \left(\int B(x) e^{\int A(x)dx} dx + C' \right)}$$

Formula (Integration by parts): $\int u\,dv = uv - \int v\,du$

Some examples:

1. $x y' + 5y = 4x^2$

$y' + \frac{5}{x} y = 4x$, $A(x) = 5/x$, $B(x) = 4x$

$$\therefore y = e^{-\int \frac{5}{x} dx} \left(\int 4x\, e^{\int \frac{5}{x} dx} + C' \right)$$

$$= e^{-5\ln x} \left(\int 4x\, e^{5\ln x} dx + C' \right)$$

2. $3y' + \frac{1}{2} y = x$, $A(x) = \frac{1}{6}$, $B(x) = y$

$$\int A(x)\,dx = \int \frac{1}{6} dx = \frac{x}{6}$$

$$\int B(x) e^{\int A(x)dx} dx = \int \frac{1}{3} x\, e^{x/6} dx = 2x e^{\frac{1}{6}x} - 12 e^{\frac{1}{6}x}$$

[Integrate by parts]

$$\therefore \quad y = e^{-\int A(x)dx}\left(\int B(x)\, e^{\int A(x)dx}\, dx\right)$$

$$= e^{-\frac{x}{6}}\left(2x\, e^{\frac{1}{6}x} - 12\, e^{\frac{1}{6}x} + c'\right)$$

$$= 2x - 12 + c' e^{-x/6}$$

2. second order linear homogeneous differential eq. with constant coefficients.

This DE is of the form

$$a\frac{d^2y(x)}{dx^2} + b\frac{dy(x)}{dx} + c\,y(x) = 0$$

where a, b & c are constants.

$$a y'' + b y' + c = 0$$

To solve this eq we try a solution of the form

$$y = e^{\alpha x}$$

$$\Rightarrow \quad a\alpha^2 e^{\alpha x} + b\alpha\, e^{\alpha x} + c\, e^{\alpha x} = 0$$

$$a\alpha^2 + b\alpha + c = 0$$

Using the quadratic formula we find

$$\alpha = -\frac{b}{2a} \pm \sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}}$$

Two roots $\alpha_+ = -\frac{b}{2a} + \sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}}$, $\alpha_- = -\frac{b}{2a} - \sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}}$

The solution of the DE is in general, given by,

$$y = A e^{\alpha_+ x} + B e^{\alpha_- x}, \text{ where } A \& B \text{ are const.}$$

or,

$$y = A e^{\left(-\frac{b}{2a} + \sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}}\right)x} + B e^{\left(-\frac{b}{2a} - \sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}}\right)x}$$

$$= e^{-\frac{b}{2a}x}\left(A e^{\sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}}\,x} + B e^{-\sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}}\,x}\right)$$

cases

1. $\left(\frac{b}{2a}\right)^2 - \frac{c}{a} > 0$:

$$y(x) = e^{-\frac{b}{2a}x}\left(A e^{\sqrt{\left(\frac{b}{2a}\right)^2-\frac{c}{a}}\,x} + B e^{-\sqrt{\left(\frac{b}{2a}\right)^2-\frac{c}{a}}\,x}\right)$$

2. $\left(\frac{b}{2a}\right)^2 - \frac{c}{a} < 0$

Now, the radical is an imaginary no. like then write

$$\sqrt{\left(\frac{b}{2a}\right)^2-\frac{c}{a}} = \sqrt{-1}\left(\frac{c}{a}-\left(\frac{b}{2a}\right)^2\right) = i\sqrt{\frac{c}{a}-\left(\frac{b}{2a}\right)^2}$$

(i.e.)

$$y = e^{-\frac{b}{2a}x}\left(A e^{i\sqrt{\frac{c}{a}-\left(\frac{b}{2a}\right)^2}\,x} + B e^{-i\sqrt{\frac{c}{a}-\left(\frac{b}{2a}\right)^2}\,x}\right)$$

$$= e^{-\frac{b}{2a}x}\left\{A\cos\sqrt{\frac{c}{a}-\left(\frac{b}{2a}\right)^2}\,x + iA\sin\sqrt{\frac{c}{a}-\left(\frac{b}{2a}\right)^2}\,x + B\cos\sqrt{\frac{c}{a}-\left(\frac{b}{2a}\right)^2}\,x - iB\sin\sqrt{\frac{c}{a}-\left(\frac{b}{2a}\right)^2}\,x\right\}$$

$$= e^{-\frac{b}{2a}x}\left\{(A+B)\cos\sqrt{\frac{c}{a}-\left(\frac{b}{2a}\right)^2}\,x + i(A-B)\sin\sqrt{\frac{c}{a}-\left(\frac{b}{2a}\right)^2}\,x\right\}$$

or,

$$y = e^{-\frac{b}{2a}x}\left(c_1\cos\sqrt{\frac{c}{a}-\left(\frac{b}{2a}\right)^2}\,x + c_2\sin\sqrt{\frac{c}{a}-\left(\frac{b}{2a}\right)^2}\,x\right)$$

where c_1 & c_2 are constants.

We can write the expressions in the parentheses as a sine or cosine function.

- To write it as a sine function

let $\frac{c_1}{c_2} = \tan\theta$ ($\sqrt{c_1^2+c_2^2}$, θ, c_2, c_1)

$$c_1 = \sqrt{c_1^2+c_2^2}\sin\theta, \quad c_2 = \sqrt{c_1^2+c_2^2}\cos\theta$$

$$\Rightarrow y = e^{-\frac{b}{2a}x}\sqrt{c_1^2+c_2^2}\left(\sin\theta\cos\sqrt{\frac{c}{a}-\left(\frac{b}{2a}\right)^2}\,x + \cos\theta\sin\sqrt{\frac{c}{a}-\left(\frac{b}{2a}\right)^2}\,x\right)$$

$$= \sqrt{c_1^2+c_2^2}\; e^{-\frac{b}{2a}x}\sin\left(\sqrt{\frac{c}{a}-\left(\frac{b}{2a}\right)^2}\,x+\theta\right)$$

or,

$$y = K e^{-\frac{b}{2a}x}\sin\left(\sqrt{\left(\frac{c}{a}\right)-\left(\frac{b}{2a}\right)^2}\,x+\theta\right)$$

Similarly,

$$y = K e^{-\frac{b}{2a}x}\cos\left(\sqrt{\frac{c}{a}-\left(\frac{b}{2a}\right)^2}\,x+\beta\right).$$

3. $\left(\frac{b}{2a}\right)^2 - \frac{c}{a} = 0$

$$y(x) = e^{-\frac{b}{2a}x}(A+B) \equiv D\,e^{-\frac{b}{2a}x}$$

However, the most general solution here is

$$y(x) = (D + \bar{E}x)\,e^{-\frac{b}{2a}x}$$

Kotebe College of Teacher Education
Department of Physics

Assignment I

Vectors

1. A person walks along a circular path of radius 5m, around one half of the circle. a, Find the magnitude of the displacement vector. b, How far did the person walk? c, What is the magnitude of the displacement if the circle is completed?

2. A vector $\bar{A}$ has a positive x component of 4 units and a negative y component of 2 units. What second vector $\bar{B}$ when added to $\bar{A}$ will produce a resultant vector three times the magnitude of $\bar{A}$ directed in the positive y direction?

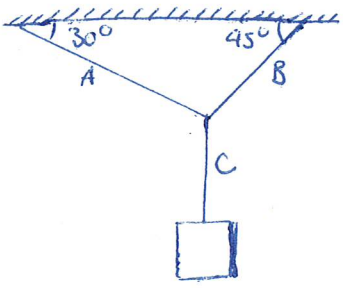
3. Use the method of rectangular resolution to find the resultant of the following set of forces and the angle it makes with the positive x-axis: 200 N, along the x-axis toward the right; 300 N, 60° above the x-axis to the right; 100 N, 45° above the x-axis to the left; 200 N, vertically down.

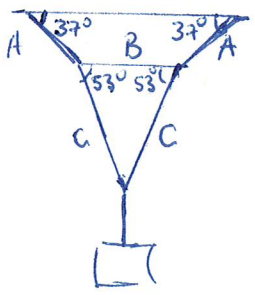
4. When two vectors $\bar{A}$ and $\bar{B}$ are drawn from a common point, the angle between them is θ. Show that the magnitude of their vector sum is given by

$$\sqrt{A^2 + B^2 + 2AB\cos\theta}.$$

Equilibrium

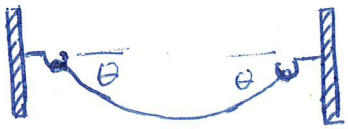
1. Find the tension in each cord in the figure below if the weight of the suspended body is 200 N.



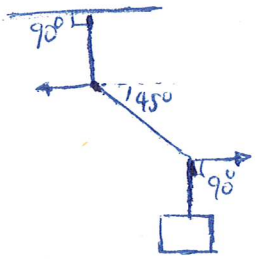


2. A flexible rope of weight W hangs between two hooks at the same height, as shown in the figure below. At each end the rope makes an angle θ with the horizontal.

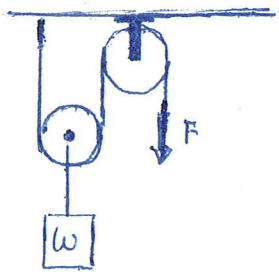
a, What is the magnitude and direction of the force $\vec{F}$ exerted by the rope on the hook at the left?
b, What is the tension T in the rope at its lowest point?



3. In the figure below, find the weight of the suspended body if the tension in the diagonal string is 20 N.



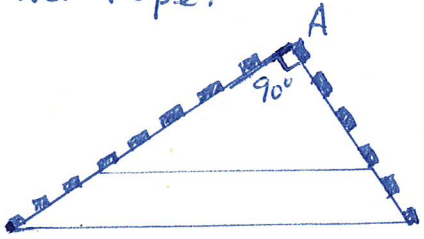
4. The diagram below shows a man lifting a weight w by pulling down on a rope with a force F. The upper pulley is attached to the ceiling by a chain, and the lower pulley is attached to the weight by another chain. If w = 400 N, find the tension in each chain and the force F if the weight is lifted at constant speed (no net force). Assume the weights of the rope, pulleys, and chain to be negligible.



5. Can a body be in equilibrium when only one force acts on it?

6. Two ladders, 4m and 3m long, respectively, are hinged at point A and tied together by a horizontal rope 0.6m above the floor, as in the figure below. The ladders weigh 400 N and 300 N, respectively, and the center of gravity of each is at its center. If the floor is frictionless, find a, the upward force at the bottom of each ladder, b, the tension in the rope, and c, the force that one ladder exerts on the other at point A. d,

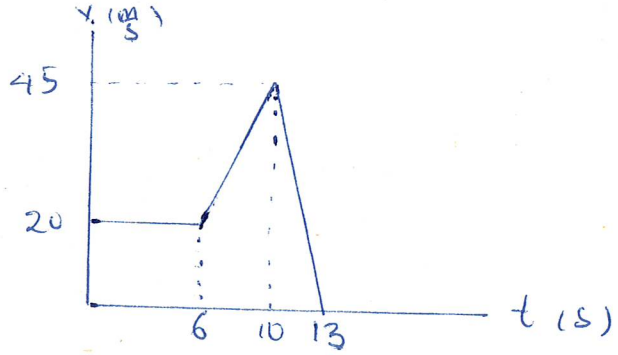
If a load of 1000 N is now suspended from point A, find the tension in the rope.



Rectilinear Motion

1. A body moves along a straight line, its distance from the origin at any instant being given by the equation $x = 8t - 3t^2$, where x is in meters and t is in seconds. Find the average velocity of the body in the interval from $t = 0$ to $t = 1s$, and in the interval from $t = 0$ to $t = 4s$.

2. The graph below shows the velocity of a body plotted as a function of time.
   a, Find the instantaneous acceleration at $t = 3s$, at $t = 7s$, and at $t = 11s$.
   b, How far does the body go in the first 5 s? The first 9 s? The first 13 s?



3. An airplane taking off from a landing field has a run of 500 m. If it starts from rest, moves with constant acceleration, and makes the run in 30 s, with what velocity did it take off?

4. A body moving with constant acceleration covers the distance between two points 60 m apart in 6 s. Its velocity as it passes the second point is 15 m/s. a, What is the acceleration? b, What is its velocity at the second point?

5. The "reaction time" of the average automobile driver is about 0.7s. (The reaction time is the interval between the perception of a signal to stop and the application of brakes.) If an automobile can accelerate at $-5\ m/s^2$, compute the total distance covered in coming to a stop after a signal is observed: a, from an initial velocity of 15 m/s, b, from an initial velocity of 30 m/s.

6. At the instant the traffic lights turn green, an automobile that has been waiting at an intersection starts ahead with a constant acceleration of $2\ m/s^2$. At the same instant a truck, traveling with a constant velocity of 10 m/s, overtakes and passes the automobile. a, How far beyond its starting point will the automobile overtake the truck? b, How fast will it be traveling?

7. A ball is thrown vertically downward from the top of a building, leaving the thrower's hand with a velocity of 10 m/s.
a, What will be its velocity after falling for 2s?
b, How far will it fall in 2s?
c, What will be its velocity after falling 10m?
d, If it moved a distance of 1m while in the thrower's hand, find its acceleration while in his hand.
e, If the ball was released at a point 40 m above the ground, in how many seconds will it strike the ground?
f, What will the velocity of the ball be when it strikes the ground?

8. A balloon, rising vertically with a velocity of 5 m/s, releases a sandbag at the instant when the balloon is 20 m above the ground. a, Compute the position and velocity of the sandbag at the following ~~distance~~ times after its release: $\frac{1}{4}$ s, $\frac{1}{2}$ s, 1s, 2s. b, How many seconds after its release will the bag strike the ground? c, With what velocity will it strike?

9. A stone is dropped from the top of a tall cliff, and 1s later a second stone is thrown vertically downward with a velocity of 20 m/s. How far below the top of the cliff will the second stone overtake the first?

10. A ball is dropped from the cornice of a building takes 0.25 s to pass a window 3m high. How far is the top of the window below the cornice?

11. The rocket-driven sled Sonic Wind NO. 2, used for investigating the physiological effects of large accelerations, runs on a straight, level track 3500 ft long. Starting from rest, it can reach a speed of 1000 mi/hr in 1.8 s.
a, Compute the acceleration, assuming it to be constant.
b, What is the ratio of this acceleration to that of a free falling body, g?
c, What is the distance covered?

12. A "moving sidewalk" in an airport terminal building moves 2m/s and is 150 m long. If a man steps on at one end walks 2m/s relative to the moving sidewalk, how much time does he require to reach the opposite end if he walks a, in the same direction the sidewalk is moving;
b, in the opposite direction?

Kotebe College of Teacher Education
Department of Science

Physics problems for Phys. 111

Work Sheet 5

Vectors

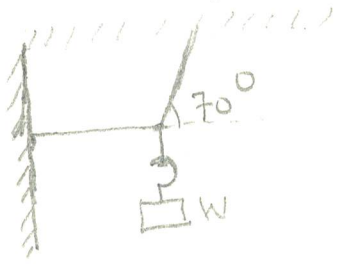
1. If an airplane flies Southwest for 100km, how far west and how far South has it gone?

2. A bug sits on a board which is floating Southward down a straight stream with a speed of 5.0cm/s. The bug now walks eastward at a speed of 3.0cm/s. What is the speed of the bug relative to the shore of the stream?

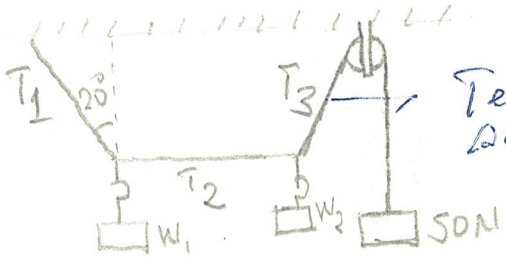
Equilibrium

1. Find the tension in the rope of the fig. below if the downward force F is of magnitude 150 lb.

2. The weight W shown in the fig. below is 200N. What is the tension in the horizontal rope? Assume the ropes to have negligible weight.



3. For the equilibrium situation shown below assume the pulley to be frictionless. Find W_1, W_2, T_1 & T_2.



Tension in the two strings is the same since the pulley is frictionless.

Rectilinear Motion

Unless otherwise stated, assum uniform acceleration. Use $g = 9.8\ m/s^2 = 32\ ft/s^2$. Neglect air friction.

1. According to the theory of continental drift, South America was once attached to Africa. They have been drifting apart for about 200 million years. Their present separation is about 400 mi. Assume these numbers to be exact, what was the average rate of separation of the continents in miles per year?

2. Suppose two cars A & B are racing a 200m distance. Car A runs at a speed of 5.0 m/s while car B runs at a speed of 3.0 m/s. TO make the race more even, A is required to start t seconds later than B. How large should t be if they are to end in a tie?

3. A bullet moving with a speed of 150 m/s strikes a tree and penetrates 3.5 cm before stopping. Find its deceleration and the time taken to stop.

4. A truck initially traveling at 50 ft/s decelerates at $-4\ ft/s^2$. Find a, how long it takes to stop
b, how far it moves in that time
c, how far the truck moves in the third second after the brakes are applied.

5. A boy throws a ball straight down from the top of a 50 ft building with a speed of 20 ft/s. How long does it take for the ball to reach the ground, and how fast is it moving just before it hits?

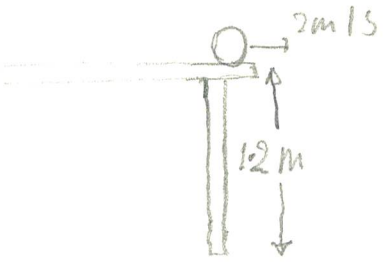
Laws of Dynamics

1. A 3600 lb car is to be pushed up a hill which is inclined 8° to the horizontal. If friction forces are negligible, what is the minimum pushing force needed?

2. A car is moving up a 15° incline at a speed of 20 m/s when the driver disengages the motor. How far up along the incline will the car coast if friction forces are negligible?

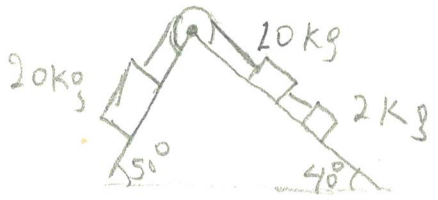
3·2f, in prob 2, The 1400 kg car [coasts] 30 m before stopping, how large is the average (assumed constant) friction force which stopped it?

4. A marble rolls off the edge of a horizontal table 1.2m high as shown. Find the horizontal distance that can be moved by the marble if it has been rolling at a speed of 2m/s. What is its velocity just before it hits the floor?



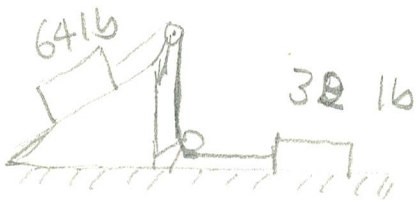
~~4~~.

[illegible] Find the tension in the various cords shown in the fig. below and the acceleration of the blocks if μ for each block is 0.20.



0.977 m/s^2
17.6 N
105 N

(5) Each of the two block in the fig. below, experiences a friction force of 5 lb. How long will it take, starting from rest, to pull the 32 lb block through a distance of 19 ft?



Department of Science

Phys. 201, Worksheet I

Vectors

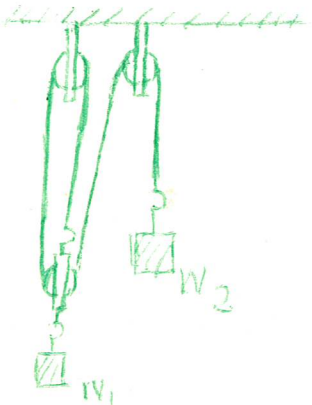
1. An airplane flies 60 km in a direction 40° north of east. What is the eastward component of the displacement? What is the northward component?

2. The following three forces act upon an object: 200 lb horizontally to the right, 150 lb downward and to the left at an angle of 30° below the horizontal, and 500 lb upward and to the right at an angle of 53° above the horizontal. Find the resultant force.

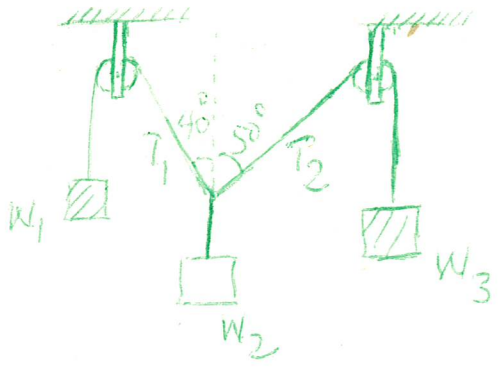
3. A 260 N force and a 360 N force are applied to the same object. If the angle between the two forces is 63°, what is the magnitude of the resultant force?

Equilibrium

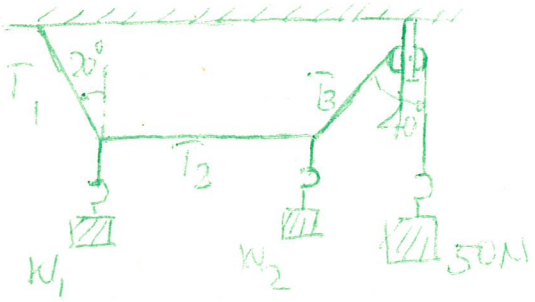
1. In the equilibrium situation shown in the figure below, the pulleys are nearly frictionless, and so the tension is the same in all parts of the cord. Assume that the weights of the pulley and cord are negligible and that the cords are essentially vertical. What is the tension in the cord in terms of W_2? What is the ratio of W_1 to W_2?



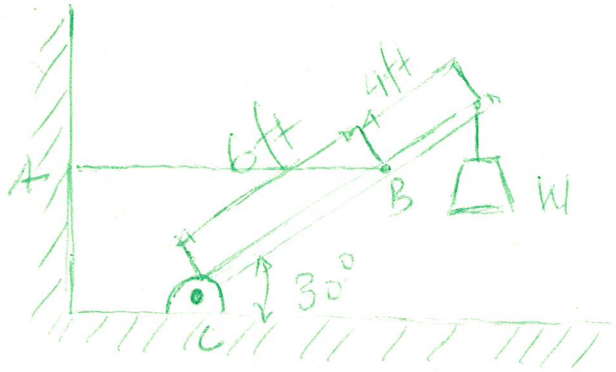
2. For the equilibrium situation shown below, find W_2 and W_3. Assume $W_1 = 100$ lb and consider the pulleys to be frictionless.



3. For the equilibrium situation shown in the following figure, assume the pulley to be frictionless. Find W_1, W_2, T_1 and T_2.



4. If the weight of the boom in the figure below is neglected, and the weight W is 200 lb, a) what is the tension in the horizontal cable? b) what are the horizontal and vertical components of the force exerted on the boom at its lower end?



Uniformly accelerated Motion

1. A can run at a top speed of 5.0 m/s while B can run at a speed of only 3.0 m/s. They are to race 200 m. To make the race more even, A is required to start 4 seconds later than B. How large should t be if they are to end in a tie?

2. A truck initially travelling at 50 ft/s decelerates at 4 ft/s². Find a, how long it takes to stop b, how far it moves in that time, and c, how far the truck moves in the third second and after the brakes are applied?

3. A boy wants to throw a can straight up in the air and then hit it with a second can. He wants the collision to occur 5.0 m above the throwing point. In addition, he knows that the time he needs between throws is 3.0 s. Assuming he throws both cans with the same speed, what must be the initial speed of the cans?

4. A stone is thrown straight up with an initial speed of 80 ft/s. How high does the stone go and how long does it take to reach its highest point?

5. A brick is thrown vertically down from the top of a bridge, and 4 s later it strikes the water with a final velocity of 60 m/s. a, what was the initial velocity of the brick? b, how high is the bridge above the water?

Projectile Motion

1. A stone is thrown horizontally from the top of

a building with an initial velocity of 2000 ft/s. At the same instant another stone is dropped from rest. a, Compute the position and velocity of the latter after 3 s. b, How far does the first stone travelled horizontally during this same time? c, How far has it travelled vertically? d, What are the horizontal and vertical components of its velocity after 3 s?

2. A bomber flying with a horizontal speed of 400 ft/s releases a bomb, which strikes the ocean below 6 s later. At what altitude is the plane flying? How far has the bomb travelled horizontally from the time of release?

Newton's 2nd law of motion

1. A 64 lb box hangs at the end of a rope. Find the tension in the rope if the acceleration is a, upward 5 ft/s^2 and b, downward at 5 ft/s^2.

2. The mass m_1, m_2 and m_3 of the figure below are 10, 8 and 6 kg. Neglecting friction, what are the acceleration of the system and the tension in each cord.

3. Find the tension in the various cords shown in the figure below and the acceleration of the blocks if μ for each block is 0.20

Work, Energy and power

1. A 16 kg load is held at a height of 30 m above the ground. a, What is the potential energy at this height? b, If the ball is dropped, what is its downward velocity

when it is 15m above the ground? c, What are the kinetic and potential energies at 25m? d, What is the total energy at this point? e, What is its kinetic energy at the ground? f, With what speed does it strike the ground?

2. A car of mass m rests on a hill of height h and length L. Show that the speed of the car when it reaches the bottom of the hill is

$$v = \sqrt{2gh - \frac{2LF}{m}}$$

Circular Motion

1. A 4 kg mass swings in a horizontal circle of radius 2m. If it revolves at 6 revs, what is its period of revolution? What is its linear speed? Find its centripetal acc and the tension in the cord?

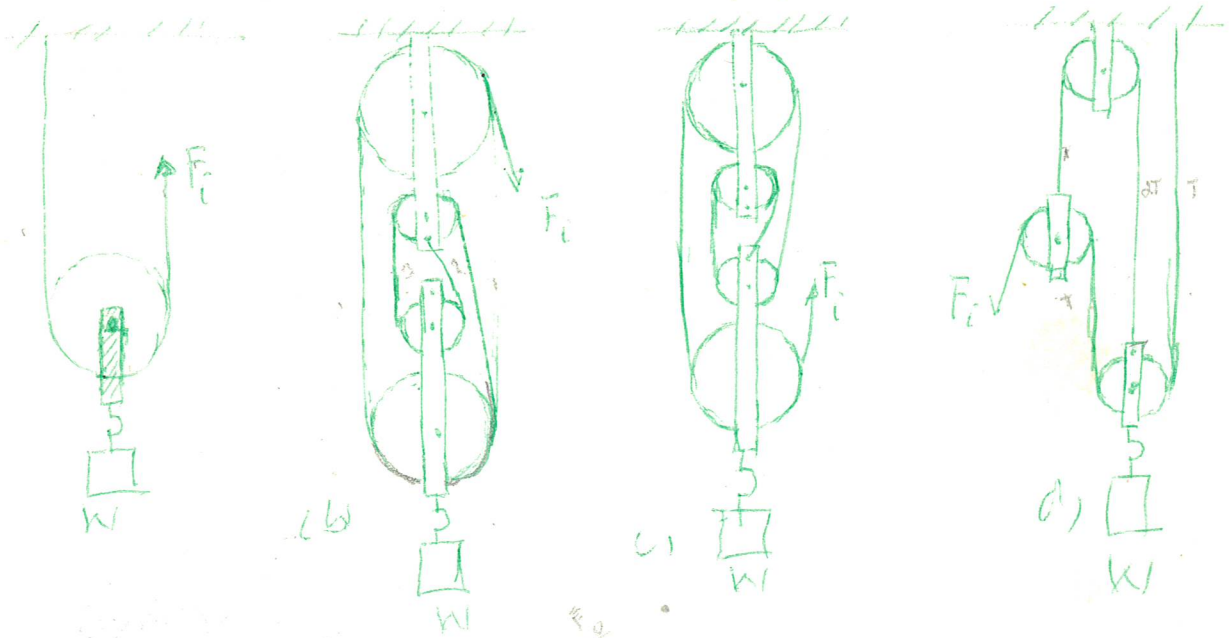
2. Do prob. (1), if the radius of the circle of rotation were 4m, what would be the centrifugal force?

Machines

1. During the operation of a 30 hp engine, energy is lost at the rate of 20 hp because of friction. What are the useful output power and the efficiency of the engine?

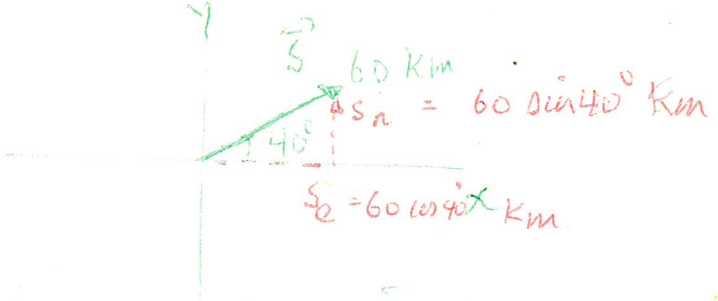
2. A 20 kg mass is to be lifted with a rod 2m long. If you can exert a downward force of 40N on one end of the rod, where should you place a block of wood to act as a fulcrum?

3. Determine the force F required to lift a 200 N load W with each of the four pulley systems shown.


(a)
F_i
W
(b)
F_i
W
(c)
F_i
W
(d)
F_i
T
2T
W

Solutions to problems & Phys 101
for
Work sheet I
Vectors

1. Introducing x-y coordinate system we find the comps of the displacement as follows:



The eastward comp. is given by

$$\frac{S_e}{S} = \cos 60^\circ$$

or $S_e = S \cos 60^\circ$

$= (60\,km)\,\frac{1}{2}$

$= 30\,km$

The northward comp of S is given by

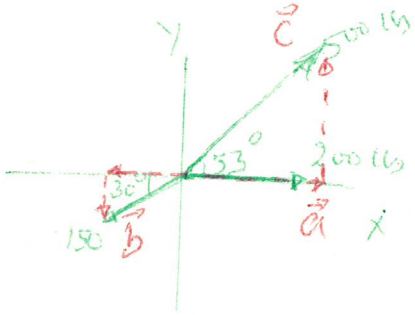
$$\frac{S_n}{S} = \sin 60^\circ$$

or $S_n = S \sin 60^\circ$

$= (60\,km)(0.866)$

$= 51.96\,km$

2. Here again we use the x-y coordinate system



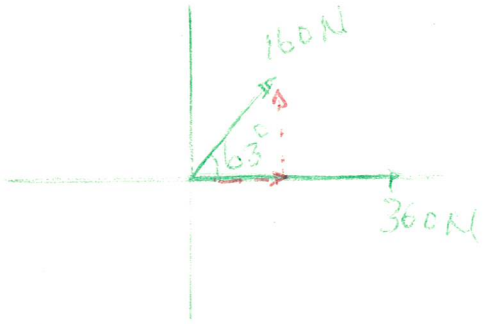
Vector	x- comp	y- comp
$\vec{a}$	+200 lb	0
$\vec{b}$	$-150 \cos 30^\circ$	$-150 \sin 30^\circ$
$\vec{c}$	$+300 \sin 53^\circ$	$+300 \sin 53^\circ$
$\vec{F}_r$	371	224 lb

$F_r = \sqrt{(371)^2 + (324)^2}$ lb $= 492.56$ lb

The angle the resultant force make with the horizontal is given as

$\theta = \tan^{-1}\left(\frac{324\text{ lb}}{371\text{ lb}}\right) = 41^\circ$, above the horizontal to the right

3.



The X-comp. of the resultant force is given by

$F_{rx} = (160\text{N})\cos 63^\circ + 360\text{N}$

$= (160\text{N})(0.454) + 360\text{N}$

$= 72.64\text{ N} + 360\text{N}$

$= 432.64\text{N}$

The Y-comp of the resultant force is given by

$F_{ry} = (160\text{N})\sin 63^\circ$ To

$= 160\text{N} \times 0.891$

$= 142.56\text{ N}$

Thus the resultant of the magnitude is given by

$F_r = \sqrt{(432.64\text{N})^2 + (142.6\text{N})^2} = 455.52\text{ N}$

Equilibrium

1. As is also stated in the problem, the tension remains the same throughout the cord. And since it is effected by the weight on the right, the tension is W_2.

Since this knows the condition of equilibrium:

$$\Sigma F_y = 0$$

$$\Rightarrow \quad 4T = W_1 + W_2$$

But $\quad T = W_2$

$$\Rightarrow \quad 4W_2 = W_1 + W_2$$

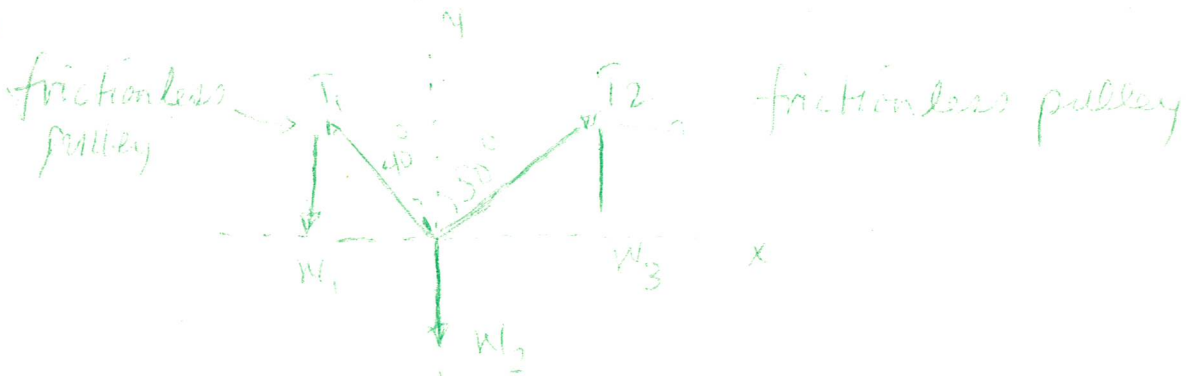
$$-W_1 = W_2 - 4W_2$$

$$-W_1 = -3W_2$$

$$W_1 = 3W_2$$

$$\text{or} \quad \frac{W_1}{W_2} = \frac{3}{1}$$

2.



Since the pulleys are frictionless so that the tension throughout the left rope is $T_1 = W_1 = 100\text{lb}$ and that throughout the right rope is $T_2 = W_3$

from the first condition of equil

$$\Sigma F_x = 0 : \quad -100\text{lb}\cos 40^\circ + W_3 \cos 50^\circ = 0$$

$$\Rightarrow \quad W_3 = \frac{100\text{lb}\cos 40^\circ}{\cos 50^\circ} = 119.2\text{lb}$$

$\Sigma F_y = 0 \cdot (100\ lb)\sin 40^\circ + 119.2\ lb \sin 50^\circ$

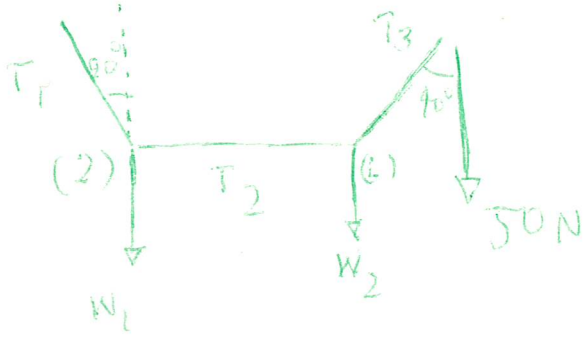
$-100\ lb - 119.2\ lb - W_2 = 0$

$64.28\ lb + 91.3\ lb - 219.2\ lb - W_2 = 0$

$-63.62\ lb - W_2 = 0$

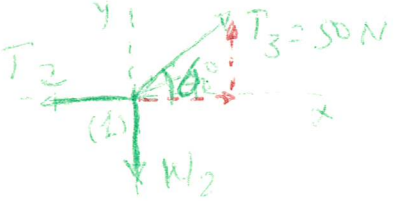
or $W_2 = -63.61\ lb$

3.



Since the pulley is frictionless $T_3 = 50\ N$.

Consider point (1) the right knot; the forces acting on it are as shown below



From the first condition of equil.

$\Sigma F_x = 0 \Rightarrow (50\ N)\cos 60^\circ - T_2 = 0$

$(50\ N)\ 0.5 - T_2 = 0$

$25.00\ N - T_2 = 0$

or $T_2 = 25.00\ N$

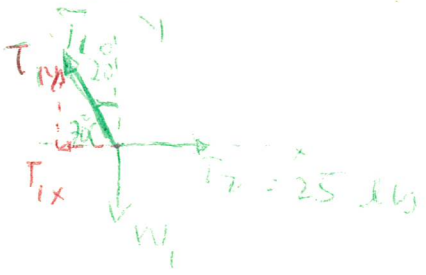
$\Sigma F_y = 0 \rightarrow (50\ N)\sin 60^\circ - W_2 = 0$

$(50\ N)\ 0.866 - W_2 = 0$

$43.30\ N - W_2 = 0$

or $W_2 = 43.30\ N$

Consider the left knot (2). The forces acting on it are shown below.



From the first condition of equilibrium

$$\Sigma F_x = 0 \Rightarrow \quad 25N - T_1 \cos 70^\circ = 0$$

$$25N - 0.342\,T_1 = 0$$

$$\text{or} \quad T_1 = \frac{25N}{0.342} = 73.1N$$

$$\Sigma F_y = 0 \Rightarrow \quad 73.2N \sin 70^\circ - W_1 = 0$$

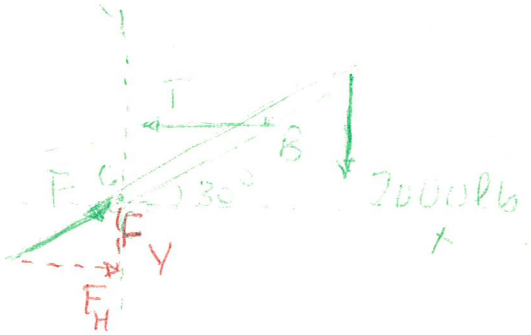
$$(73.1N)(0.940) - W_1 = 0$$

$$68.69N - W_1 = 0$$

$$\text{or} \quad W_1 = 68.69N$$

Therefore, the weights W_1, W_2, and the tensions T_2, T_1 are respectively 68.69 N, 413.3 N, 73.1 N, and 25 N

4).



From the first condition of equilibrium

$$\Sigma F_x = 0: \quad F_H - T = 0 \qquad \text{---------} \quad (1)$$

$$\Sigma F_y = 0: \quad F_V - 2000\,lb = 0 \qquad \text{---------} \quad (2)$$

Considering C as an axis of rotation and taking torques about this point of that, ccw torques are positive we have that.

$$\Sigma C = 0: \quad T\,lb \sin 30^\circ) - 2000\,lb(lb \sin 30^\circ) = 0 \qquad \text{---} \quad (3)$$

Then from the 3rd equation.

$$T = 5773\ N$$

Using this value in (1) the horizontal comp. of F is given to be

$$F_H = 5773\ N$$

And from the 2nd eq. the vertical comp. is

$$F_V = 2000\ lbs.$$

To find torques one can use point B also as an axis of rotation. Then using this point,

$$F_H \cdot 6\sin 30^\circ - F_V \cdot 6\cos 30^\circ - 2000\ lb \cdot 4\cos 30^\circ$$

$$\Rightarrow 3F_H - 5.20F_V - 6928\ lb = 0 \quad \text{---- } 3'$$

once again from (2) $F_V = 2000\ lb.$

$$\Rightarrow 3F_H - 5.2(2000\ lb) - 6928\ lb = 0$$

or $F_H = 5776\ N$, which is the same inspite of approximation errors.

uniformly accelerated Motion

1. The time taken by A to run 200 m at 5.0 m/s is

$$t_A = \frac{200\ m}{5.0\ m/s} = 40\ s$$

since A starts t seconds later than B and they end in tie the time taken by B is

$$t_B = t_A + t = 40\ s + t$$

But $t_B = \frac{200\ m}{3\ m/s} = 66.67\ s$

$$\Rightarrow 40\ s + t = 66.67\ s$$

or $t = 26.67\ s$

2. a) The time taken is given by

$$t = \frac{v_f - v_i}{a}$$

or $a = \dfrac{0 - 50\,ft/s}{-4\,ft/s^2} = 12.50\,s$

b) $v_f^2 - v_0^2 = 2as$

$\Rightarrow s = \dfrac{-v_0^2}{2a}$

or $s = \dfrac{-(50\,ft/s)^2}{2(-4\,ft/s^2)} = \dfrac{2500\,ft^2/s^2}{8\,ft/s^2} = 312.5\,ft$

by Antenan Biru

$S_{th} = V_0 + \frac{1}{2}a(2t-1)$

$S_{3rd} = 50\,ft/s \times t + \frac{1}{2}(-4\,ft/s^2)(2(3)-1)$

$S_{3rd} = 50\,ft - 10\,ft$

$S_{3rd} = 40\,ft.$

c) In 3s the dist. moved is

$S_3 = 50\,ft/s \times 3s - \frac{1}{2} \times 4\,ft/s^2 (3s)^2$

$= 150\,ft - 18\,ft$

$= 132\,ft$

In 2s the dist. moved is

$S_2 = 50\,ft/s \times 2s - \frac{1}{2} \times 4\,ft/s^2 (2s)^2$

$= 100\,ft - 8\,ft$

$= 92\,ft$

Thus the dist. moved in the 3rd second is

$S = 132\,ft - 92\,ft = 40\,ft$

3. The cans hit each other when the first can starts its way down, the 2nd can that thrown later, moves so that

$h_2 = 5\,m$, till it hits and $a = g = -10\,m/s^2$

But $h_2 = v_0 t_2 + \frac{1}{2} g t_2^2$

or $5m = v_0 t_2 - 4.9 t_2^2$ ------ (1)

The distance moved when the first can reaches its max. position is

$y_{max} = v_0 t_1 + \frac{1}{2} g t_1^2$

But at this position the final velocity of the can is zero that

$$0 = v_0 + g t_1$$

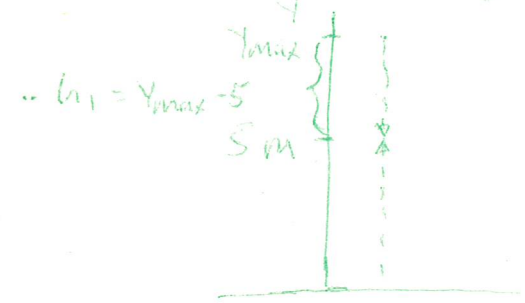
$$\Rightarrow t_1 = -\frac{v_0}{g} \quad \text{---------- (2)}$$

$$\Rightarrow y_{max} = v_0\left(-\frac{v_0}{g}\right) + \frac{1}{2}g\left(-\frac{v_0}{g}\right)^2$$

or $$y_{max} = -\frac{v_0^2}{2g} \quad \text{---------- (3)}$$

This can be proved by the formula I proved i.e $v_0 = \sqrt{\frac{(tg)^2 - 2gh}{4}}$ ... the paper I put my proofs about this can be found on ~~to~~ found on the Volume 2.

Look the figure below



h_1 is the distance moved by the first can on its way down until it collides.

$$h_1 = \frac{1}{2}g t_1'^2$$

$$\Rightarrow t_1' = \sqrt{\frac{2h_1}{g}} = \sqrt{\frac{2(y_{max}-5)}{g}}$$

Using the 2nd relation

$$t_1' = \sqrt{\frac{2(+\frac{v_0^2}{2g} - 5)}{g}}$$

$$\text{or } t_1' = \sqrt{\frac{+(v_0^2 + 10g)}{g^2}} = \frac{1}{g}\sqrt{+(v_0^2+10g)} \quad \text{---- (4)}$$

The total time taken by the first can for its motion until it collides is $(t_1 + t_1')$ and that taken for the second can is t_2. on the other hand, we have

$$t_1 + t_1' - t_2 = 3s$$

$$\Rightarrow t_2 = t_1 + t_1' - 3s = -\frac{v_0}{g} + \frac{1}{g}\sqrt{(+v_0^2+10g)} - 3$$

$$t_2 = \frac{-v_0 + \sqrt{2(v_0^2 + 10g_s)} - 3g_s}{g_s} \quad \text{---} \quad 5$$

From eq (2)

$$5 = v_0 t_2 - \tfrac{1}{2} g t_2^2$$

Then using the expression for t_2

$$5 = v_0\left[\frac{-v_0 + \sqrt{2(v_0^2 + 10g_s)} - 3g_s}{g_s}\right] + \tfrac{1}{2}g\left[\frac{-v_0 + \sqrt{2(v_0^2 + 10g_s)} - 3g_s}{g_s}\right]^2$$

$$= \frac{-v_0^2 + v_0\sqrt{2(v_0^2 + 10g_s)} - 3g_s v_0}{g_s} + \frac{g_s}{2}\left[\frac{(v_0 + 3g_s) - \sqrt{2(v_0^2 + 10g_s)}}{g_s}\right]^2$$

$$= \frac{(v_0 + 3g_s)\sqrt{2(v_0^2 + 10g_s)} + g_s\left[2(v_0^2 + 10g_s)\right]}{g_s}$$

$$= \frac{-v_0^2 + v_0\sqrt{2(v_0^2 + 10g_s)} - 3g_s v_0}{g_s} + \frac{g_s}{2}\left(\frac{v_0 + 3g_s}{g_s}\right)^2$$

$$- \frac{v_0\sqrt{2(v_0^2 + 10g_s)}}{g_s} - 3\sqrt{2(v_0^2 + 10g_s)} + \left[\frac{v_0^2 + 10g_s}{2g_s}\right]$$

$$= \frac{-(v_0^2 + 3g_s v_0)}{g_s} + \frac{g_s}{2}\left(\frac{v_0^2 + 6g_s v_0 + 9g_s^2}{g_s^2}\right)$$

$$- 3\sqrt{2(v_0^2 + 10g_s)} + \left(\frac{v_0^2 + 10g_s}{2g_s}\right)$$

$$= \frac{-2(v_0^2 + 3g_s v_0) + (v_0^2 + 6g_s v_0 + 9g_s^2) + 2(v_0^2 + 10g_s)}{2g_s}$$

$$- 3\sqrt{2(v_0^2 + 10g_s)}$$

$$= \frac{+ 9g_s^2 + 10g_s}{2g_s} - 3\sqrt{2(v_0^2 + 10g_s)}$$

$$\text{Assume } g_s = -10\ m/s^2$$

$$5 = \frac{9(-10)^2 + 10\times(-10)}{2\times(-10)} - 3\sqrt{v_0^2 + 10\times(-10)}$$

$$5 = \frac{900-100}{-20} - 3\sqrt{v_0^2+100}$$

$$= \frac{800}{-20} - 3\sqrt{v_0^2-100}$$

$$= -40 - 3\sqrt{v_0^2-100}$$

$$45 = -3\sqrt{v_0^2-100}$$

$$15 = -\sqrt{v_0^2-100}$$

$$225 = v_0^2 - 100$$

$$v_0^2 = 225 + 100$$

$$= 325$$

$$\text{or } v_0 = \pm\sqrt{325}$$

$$\approx \pm 18.0 \text{ m/s}$$

Since the cans were moving upwards we drop out the value -18 m/s for the velocity, then

$$v_0 = 18.0 \text{ m/s}$$

which is the initial velocity of the balls.

4. At the stone reaches it maximum height its speed reduces to zero i.e., $v_f = 0$

$$v_f^2 - v_0^2 = 2gh$$

$$\Rightarrow h = \frac{v_f^2 - v_0^2}{2g}$$

$$= -\frac{v_0^2}{2g}$$

Suppose up is positive.

Then,

$$h = -\frac{(80\,ft/s)^2}{2(-32\,ft/s^2)} = 200\,ft$$

$$0 = v_0 + gt$$

$$\Rightarrow t = -\frac{v_0}{g}$$

or $$t = -\frac{80\,ft/s}{-32\,ft/s^2}$$

$$= 2.5\,s$$, which is the time taken by the stone to reach its highest point.

5. a) $$v_f = v_0 + gt$$

$$\Rightarrow v_0 = v_f - gt$$

Suppose down is positive.

$$v_0 = 60\,m/s - (9.8\,m/s)(4s)$$

$$= 60\,m/s - 39.2\,m/s$$

$$= 20.8\,m/s$$

b,

$h = v_0 t + \frac{1}{2} g t^2$

or $h = (20.8 m/s)(4s) + \frac{1}{2}(9.8 m/s^2)(9s)^2$, down is positive

$= 261.6 m$

Projectile Motion

1. a, Since the stone is dropped from rest $v_0 = 0$.

$h = \frac{1}{2} g t^2$

or $h = \frac{1}{2}(9.8 m/s^2)(3s)^2$; down is positive

$= 44.1 m$

In meters feet

$h = \frac{1}{2}(32 ft/s^2)(3s)^2$

$= 149 ft$

if down is negative $h = -44.1 m = -144 ft$

$v = g t$

or $v = (32 ft/s^2)(3s) = -96 ft/s$

b, $x = v_{0x} t = v_x t$

or $x = (100 ft/s)(3s) = 300 ft$

c, $y = v_{0y} t + \frac{1}{2} g t^2$

$= \frac{1}{2} g t^2$, $v_{0y} = 0$

or $y = \frac{1}{2}(-32 ft/s^2)(3s)^2$, down is negative

$= -144 ft$

d, $v_x = v_{0x} = 100 ft/s$

$$v_y = v_{0y} + g t = \frac{v}{5} t$$

or $y = (-32\ ft/s)(3s) = -96\ ft$

2. When released the bomb will have the same horizontal velocity 400 ft/s with the plane - the altitude at which the plane is found by considering the motion of the bomb, i.e.,

$$y = \tfrac{1}{2} g t^2$$

or $y = \tfrac{1}{2}(-9.8\ m/s^2)(6s)^2$

$= -176.4\ m = -576\ ft$

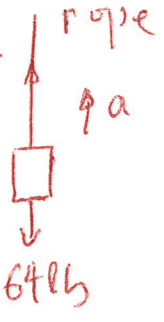
for the plane $y_0 = 176.4\ m = 576\ ft$

$x = v_{0x} t$

or $x = (400\ ft/s)(6s) = 2400\ ft$

Newtons 2nd law of motion

a,



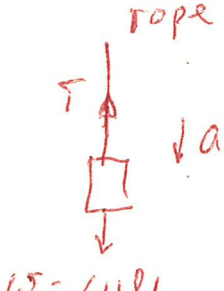
$T - w = ma$

$\Rightarrow T = w + ma$

or $T = 64\ lb + \frac{64\ lb}{32\ ft/s^2} \cdot 5\ ft/s^2$

$= 74\ lb$

b,



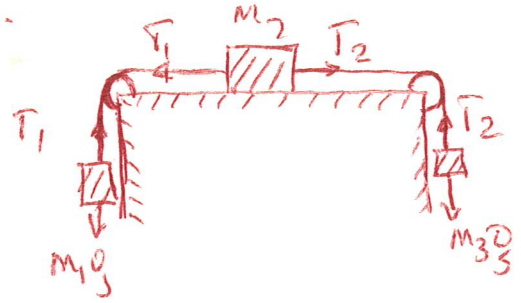
$w - T = ma$

$\Rightarrow T = ma - w$

or $T = -\frac{64\ lb}{32\ ft/s^2} \cdot 5\ ft/s^2 + 64\ lb$

$= 54\ lb$

2.



Apply Newtons 2nd law of motion to each mass

$m_1 g - T_1 = m_1 a$ --- (1)

$T_1 - T_2 = m_2 a$ --- (2)

$T_2 - m_3 g = m_3 a$ --- (3)

Adding the first two equations we obtain

$m_1 g - T_2 = (m_1 + m_2) a$ --- (4)

Adding this and the last eq. we obtain

$(m_1 - m_3) g = (m_1 + m_2 + m_3) a$

$\Rightarrow a = \dfrac{m_1 - m_3}{m_1 + m_2 + m_3} g$

or $a = \dfrac{10 - 6}{10 + 8 + 6} 9.8\ m/s^2$

$= 1.63\ m/s^2$

From the 4th eq.

$T_2 = -(m_1 + m_2) a + m_1 g$

or $T_2 = -(10kg + 8kg)(1.63\ m/s^2) + (10kg \times 9.8\ m/s^2)$

$= 68.6\ N$

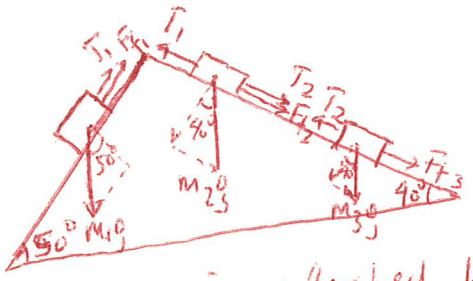
From the first eq.

$T_1 = (m_1 g - m_1 a) = (g - a) m_1$

or $T_1 = (9.8 m/s^2 - 1.63 m/s^2) 10kg$

$= 81.67\ N$

3.



$m_1 = 20kg$

$m_2 = 10kg$

$m_3 = 2kg$

$\mu_K = 0.2$

The motion is effected by m_1 since it has a greater mass. Then we apply Newtons 2nd law to each of the blocks.

$m_1 g \sin 50^\circ - T_1 - F_{f_1} = m_1 a$

But $F_{f_1} = \mu_K m_1 g \cos 50^\circ$

$\Rightarrow m_1 g \sin 50^\circ - T_1 - \mu_K m_1 g \cos 50^\circ = m_1 a$

$\Rightarrow m_1 g (\sin 50^\circ - \mu_K \cos 50^\circ) - T_1 = m_1 a$ --- (1)

$T_1 - m_2 g \sin 40^\circ - T_2 - F_{f_2} = m_2 a$

But $F_{f_2} = \mu_K m_2 g \cos 40^\circ$

$\Rightarrow T_1 - m_2 g \sin 40^\circ - T_2 - \mu_K m_2 g \cos 40^\circ = m_2 a$

$T_1 - m_2 g(\sin 40^\circ + \mu_k \cos 40^\circ) - T_2 = m_2 a$ ... 2

$T_2 - m_3 g \sin 40^\circ - F_{f_3} = m_3 a$

But $F_{f_3} = \mu_k m_3 g \cos 40^\circ$

$\Rightarrow T_2 - m_3 g \sin 40^\circ - \mu_k m_3 g \cos 40^\circ = m_3 a$

$T_2 - m_3 g(\sin 40^\circ + \mu_k \cos 40^\circ) = m_3 a$ - 3

Taking these three equations we solve them simultaneously.

$$\begin{cases} m_1 g(\sin 50^\circ - \mu_k \cos 50^\circ) - T_1 = m_1 a \\ T_1 - m_2 g(\sin 40^\circ + \mu_k \cos 40^\circ) - T_2 = m_2 a \\ T_2 - m_3 g(\sin 40^\circ + \mu_k \cos 40^\circ) = m_3 a \end{cases}$$

Adding the first two equations we get

$$m_1 g(\sin 50^\circ - \mu_k \cos 50^\circ) - m_2 g(\sin 40^\circ + \mu_k \cos 40^\circ) - T_2 = (m_1 + m_2) a$$

Adding this equation with the last of the above three equations we obtain

$$m_1 g(\sin 50^\circ - \mu_k \cos 50^\circ) - m_2 g(\sin 40^\circ + \mu_k \cos 40^\circ) - m_3 g(\sin 40^\circ + \mu_k \cos 40^\circ) = (m_1 + m_2 + m_3) a$$

$$\Rightarrow m_1 g(\sin 50^\circ - \mu_k \cos 50^\circ) - (m_2 + m_3) g(\sin 40^\circ + \mu_k \cos 40^\circ) = (m_1 + m_2 + m_3) a$$

$$\Rightarrow a = \frac{m_1}{m_1 + m_2 + m_3} g(\sin 50^\circ - \mu_k \cos 50^\circ) - \frac{m_2 + m_3}{m_1 + m_2 + m_3} g(\sin 40^\circ + \mu_k \cos 40^\circ)$$

or $a = \frac{20\,kg}{20\,kg + 10\,kg + 12\,kg}(9.8\,m/s^2)(0.766 - 0.1286) - \frac{10\,kg + 12\,kg}{20\,kg + 10\,kg + 12\,kg}(9.8\,m/s^2)(0.643 + 0.153)$

$= 0.98\,m/s^2$, which is the acceleration of the blocks.

From the first equation

$$T_1 = m_1 g(\sin 50^\circ - \mu_k \cos 50^\circ) - m_1 a$$

or $T_1 = 20kg(9.8m/s^2)(0.766 - 0.1286) - 20kg(0.98m/s^2)$

$= 205.33\ N$

From the last eq.

$$T_2 = m_3 a + m_3 g(\sin 40^\circ + \mu_k \cos 40^\circ)$$

or $T_2 = 2kg(0.98m/s^2) + 2kg(9.8m/s^2)(0.643 + 0.153)$

$= 17.56\ N$

Work, Energy and Power

1. a) $E_p = mgh$

or $E_p = (16kg)(9.8m/s^2)(30m) = 4704\ J$

And this is the maximum.

b) Let its velocity at this point be v.

$$mgh' + \tfrac{1}{2}mv^2 = E_{p,max}, \quad h' = 15m$$

$$\Rightarrow v = \sqrt{\frac{2(E_{p,max} - mgh')}{m}}$$

$$= \sqrt{\frac{2(mgh - mgh')}{m}}$$

$$= \sqrt{2g(h - h')}$$

or $v = \sqrt{2(9.8m/s^2)(30m - 15m)}$

$= 27.15\ m/s$

c) $E_p' = mgh'$

or $E_p = (16kg)(9.8m/s^2)(15m) = 2352\ J$

$E_k = \tfrac{1}{2}mv^2$

or $E_K = \frac{1}{2}(16kg)(17.15m/s)^2 = .2352\,J$

or $E_K = E_{p,max} - E_p' = 4704\,J - 2352\,J$

$= 2352\,J$

d,

$E_{tot} = E_p' + E_K = 2352\,J + 2352\,J$

$= 4704\,J$

e,

At the ground the ball does not have P.E. since here the height is reduced to zero. Therefore have

$E_{K_{max}} = E_{p,max} = 4704\,J$

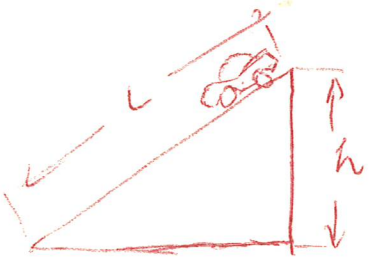
f,

$E_{K,max} = \frac{1}{2} m v_{max}^2$

$\Rightarrow v_{max} = \sqrt{2E_{K,max}}$

or $v_{max} = \sqrt{2(4704\,J)} = 96.99\ m/s$

2.



From the work-Energy theorem we have that

$\Delta E_K + \Delta E_p = \text{Work}_{dec} - \text{Work}_{inc}$

$\Rightarrow \left(\frac{1}{2}mv_0^2 - \frac{1}{2}mv_f^2\right) + (mgh_0 - mgh_f) = LF$

$-\frac{1}{2}mv_f^2 + mgh_0 = LF$

There is nothing Work done that increases the K.E. & P.E. of the car. But friction decreases the K.E. of the car

$\frac{1}{2}mv_f^2 = mgh_0 - LF$

But $h_0 = h$,

$\Rightarrow \frac{1}{2}mv_f^2 = mgh - LF$

or $v_f = \sqrt{2gh - \frac{2LF}{m}}$

Circular Motion

$\omega = 6\ rev/s = 37.70\ rad/s$

then $T = 2\pi/\omega = \frac{2\pi\ rad}{2\pi 6\ rad/s} = 0.267\ sec$

$v = \omega r$

or $v = (37.7\ rad/s)(2m)$

$= 75.4\ m/s$

$a_c = v^2/r$, or $a_c = \frac{(75.4\ m/s)^2}{(2m)} = 2842.58\ m/s^2$

The tension in the cord equals the centripetal force. Hence

$T = m a_c$ or $T = 4kg(2842.58 m/s^2) = 11370.32\ N$

2. $F_{ct} = -F_c = -m\frac{v^2}{r}$

Then $v = \omega r = (37.7\ rad/s)(4m) = 150.8\ m/s$

Then $F_c = -4kg\left(\frac{(150.8\ m/s)^2}{4m}\right)$

$= -22{,}740.64\ N$

Machines

1. Work input = (Work done against friction) + work output

$\Rightarrow$ power input = (power loss due to friction) + power out put

Then, for our case, the out put power is

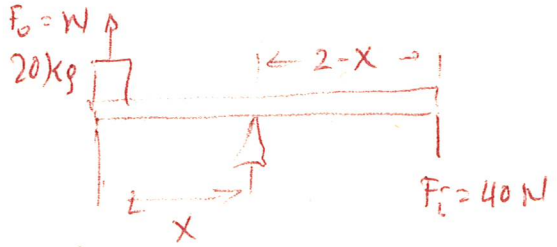
$P_{output} = 300\ hp - 200\ hp = 100\ hp$

Efficiency = $\frac{\text{out put work}}{\text{input work}} = \frac{\text{out put power}}{\text{input power}}$

or $E = \frac{100\ hp}{300\ hp} = 1/3 = 0.33$

or $E = 33\%$

2.



This is the case of a lever and for lever we have that

$\frac{S_i}{S_o} = \frac{F_o}{F_i}$

$s_i = x$, $s_o = 2 - x$

See it quite good

$s_i = x$, $s_o = 2-x$
$x = 1.66\,m$
so $s_o = (2 - 1.66)\,m$
$= 0.44\,m$
∧ would be placed 0.44m from the Load

$$\Rightarrow \quad \frac{x}{2m-x} = \frac{20 \times 9.8}{40}$$

$$\frac{x}{2m-x} = \frac{196}{40} = 4.9$$

$$\Rightarrow \quad x = (2m - x)\,4.9 = 9.8m - 4.9x$$

$$x + 4.9x = 9.8\,m$$

$$5.9x = 9.8\,m$$

$$\text{or} \quad x = \frac{9.8\,m}{5.9} = 1.66\,m$$

i.e., the block of wood has to be located 1.66 m from the load.

3. For a system of frictionless pulleys we have that

$$F_i s_i = F_o s_o$$

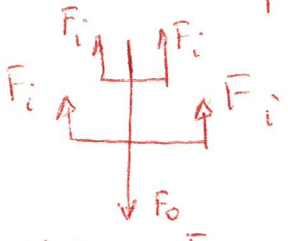
$$\text{or} \quad s_i/s_o = F_o/F_i .$$

a) two ropes are supporting the moveable pulley

$$F_i\,2 = F_o$$

$$\text{or } F_i = F_o/2 = \frac{200N}{2} = 100\,N$$

b) The movable pulley is supported by four ropes

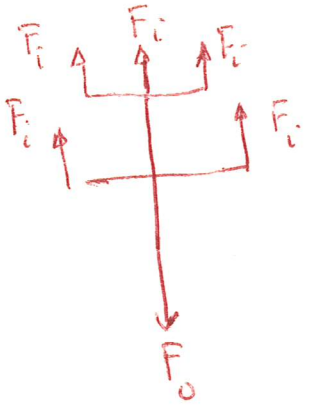


Then $4F_i = F_o$

$$\Rightarrow F_i = F_o/4$$

$$\text{or} \quad F_i = \frac{200N}{4} = 50\,N$$

c. Here we have five ropes supporting the movable pulley.

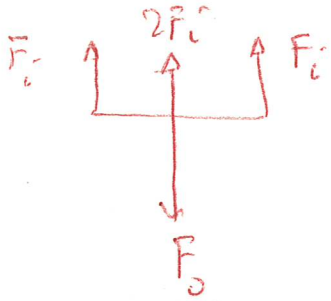


$5F_i = F_o$

$\Rightarrow F_i = F_o/5$

or $F_i = \frac{200N}{5} = 40N$

d. In this case the middle rope supports twice as great as the input force.



$4F_i = F_o$

$\Rightarrow F_i = F_o/4$

or $F_i = \frac{200\,N}{4} = 50N$

Examples

1. Prove that when a body starts from rest and rotates about a fixed axis with constant angular acc., the radial acc. of a point in the body is directly proportional to its angular displacement.

Solution:

$$a_c = \frac{v^2}{r} = \omega^2 r$$

But $\omega = \alpha t$ and $\theta = \frac{1}{2}\alpha t^2$, so that elimination of t between these two eqns gives

$$\omega = \alpha \sqrt{\frac{2\theta}{\alpha}}.$$

That is

$$a_c = \alpha^2 \frac{2\theta}{\alpha} r = 2\alpha r \theta.$$

2. A grindstone in the form of a solid cylinder has a radius of 0.5 m and a mass of 50 kg.

a) What torque will bring it from rest to an angular velocity of 300 rev/min in 10 s.

b) What is the K.E. when it is rotating at 300 rev/min?

Solution:

a) The resultant torque acting on a body is given by

$$\tau = I\alpha = I\left(\frac{\omega - \omega_0}{t}\right)$$

But for the grindstone of solid cylinder form

$I = \frac{1}{2} m R^2$, R is the radius of the cylinder.

$$\Rightarrow \quad \tau = \frac{mR^2}{2}(\omega - \omega_0)$$

or,

$$\tau = \frac{50\text{kg}(0.5\text{m})^2}{2}\left(\frac{31.4\ \text{rad/s}}{10\ \text{s}}\right) = 19.625\ \text{kg m}^2\text{s}^{-2} = 19.625\ \text{Nm}$$

b)

$$K = \frac{1}{2} I \omega^2 = \frac{1}{2} \times \frac{1}{2} \times 50\text{kg}(0.5\text{m})^2 (31.4\ \text{rad/s})^2 =$$

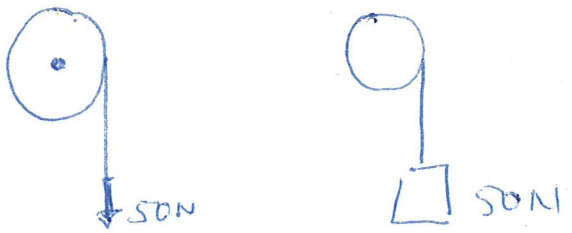
3. A cord is wrapped around the rim of flywheel 0.5m in radius, and a steady pull of 50N is exerted on the cord, as in fig. (a) below. The wheel is mounted in frictionless bearings on a horizontal shaft through its center. The moment of inertia of the wheel is $4\ kg \cdot m^2$.

a. Compute the angular acc. of the wheel.

b. Show that the WORK done in unwinding 5m of cord equals the gain in K.E. of the wheel.

c. If a mass having a weight of 50N hangs from the cord, as in fig. (b), Compute the angular acc of the wheel. [Why is ~~not~~ this not the same as in part (a).]

Solution



(a) (b)

Solution:

a, Now the torque which accelerates the flywheel is due to the 50N force so that

$$\tau = I\alpha$$

Gives

$$50N \times 0.5m = 4\ kg \cdot m^2 \alpha \quad \text{or} \quad \alpha = 6.25\ rad/s^2.$$

b, The WORK done in unwinding 5m cord from the wheel is

$$W = \tau \Delta\theta = 50N \times 0.5m \times \frac{5m}{0.5m} = 250\ J.$$

But the angular velocity after unwinding this length of the cord is

$$\omega = \sqrt{2\alpha\theta} = \sqrt{2 \times 6.25 \times \frac{5}{0.5}}\ rad/s = \sqrt{125}\ rad/s.$$

Assuming the wheel starts from rest

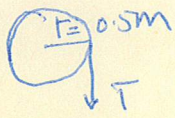
$$\Delta K = \tfrac{1}{2} I \omega^2 = \tfrac{1}{2} \times 4\ \text{kg}\cdot\text{m}^2 \times 125\ \text{s}^{-2} = 250\ \text{J}$$

That is,

$$W = \Delta K.$$

C, In this case the hanging body accelerates downward with an acc. that equals the tangential acc. of a point on the rim of the flywheel.

$$W - T = ma$$



$$Tr = I\alpha$$

Since $a = r\alpha$ we have

and $$W - T = mr\alpha$$

$$Tr = I\alpha$$

or,

$$+\begin{cases} W - T = mr\alpha \\ T = \frac{I}{r}\alpha \end{cases} \qquad W = \left(mr + \frac{I}{r}\right)\alpha$$

$$\alpha = \frac{W}{mr + I/r}$$

Or,

$$\alpha = \frac{50\ \text{N}}{\frac{50}{10}\text{kg} \times 0.5\text{m} + \frac{1/4\ \text{kg}\cdot\text{m}^2}{0.5\text{m}}} = \frac{50\text{N}}{10.5\ \text{kg}\cdot\text{m}} = 4.74\ \text{rad/s}^2.$$

This acc. is less than that in (a) because the tension is less than the accelerating force we have in (a).

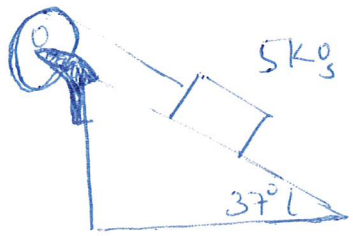
4. A block of mass $m = 5$kg slides down a surface inclined 37° to the horizontal, as shown in the figure below. The coeff. of sliding friction is 0.25. A string attached to the block is wrapped around

A flywheel on a fixed axis at O. The flywheel has a mass $M = 20\,kg$, an outer radius $R = 0.2m$, and a moment of inertia wrt the axis of $0.2\,kg \cdot m^2$.

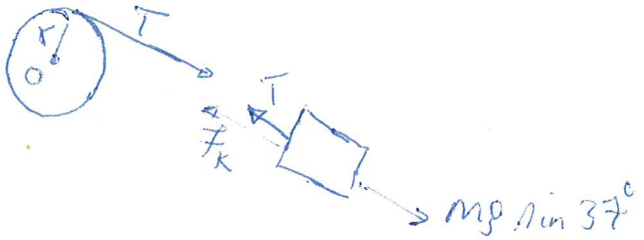
a, What is the acc. of the block down the plane?

b, What is the tension in the string?

SOLUTION



Solution.



$$mg \sin 37^\circ - T - f_k = ma$$

$$T r = I\alpha = I\frac{a}{r}$$

$$\begin{cases} mg \sin 37^\circ - T - \mu mg \cos 37^\circ = ma \\ T = I/r^2 \end{cases}$$

$$mg(\sin 37^\circ - \mu \cos 37^\circ) = (m + I/r^2)a$$

or,

$$a = \frac{mg(\sin 37^\circ - \mu \cos 37^\circ)}{m + I/r^2} = \frac{50(0.6 - 0.25 \times 0.8)}{5 + 0.2/0.04}\ \frac{m}{s^2}$$

$$= 2\ m/s^2$$

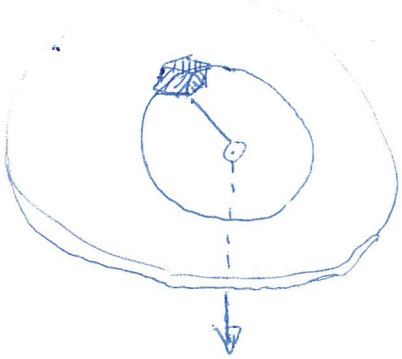
b,

$$T = \frac{I\alpha}{r^2} = \frac{0.2 \times 2}{0.04}\,N = 10N.$$

5. A block of mass 0.05kg is attached to a cord passing through a hole in a horizontal frictionless surface, as in the figure below. The block is originally revolving at a dist. 0.2m from the hole, with an angular velocity of 3 rad/s. The cord is then pulled from below, shortening the radius of the circle in which the block revolves to 0.1m. The block may be considered a point mass.

a) What is the new angular velocity?

b) Find the change in K.E. of the block.



Solution:

a) In this prob. the downward force does not give a torque about the axis of rotation which passes through the hole. This is so because its line of action passes right through the hole. Thus, angular mom. is conserve so that

$$L_2 = L_1 \quad \text{or} \quad I_2\omega_2 = I_1\omega_1$$

whence

$$\omega_2 = \frac{I_1}{I_2}\omega_1 .$$

or,

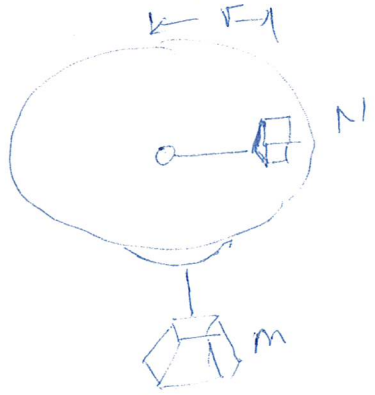
$$\omega_2 = \frac{0.05 \times 0.2^2}{0.05 \times 0.1^2} \times 3 \text{ rad/s} = 12 \text{ rad/s}.$$

b,

$$\Delta K = \tfrac{1}{2} I\omega_2^2 - \tfrac{1}{2} I_1 \omega_1^2 = \frac{1}{2}\left(.005 \times 0.2^2 \times 12^2 - .05 \times 0.1^2 \times 3^2\right) J$$

$$= 0.027 J.$$

6 A block of mass M rests on a turntable ~~is attached to a cord passing through a hole in a horizontal frictionless surface. The block is originally revolving in a circle of radius 0.5m about the hole~~ that is rotating at a const. angular velocity ω. A smooth cord runs from the block through a hole in the center of the table down to a hanging block of mass m. The coeff. of friction between the first block and the turntable is μ. (See the figure below) Find the largest and smallest values of the radius r for which the first block will remain at rest relative to the turntable.



Solution:

If the radius of the circle of rotation has to be large M must move to the edge of the turntable. In that case the force of friction will act to the center of the turntable and we have

$$mg + \mu Mg = M\omega^2 r \quad \text{or} \quad r = \frac{mg + \mu Mg}{\omega^2}$$

For a smaller radius M must move toward the center of the turntable in which case the force of friction will act outward and the centripetal force is given by

$$mg - \mu Mg = M\omega^2 r \quad \text{or} \quad r = \frac{mg - \mu Mg}{\omega^2}$$

Thus

$$r = \frac{mg \pm \mu Mg}{\omega^2}$$

Assignment II

Newton's Laws

1. If action and reaction are always equal in magnitude and opposite direction, why don't they cancel each other and leave no net force for acceleration of the body?

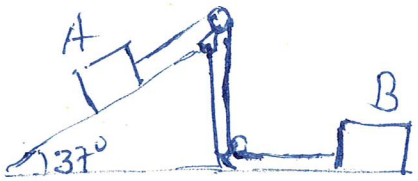
2. A 0.22 rifle bullet, traveling at 360 m/s, strikes a block of soft wood, which it penetrates to a depth of 0.1m. The mass of the bullet is 1.8 g. Assume a constant retarding force.
   a, How much time was required for the bullet to stop?
   b, What was the accelerating force, in newtons? in pounds?

3. A hockey puck leaves a player's stick with a velocity of 10 m/s and slides 40m before coming to rest. Find the coefficient of friction between the puck and the ice.

4. A 5kg block slides down a plane inclined at 30° to the horizontal. Find the acceleration of the block a, if the plane is frictionless; b, if the coefficient of kinetic friction is 0.2.

5. A 160 lb man stands on a bathroom scale in an elevator. As the elevator starts moving, the scale reads 200 lb.
   a, Find the acceleration of the elevator (magnitude and direction).
   b, What is the acceleration if the scale reads 120 lb?
   c, If the scale reads zero, should the man worry?

6. Two blocks, each having mass 20 kg, rest on frictionless surfaces, as shown below. Assuming the pulleys to be light and frictionless, compute a, the time required for block A to move 1m down the plane, starting from rest; b, the tension in the cord connecting the blocks.



7 Two 0.2 kg blocks hang at the ends of a light flexible cord passing over a small frictionless pulleys, as in the figure. A 0.1 kg block is placed on the right, and removed after 2s.

a, How far will each block move in the first second after the 0.1 kg block is removed?

b, What was the tension in the cord before the 0.1kg block was removed? After it was removed?

c, What was the tension in the cord supporting the pulley before the 0.1kg block was removed? Neglect the weight of the pulley.

Work and Energy

1. A 40 kg block is pushed a distance of 5m along a level floor at constant speed by a force at an angle of 30° below the horizontal. The coefficient of friction between block and floor is 0.25. How much work is done by the force? By friction?

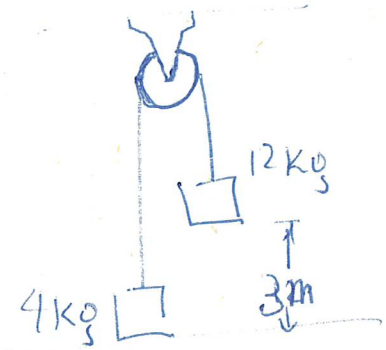
2. A body of mass 2kg is initially at rest on a horizontal frictionless plane. It is then pulled 4m by a horizontal force of magnitude 25N. Find its final speed.

3. A 5kg block is lifted vertically at a constant velocity of 4m/s through a height of 12m.

a, How great a force is required?

b, How much work is done? What becomes of this work?

4. The system in the figure below is released from rest with the 12kg block 3m above the floor. Use the principle of conservation of mechanical energy to find the velocity with which the block strikes the floor. Neglect friction and inertia of the pulley.

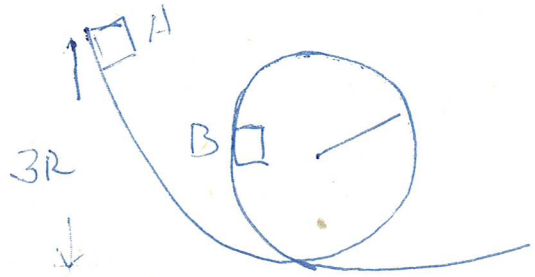


5. The spring of a spring gun has a force constant of 500 N/m. If it is compressed by 0.05m and a ball of mass 0.01kg is placed in the barrel against the compressed spring.
a, Compute the maximum velocity with which the ball leaves the gun when released.
b, Determine the maximum velocity with which the ball leaves if a constant resisting force of 10N acts on the ball.

6 (Figure below). A block of mass 2kg is released from rest at point A on a track that is one quadrant of a circle of radius 1m (below). It slides down the track and reaches point B with a speed of 4m/s. From point B it slides on a level surface a distance of 3m to point C, where it comes to rest.
a, What was the coefficient of sliding friction on the horizontal surface?
b, How much work was done against friction as the body slid down the circular arc from A to B?

7. A small body of mass m slides without friction around the loop-the-loop apparatus shown below. It starts from rest at point A at a height 3R above the bottom of the loop. When it reaches point B at the end of a horizontal diameter of the loop, compute a, its radial acceleration, b, its tangential acceleration, and c, its resultant acceleration.

8. (Almeter)



8. A meter stick, pivoted about a horizontal axis through its center, has a body of 2 kg attached to one end and a body of 1 kg attached to the other end. The mass of the meter stick can be neglected. The system is released from rest with the stick horizontal. What is the velocity of each body as the stick swings through a vertical position?

9. A 50 kg man jumps from a height of 2m onto a platform mounted on springs. As the springs compress, the platform is pushed down a maximum distance of 0.2m below its initial position, and it then rebounds.
a, What is the mans speed at the instant the platform is depressed 0.1m?
b, If the man had just stepped gently onto the platform how far would it have been pushed down?

10. A skier starts at the top of a large frictionless spherical snowball, with very small initial velocity, and skis straight down the side. At what point does he lose contact with the snowball and fly off at a tangent? That is, at the instant he loses contact with the snowball, what angle does a radial line from the center to the skier make with the vertical?

11. A 2000 kg elevator starts from rest and is pulled upward with a constant acceleration of $4 m/s^2$.
a, Find the tension in the supporting cable.
b, What is the velocity of the elevator after it has risen 15 m?

12. An automobile of mass 1200 kg has a speed of 30 m/s on a horizontal road when the engine is developing 50 hp. What is its speed, with the same horsepower, if the road rises 1m in 20 m? Assume all friction forces to be constant.

13. a, If 20 hp are required to drive a 1200 kg automobile at 50 km/hr on a level road, what is the total

retarding force due to friction, windage, etc.

b, What power is necessary to drive the car at 50 km/hr up a 10 percent grade (i.e., one rising 10 m vertically in 100 m horizontally)?

c, What power is necessary to drive the car at 50 km/hr down a 2 percent grade?

d, Down what percent grade would the car coast at 50 km/hr?

Impulse and Momentum

1. A golf ball of mass 0.10 kg is initially at rest when it is struck by a club and given a speed of 50 m/s. If the club and ball are in contact for 0.002 s, what average force acted on the ball?

2. A tennis ball approaches a player's racket horizontally at 10 m/s. After it is struck its velocity is horizontal, in the opposite direction, with magnitude 20 m/s. The ball has mass 0.06 kg, and it is in contact with the racket for 0.01 s. What average force acted on the ball?

3. A steel ball of mass 0.5 kg is dropped from a height of 4 m onto a horizontal steel slab. The collision is "elastic", and the ball rebounds to its original height.

a, Calculate the impulse delivered to the ball during the impact.

b, If the ball is in contact with the slab for 0.002 s, find the average force on the ball during impact.

4. A bullet of mass 20 g, traveling in a horizontal direction with a velocity of 500 m/s, is fired into a wooden block of mass 1 kg, initially at rest on a level surface. The bullet passes through the block and emerges with its velocity reduced to 100 m/s. The block slides a distance of 20 cm along the surface from its initial position.

a, What was the coefficient of sliding friction between block and surface?

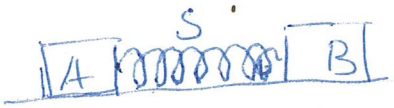
b, What was the ~~appl~~ decrease in K.E. of the bullet?
c, What was the K.E. of the block at the instant after the bullet passed through it?

5. A 5g bullet is fired horizontally into a 3kg wooden block resting on a horizontal surface. The coefficient of sliding friction between block and surface is 0.20. The bullet remains embeded in the block, which is observed to slide 25cm along the surface before stopping. What was the velocity of the block?

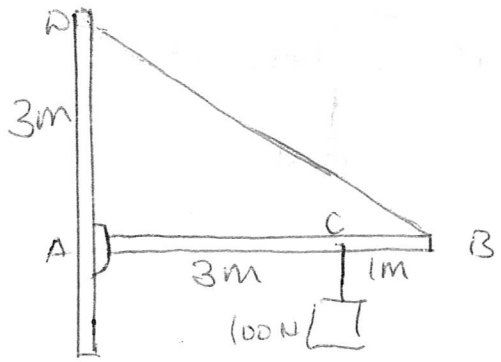
6. A rifle bullet of mass 0.01kg strikes and embeds itself in a block of mass 0.99kg which rests on a horizontal frictionless surface and is attached to a coil spring, as shown below. The impact compresses the spring 10cm. Calibration of the spring shows that a force of 1N is required to compress the spring 1cm.
a, Find the maximum P.E. of the spring.
b, Find the velocity of the block just after the impact.
c, What was the initial velocity of the bullet?

7. A small steel ball moving with speed v_0 in the positive x-direction makes a perfectly elastic, noncentral collision with an identical ball originally at rest. After impact, the first ball moves with speed v_1 in the first quadrant at an angle θ_1 with the x-axis and the second with speed v_2 in the fourth quadrant at an angle θ_2 with the x-axis.
a, Write the equations expressing conservation of linear momentum in the x-direction and in the y-direction.
b, Square these equations and add them.
c, At this point, introduce the fact that the collision is perfectly elastic.
d, Prove that $\theta_1 + \theta_2 = \pi/2$

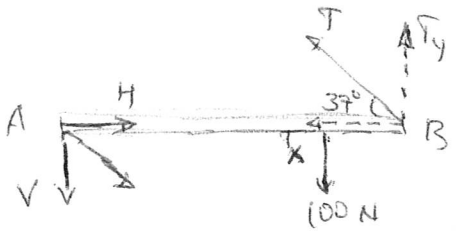
8. Block A in the figure below has a mass of 1kg and block B has a mass of 2kg. The blocks are forced together, compressing a spring S between them, and the system is released from rest on a level frictionless surface. The spring is not fastened to either block and drops to the surface after it has expanded. Block B acquires a speed of 0.5m/s. How much P.E. was stored in the compressed spring?



5 Find the tension in the cable BD in the figure below, and the horizontal and vertical comps of the force exerted on the strut AB at pin A, using:
a, the first cond. of equil. only, taking torques about an axis through A, then about an axis through B, and finally the 1st & 2nd conditions of equil. ($\Sigma F_x = 0$, $\Sigma F_y = 0$, $\Sigma \tau = 0$), taking torques about an axis through pin A ⊥ to the plane of the diagram.
b, the 2nd cond. of equil. only, taking torques about an axis through A, then about an axis through B, and finally about an axis through D. The weight of the strut can be neglected.
c, Represent the computed forces by vectors in a diagram, and show that the lines of action of the forces exerted on the strut at points A, B & C intersect at a common point.



a,



$T_x = -T\cos 37^\circ = -0.8T$
$T_y = T\sin 37^\circ = 0.6T$

$\Sigma F_x = H - 0.8T = 0$
$\Sigma F_y = 0.6T - V - 100\,N = 0$
$\Sigma \tau_A = (0.6T)4m - 100\,N \times 3m = 0$, CCW torques are taken "+".

The last eq. gives

$$T = \frac{100\,N}{0.2 \times 4} = 125\,N$$

From the 1st eq.

$H = 0.8 \times 125N = 100\,N$, correct choice of direction

From the 2nd eq.

$$V = 0.6T - 100N = 0.6 \times 125N - 100N = -25N,$$

i.e., V is directed upward.

b, Taking moment torques about end A we have

$$\Sigma \tau_A = 0.6T\,(4m) - 100N \times (3m) = 0, \text{ or } T = 125N.$$

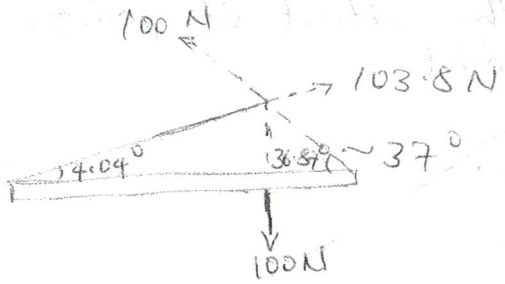
Taking torques about end B we have

$$\Sigma \tau_B = 100N \times 1m + V \times 4m = 0, \text{ or } V = -\frac{100}{4}N = -25N, \text{up.}$$

Taking torques about point D we have

$$\Sigma \tau_D = H \times 3m - 100N \times 3m = 0, \text{ or } H = 100N.$$

C,



(*) Problems on Rotation

1. A long uniform rod of length L and mass M is pivoted about a horizontal, frictionless pin through one end. The rod is released from rest in a vertical position as in the figure below. At the instant the rod is horizontal find a, the angular velocity of the rod b, its angular acc. c, the x & y comps of the acc. of its c.m., and d, the comps of the reaction force at the pivot.

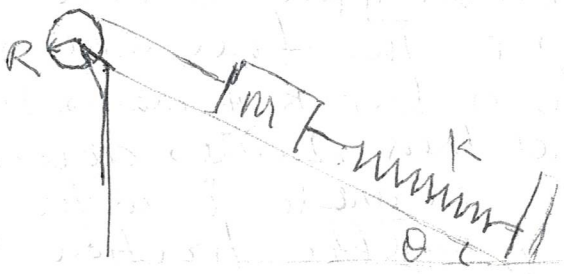
- When the rod makes 53° from the horizontal.
- pivot $\frac{1}{4}$ from one end

a, $\sqrt{3g/L}$ b, $3g/2L$

c, $a_x = \frac{3}{2}g$, $a_y = \frac{3}{4}g$

d, $R_x = \frac{3}{2}Mg$ $R_y = \frac{1}{4}Mg$

2. The pulley shown in the figure below has a radius R and moment of inertia I. One end of the mass m is connected to a spring of force const. K, and the other end is fastened to a cord wrapped around the pulley. The pulley, axle & the incline are smooth. If the pulley is wound ccw so as to stretch the spring a dist. d from its unstretched position and then released from rest, find a, the angular velocity of the pulley when the spring is again unstretched.



3. Def.

The Radius of gyration k of a body is given by

$$I = Mk^2$$

where I is the moment of inertia of the body and M its mass. Using this expression calculate k for a solid disk of radius R, a uniform rod

of length L and a solid sphere of radius R. Assume all are rotating about a central axis.

4. A uniform horizontal plank of mass M and length L is supported at each end by vertical ropes. At the instant one of the ropes breaks, show that a) the angular acc. of the plank is $3g/2L$, b) the acc. of the c.m. is $3g/4$ and c) the tension in the supporting rope is $Mg/4$.

5. A mass m is supported by a cord that is wound around a spool of radius r and mass M. A motor accelerates the spool to angular velocity ω_0 and then disengages from the spool. a) How far will the mass m rise (in terms of m, M, r, ω_0 & g) after the motor has been disengaged? $h = \frac{r^2\omega_0^2}{2g}\left(1+\frac{M}{2m}\right)$

b) Show that the acc. of the mass m is

$$a = -\frac{m}{m+\frac{1}{2}M}g.$$

c) Show that the mass m rises to a max. height in time

$$t = \left(1+\frac{M}{2m}\right)\frac{r\omega_0}{g}.$$

~~6~~. A cord is wrapped around a pulley of mass m and radius r. The free end of the cord is connected to a block of mass M. The block starts from rest and then slides down a rough incline which makes an angle θ with the horizontal. The coeff. of kinetic friction between the block and the incline is μ. a) Use the W-E theorem to show that the block's velocity is v as a function of the dist. d down the incline is

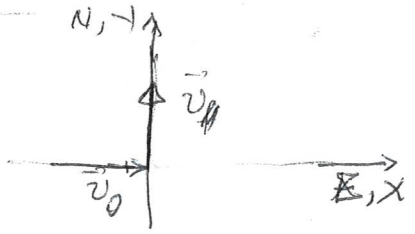
$$v = \left[4gd\left(\frac{M}{m+2M}\right)(\sin\theta - \mu\cos\theta)\right]^{1/2}$$

Problems

I. Impulse & Momentum

1. A 1500 kg automobile travels eastward at a speed of 8 m/s. It makes a 90° turn to the north in a time of 3 s and continues with the same speed. Find a, the impulse delivered to the car as a result of the turn and b, the average force exerted on the car during the turn.

a,



$$\vec{v}_0 = 8\,\frac{m}{s}\,\vec{i}\,, \quad \vec{v}_f = 8\,\frac{m}{s}\,\vec{j}$$

$$\vec{I} = m\,\vec{v}_f - m\vec{v}_0 = m(\vec{v} - \vec{v}_0)$$

or,

$$\vec{I} = 1500\ kg\,(8\,\vec{j} - 8\,\vec{i})\,\frac{m}{s} = 1.2\times10^4\,(\vec{j} - \vec{i})\,kg\,\frac{m}{s}$$

Its magnitude is

$$I = 1.2\times10^4\ kg\,\frac{m}{s} = 1.2\times10^4\ N\,s.$$

b, But

$$\vec{F}\,t = \vec{I} \quad \text{or} \quad F\,t = I$$

So that

$$\vec{F} = \frac{\vec{I}}{t} = \frac{1.2\times10^4\ N\,s}{3\ s}\,(\vec{j} - \vec{i}) = 4\times10^3\,(\vec{j} - \vec{i})\ N$$

or, the magnitude is

$$F = 5.6\times10^3\ N.$$

2. A 200 kg projectile, fired with a speed of 100 m/s at a 60° angle, breaks into three pieces of equal mass at the highest point of its arc. Two of the fragments move with the same equal speed right after the explosion as the entire projectile had just before the explosion. One

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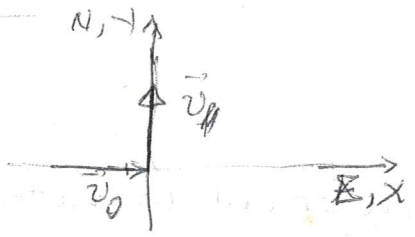
a.



$$\vec{v}_0 = 8\,\frac{m}{s}\,\vec{i}\,, \quad \vec{v}_f = 8\,\frac{m}{s}\,\vec{j}$$

$$\vec{I} = m\vec{v}_f - m\vec{v}_0 = m(\vec{v} - \vec{v}_0)$$

or,

$$\vec{I} = 1500\ kg\left(8\,\vec{j} - 8\,\vec{i}\right)\frac{m}{s} = 1.2\times10^4\left(\vec{j} - \vec{i}\right)kg\,\frac{m}{s}$$

Its magnitude is

$$I = 1.2\times10^4\ kg\,\frac{m}{s} = 1.70\times10^4\ N\,s.$$

b. ~~From the impulse ...~~ But

$$\vec{F}\,t = \vec{I} \quad \text{or} \quad Ft = I$$

So that

$$\vec{F} = \frac{\vec{I}}{t} = \frac{1.2\times10^4\ N\,s}{3\ s}\left(\vec{j} - \vec{i}\right) = 4\times10^3\left(\vec{j} - \vec{i}\right)N$$

or, the magnitude is

$$F = 566\times10^3\ N.$$

2. A 200 kg projectile, fired with a speed of 100 m/s at a 60° angle, breaks into three pieces of equal mass at the highest point of its arc. Two of the fragments move with the same equal speed right after the explosion as the entire projectile had just before the explosion. One

$$\Delta K = \frac{1}{2}\frac{m}{3}\left(\frac{100^2+50^2}{3}+\frac{50^2}{1}+\frac{50^2}{2}\right)-\frac{1}{2}m\cdot 50^2$$

$$= \frac{1}{2}\frac{m}{3}\left(\frac{100^2}{18}+3\cdot 50^2\right)-\frac{1}{2}m\cdot 50^2$$

$$= \frac{1}{2}m\left(\frac{100^2}{3}+50^2\right)-\frac{1}{2}m\cdot 50^2 = \frac{1}{2}\cdot m\,\frac{100^2}{3}$$

or,

$$\Delta K = \frac{1}{2}\times 200\times\frac{100^2}{3}J = \frac{100^6}{3}J = 3.33\times 10^5 J.$$

II Rotation

1. A wheel rotates in such a way that its angular displacement in time t is given by $\theta = at^2 + bt^3$, where a & b are constants. Determine a) the angular speed & b) the angular acc., both as functions of time.

Ans. a) $\omega = 2at + 3bt^2$, b) $\alpha = 2a + 6bt$.

2. Two wheels A & B are connected by a belt and are free to rotate about fixed axes through their centers. Wheel A is set from rest in rotational motion with angular acc. of 10 rad/s^2.

a) Find the angular acc of wheel B.

b) If the rotational K.E. of wheel B at time $t = 2$ s is 50 J, what is its moment of inertia about the axis of rotation?

$r_A = 0.2$ m

$r_B = 0.8$ m.

(a) Along the belt, Here tangential quantities are equal so that

$$a_A = a_B \text{ or } r_A\alpha_A = r_B\alpha_B$$

$$\Rightarrow \alpha_B = \frac{r_A}{r_B}\alpha_A = \frac{0.2}{0.8}\times 10\ \frac{\text{rad}}{s^2} = 2.5\ \frac{\text{rad}}{s^2}$$

6.

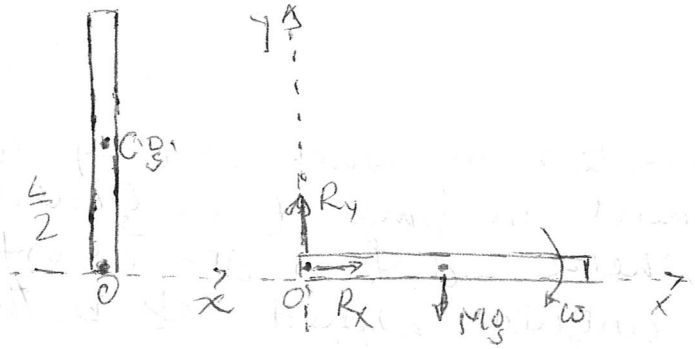
$$K_B = \frac{1}{2} I_B \omega_B^2$$

But $\omega_B = \alpha_B t$ so that $K_B = \frac{1}{2} I_B \alpha_B^2 t^2$. or,

$$I_B = \frac{2K_B}{\alpha_B^2 t^2} = \frac{2 \times 50}{2.5^2 \times 2^2} \text{ kg-m}^2 = 4 \text{ kg-m}^2.$$

Solution to problems of (*)

1→3.



a, M.E. is conserved

$$Mg\frac{L}{2} = \frac{1}{2} I_O \omega^2 = \frac{1}{2}\left(\frac{1}{3}ML^2\right)\omega^2$$

$$\Rightarrow \omega = \sqrt{3g/L}$$

b, In the horizontal position, the torque about O is due to its weight, i.e.)

$$\tau = I_O \alpha \Rightarrow Mg\frac{L}{2} = \frac{1}{3}ML^2\alpha \text{ or } \alpha = 3g/2L.$$

c, At the horizontal position the velocity of the cm which is performing circular motion about O is tangential given by

$$v = r\omega = \frac{L}{2}\sqrt{3g/L} = \frac{\sqrt{3}}{2}\sqrt{gL}.$$

The x-acc. of the c.m. is centripetal given by

$$a_x = \frac{v^2}{r} = \frac{(3/4)gL}{L/2} = \frac{3}{2}g.$$

The y-acc. of the c.m. is tangential given by

$$a_y = r\alpha = \frac{L}{2}\cdot\frac{3g}{2L} = \frac{3}{4}g.$$

d, R_x gives the centripetal force so that

$$R_x = M a_x = \frac{3}{2}Mg.$$

In the y-direction

$$Mg - R_y = M a_y = \frac{3}{4}Mg$$

$$\Rightarrow R_y = Mg - \frac{3}{4}Mg = \frac{1}{4}Mg.$$

2 → 4. M.E

M.E. is conserved here.

$$\tfrac{1}{2}kd^2 + mgh_1 = mgh_2 + \tfrac{1}{2}I\omega^2 + \tfrac{1}{2}mv^2$$

$$\tfrac{1}{2}kd^2 + mg(h_1 - h_2) = \tfrac{1}{2}I\omega^2 + \tfrac{1}{2}mv^2$$

$$\tfrac{1}{2}kd^2 + mgd\sin\theta = \tfrac{1}{2}I\omega^2 + \tfrac{1}{2}mv^2$$

$$= \tfrac{1}{2}I\omega^2 + \tfrac{1}{2}MR^2\omega^2, \quad v = R\omega$$

or,

$$\omega^2 = \frac{kd^2 + 2mgd\sin\theta}{I + MR^2}$$

or,

$$\omega = \left(\frac{kd^2 + 2mgd\sin\theta}{I + mR^2}\right)^{1/2}.$$

3 → 5.

- Solid disk about central axis

$$Mk^2 = I = \tfrac{1}{2}MR^2 \Rightarrow k = R/\sqrt{2}$$

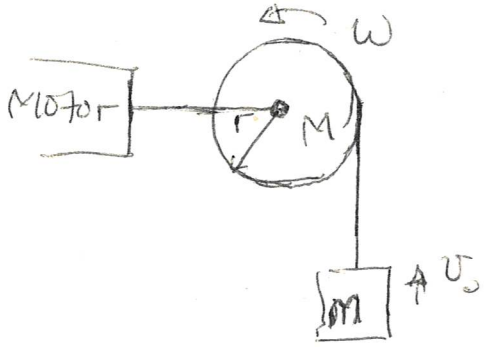
- Uniform rod of length L about central axis

$$Mk^2 = I = \frac{1}{12}ML^2 \Rightarrow k = \frac{L}{2\sqrt{3}}.$$

- Solid sphere about central axis

$$Mk^2 = I = \tfrac{2}{5}MR^2 \Rightarrow k = \sqrt{2/5}\,R.$$

5-6.



a. M.E. is conserved

$$\tfrac{1}{2}I\omega^2 + \tfrac{1}{2}mv^2 = mgh$$

But $I = \tfrac{1}{2}Mr^2$, $v_0 = r\omega$

$$\tfrac{1}{2}\left(\tfrac{1}{2}Mr^2\omega^2 + \tfrac{1}{2}mr^2\omega^2\right) = mgh$$

or,

$$h = \frac{r^2\omega^2}{2g}\left(\frac{M}{2m} + 1\right)$$

b) This is motion with const. force and hence with const. acc. then,

$$v^2 - v_0^2 = 2a_y h \Rightarrow a_y = -\frac{v_0^2}{2h} = -\frac{r^2\omega^2}{2h}$$

or,

$$a_y = -\frac{g}{\frac{M}{2m} + 1} = -\frac{m}{m + \frac{1}{2}M}g.$$

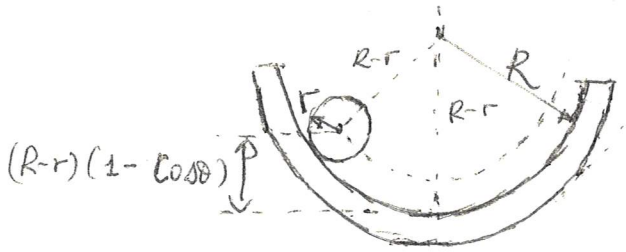
c,

$$h = \frac{v_0 + v}{2} t \rightarrow t = \frac{2h}{v_0} = \frac{2h}{r\omega}$$

or,

$$t = \frac{2}{r\omega} \cdot \frac{r^2\omega^2}{2g}\left(1 + \frac{M}{2m}\right) = \frac{r\omega}{g}\left(1 + \frac{M}{2m}\right)$$

6-7 A uniform solid sphere of radius r is placed on the inside surface of a hemispherical bowl of radius R. The sphere is released from rest at an angle θ to the vertical and rolls without slipping (Fig. below). Determine the angular speed of the sphere when it reaches the bottom of the bowl.



In rolling without slipping M.E. is conserved so that

$$Mg(R-r)(1-\cos\theta) = \frac{1}{2}Mv_{cm}^2 + \frac{1}{2}I\omega^2$$

where the 1st term on the right is the K.E. of the c.m. and $\frac{1}{2}I\omega^2$ is the K.E. of rotation of the sphere at its c.m. Since $v_{cm} = r\omega$ we write

$$Mg(R-r)(1-\cos\theta) = \frac{1}{2}mr^2\omega^2 + \frac{1}{2}\cdot\frac{2}{5}mr^2\omega^2 = \frac{7}{10}mr^2\omega^2$$

i.e,

$$\omega^2 = \frac{10}{7}\frac{g}{5}\frac{R-r}{r^2}(1-\cos\theta)$$

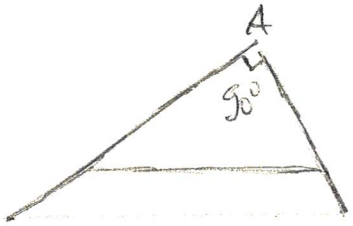
or,

$$\omega = \left[\frac{10}{7}\frac{g}{5}\frac{R-r}{r^2}(1-\cos\theta)\right]^{1/2}.$$

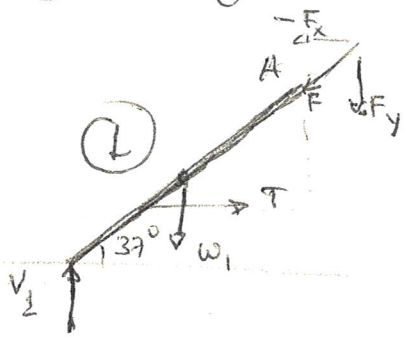
III Equilibrium of a Rigid Body

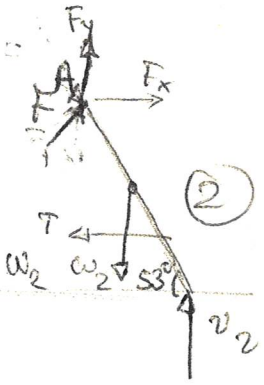
1. Two ladders 4m & 3m long, respectively, are hinged at point A and tied together by a horizontal

rope 0·6m above the floor, as in the figure below. The ladders weigh 400 N & 300 N, resp., and the center of gravity of each is at its center. If the floor is frictionless find a, the upward force at the bottom of each ladder b, The tension in the rope, and c, the force that one ladder exerts on the other at point A. d, If a load of 1000 N is now suspended from point A, find the tension in the rope.



a, Consider each ladder separately and draw two free body diagram of each.





Ladder 1

$\Sigma F_x = 0$: $T - F_x = 0$

$\Sigma F_y = 0$: $V_1 - w_1 - F_y = 0$

Ladder 2

$F_x - T = 0$

$V_2 + F_y - w_2 = 0$

$\Sigma \tau_A = 0$: $T(4 \sin 37^\circ - 0.6)m + w_1 2 \cos 37 - V_1 4 \cos 37^\circ = 0$, for (1)

$1.8T + 1.6 w_1 - 3.2 V_1 = 0$

$\Sigma \tau_A = 0$: $V_2 3 \cos 53^\circ - w_2 \frac{3}{2} \cos 53^\circ - T(3 \sin 53^\circ - 0.6) = 0$, for (2)

$-1.8T - 0.9 w_2 + 1.8 V_2 = 0$

These equations are sufficient to calculate the unknowns.

Adding the last two equations above we obtain

$1.6\,\omega_1 - 0.9\,\omega_2 + 1.8\,V_2 - 3.2\,V_1 = 0$

$\Rightarrow \quad 1.6 \times 400 - 0.9 \times 300 + 1.8\,V_2 - 3.2\,V_1 = 0$

$370 + 1.8\,V_2 - 3.2\,V_1 = 0$

or,

$3.2\,V_1 - 1.8\,V_2 = 370$

Adding the eqn $\Sigma F_y = 0$ for the two ladder above in the two cases a

$V_1 + V_2 - \omega_1 - \omega_2 = 0$

or,

$V_1 + V_2 = \omega_1 + \omega_2 = 400 + 300 = 700$

Now,

$$+ \quad 1.8 \begin{cases} 3.2\,V_1 - 1.8\,V_2 = 370 \\ V_1 + V_2 = 700 \end{cases}$$

$5\,V_1 = 370 + 1260 = 1630$

or,

$V_1 = \frac{1630}{5}\,N = 326\,N$

and,

$V_2 = 700 - V_1 = 374\,N.$

b, Using the torque eq. for ladder 1

$1.8\,T + 1.6\,\omega_1 - 3.2\,V_1 = 0$

or,

$T = \frac{3.2\,V_1 - 1.6\,\omega_1}{1.8} = \frac{3.2 \times 326 - 1.6 \times 400}{1.8}\,N$

$= \frac{3.2}{1.8}\,N = 224\,N.$

c, From the x-eq of lader 1

$F_x = T = 224\,N$

and from the y- eq

$F_y = V_1 - \omega_1 = 326N - 400N = -74N$, up.

i.e,

$F = \sqrt{F_x^2 + F_y^2} = \sqrt{224^2 + 74^2}\,N \simeq 236\,N.$

d, Will now have

$$V_1 + V_2 = 1700$$

$$3.2V_1 - 1.8V_2 = 1.6w_1 + 0.9w_2 = 370$$

from part (a). This given

$$5V_1 = 1.8 \times 1700 + 370 = 3430$$

or,

$$V_1 = \frac{3430}{5} N = 686 N.$$

Then, we had

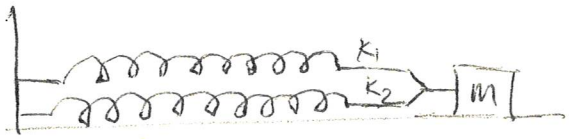
$$T = \frac{3.2V_1 - 1.6w_1}{1.8} = \frac{3.2 \times 686 - 1.6 \times 400}{1.8} N$$

$$= 864 N.$$

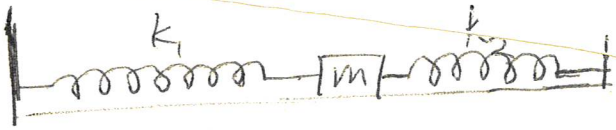
IV. Periodic Motion

✓1. Two springs with the same unstretched length but diff. force constants k_1 & k_2 are attached to a block of mass m on a level frictionless surface. Calculate the effective force const. in each of the three cases (a), (b) & (c) shown below. d, A body of mass m, suspended from a spring with a force const. k, vibrates with a frequency f_1. When the spring is cut in half and the same body is suspended from one of the halves, the frequency is f_2. What is the ratio f_2/f_1?

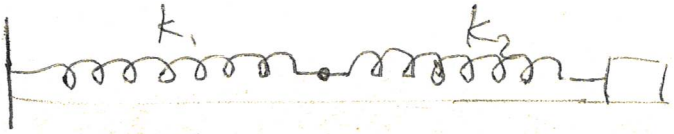
a,



b,



c,



a, Here the displacement X of the mass equals the amount by which either spring is stretched or compressed. Hence

$$F = -KX = -k_1 X - k_2 X = -(k_1 + k_2)X$$

(i.e.)

$$K = k_1 + k_2$$

b, Again, here the displacement X of the body is the same as the amount by which either spring is stretched or compressed. Then, since the forces due to the two springs are acting on the body in the same direction, we write

$$F = -KX = -k_1 X - k_2 X = -(k_1 + k_2)X$$

(i.e.)

$$K = k_1 + k_2$$

c, Let the displacement of the body be X. Then each spring is stretched or compressed by a diff. amount due to the difference in its spring const. from the other. It means these lengths are x_1 & x_2 for the spring of force const k_1 & spring of force const k_2), resp.

$$X = x_1 + x_2$$

The force on the body is

$$F = -k_2 x_2 - k x = -k_2 x_2$$

From Newtons third law $-k_1 x_1 = -k_2 x_2$

$$x_2 = X - x_1 = X - \frac{k_2}{k_1} x_2$$

$$x_2 + \frac{k_2}{k_1} x_2 = X \quad \text{or} \quad x_2 = \frac{k_1}{k_1 + k_2} X$$

(i.e.)

$$F = -KX = -\frac{k_1 k_2}{k_1 + k_2} X$$

$$\therefore \quad K = \frac{k_1 k_2}{k_1 + k_2}$$

$f_1 = \frac{1}{2\pi}\sqrt{\frac{K}{m}}$

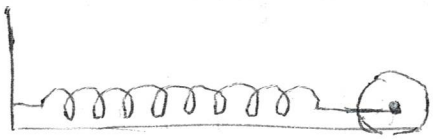
$f_2 = \frac{1}{2\pi}\sqrt{\frac{2K}{m}}$

$\Rightarrow f_2/f_1 = \sqrt{2}$.

d, For the whole spring $mg = K_1 \Delta\ell \Rightarrow k_1 = mg/\Delta\ell$

For one of the halves $mg = k_2' \Delta\ell/2 \Rightarrow k_2' = 2\frac{mg}{\Delta\ell} = 2K$

2. A uniform solid cylinder of mass M and radius R is attached to a spring of force const. k on a horizontal surface as shown in the figure below. The cylinder rolls without slipping on the horizontal surface and its c.m. executes SHM. The amplitude of oscillation is A. Determine a, the speed of the cylinder when it is at a dist. x from its equil. position. b, the period of SHM.



a) M.E. is conserved in pure rolling so that

$$\frac{1}{2}kA^2 = \frac{1}{2}kx^2 + \frac{1}{2}mv_{cm}^2 + \frac{1}{2}I\omega^2$$

$$= \frac{1}{2}kx^2 + \frac{1}{2}mv_{cm}^2 + \frac{1}{2}\cdot\frac{1}{2}mr^2\left(\frac{v_{cm}}{r}\right)^2$$

$$= \frac{1}{2}kx^2 + \frac{3}{4}mv_{cm}^2$$

$$\Rightarrow \quad \frac{3}{4}mv_{cm}^2 = \frac{1}{2}k(A^2-x^2)$$

$$v_{cm}^2 = \frac{2}{3}\frac{k}{m}(A^2-x^2)$$

i.e.)

$$v_{cm} = \sqrt{\frac{2k}{3m}(A^2-x^2)}$$

b) The cylinder rotates about its c.m.

$$\tau = I\alpha \Rightarrow fr = I\alpha = I\frac{a}{r}$$

or)

$$f = I\frac{a}{r^2} = \frac{1}{2}mr^2\frac{a}{r^2} = \frac{1}{2}ma \quad -- (1)$$

For the translational motion of the c.m

$$-f - kx = ma \quad \text{---------- (2)}$$

1+2 $\Rightarrow$ $\quad -kx = \frac{3}{2}ma$

$$a = -\frac{2k}{3m}x \Leftrightarrow a = -\omega^2 x \text{ for SHM.}$$

$$T = 2\pi/\omega = 2\pi\sqrt{\frac{3m}{2k}}.$$

3. A block vibrates at the end of a spring with SHM along the x-axis on horizontal frictionless surface according to the relation

$$x = 2\cos\left(4\pi t + \frac{\pi}{6}\right)$$

Where x is in m, t is in s, and the phase angle is in radians.

a) Determine the amplitude, frequency & period of the motion.

b) Determine the velocity & acc. of the body at any time t.

---

a) Compare this function with $x = A\cos(\omega t + \theta_0)$.

Then,

- $A = 2\,m$
- $\omega = 4\pi$ rad $\Rightarrow f = \frac{2\pi}{\omega} = \frac{2\pi}{4\pi} = \frac{1}{2}$ s.
- $T = \frac{1}{f} = 2\,s$.

b) $v = -\omega A \sin(\omega t + \theta_0)$

$\Rightarrow v = -4\pi \times 2 \sin(4\pi t + \pi/4) = -8\pi \sin(4\pi t + \pi/4)\ \frac{m}{s}$

$a = -\omega^2 A \cos(\omega t + \theta_0)$

$\Rightarrow a = -(4\pi)^2 2\cos(4\pi t + \pi/4) = -32\pi^2 \cos(4\pi t + \pi/4)\ \frac{m}{s^2}$

4. Initial conditions in SHM

$$x = A\cos(\omega t + \theta_0)$$

$$v = -\omega A \sin(\omega t + \theta_0)$$

$\Rightarrow$ $x_0 = x(0) = A\cos\theta_0$

$v_0 = v(0) = -\omega A \sin\theta_0$

Then,

$$\frac{v_0}{x_0} = -\omega \tan\theta_0 \quad \text{or} \quad \tan\theta_0 = -\frac{v_0}{\omega x_0}$$

If one knows the initial position x_0, and initial velocity v_0 and angular frequency ω The amplitude A is given by

$$A^2\cos^2\theta_0 + A^2\sin^2\theta_0 = x_0^2 + \frac{v_0^2}{\omega^2}$$

$$\Rightarrow \quad A^2 = x_0^2 + v_0^2/\omega^2$$

or,

$$A = \sqrt{x_0^2 + v_0^2/\omega^2}\,.$$

Consider the following initial conditions

1. $\omega = 2\,\text{rad/s}$, $x_0 = 1\text{m}$ & $v_0 = 2\text{m/s}$

Then,

$$A = \sqrt{1^2 + \left(\frac{2}{2}\right)^2}\text{m} = \sqrt{2}\ \text{m}$$

It follows that

$$\left.\begin{aligned} 1 &= \sqrt{2}\cos\theta_0 \Rightarrow \theta_0 = \cos^{-1}\left(\tfrac{1}{\sqrt{2}}\right) \\ 2 &= -2\sqrt{2}\sin\theta_0 \Rightarrow \theta_0 = -\sin^{-1}\left(\tfrac{1}{\sqrt{2}}\right) \end{aligned}\right\} \theta_0 = -\frac{\pi}{4} = \frac{7}{4}\pi$$

If $x_0 = -1\text{m}$,

$$\left.\begin{aligned} -1 &= \sqrt{2}\cos\theta_0 \Rightarrow \theta_0 = \cos^{-1}\left(-\tfrac{1}{\sqrt{2}}\right) \\ -2 &= -2\sqrt{2}\sin\theta_0 \Rightarrow \theta_0 = \sin^{-1}\left(\tfrac{1}{\sqrt{2}}\right) \end{aligned}\right\} \theta_0 = 225^\circ = \frac{5}{4}\pi.$$

If $x_0 = 1\text{m}$ & $v_0 = -2\text{m/s}$

$$\left.\begin{aligned} 1 &= \sqrt{2}\cos\theta_0 \Rightarrow \theta_0 = \cos^{-1}\left(\tfrac{1}{\sqrt{2}}\right) \\ -2 &= -2\sqrt{2}\sin\theta_0 \Rightarrow \theta_0 = +\sin^{-1}\left(\tfrac{1}{\sqrt{2}}\right) \end{aligned}\right\} \theta_0 = \pi/4$$

If $x_0 = -1\text{m}$ & $v_0 = -2\text{m/s}$

$$\left.\begin{aligned} -1 &= \sqrt{2}\cos\theta_0 \Rightarrow \theta_0 = \cos^{-1}\left(-\tfrac{1}{\sqrt{2}}\right) \\ -2 &= -2\sqrt{2}\sin\theta_0 \Rightarrow \theta_0 = \sin^{-1}\left(\tfrac{1}{\sqrt{2}}\right) \end{aligned}\right\} \theta_0 = \frac{3}{4}\pi.$$

Solution to Worksheet IV

1. a, As the body moves up its acc (in the absence of friction) is $-g\sin\theta$. So for the this motion the time taken is

$$t_1 = \frac{v_f - v_0}{g\sin\theta} = \frac{-v_0}{-g\sin 30^\circ} = \frac{40\,m/s}{(10\,m/s^2)\,1/2} = 8\,sec$$

For the downward movement, the conservation of energy implies $v_f = 40\,m/s$ at the initial position, (i.e.,)

$$t_2 = \frac{v_f - v_0}{g\sin\theta} = \frac{40\,m/s}{10\,m/s^2 \times 1/2} = 8\,sec$$

$\therefore$ the total time taken is $t = (8+8)s = 16\,s$.

b, Consider the upward movement,

$$s = \frac{v_f^2 - v_0^2}{2a} = \frac{-v_0^2}{-2g\sin 30^\circ} = \frac{(40\,m/s)^2}{2\times 10\,m/s \times 1/2} = 160\,m$$

2.

a, $F = ma \;\therefore\; a = \frac{F}{m} = \frac{30\,N}{15\,kg} = 2\,m/s^2$

b, since the body starts from rest.

$$s = \tfrac{1}{2}at^2 = \tfrac{1}{2}(2\,m/s^2)(10\,s)^2 = 100\,m$$

c, $v = at = 2\,m/s^2 \times 10\,sec = 20\,m/s$

d, From c, $v = 20\,m/s$ at the end of the 10^{th} second. since $F = 0$ also $a = 0$ i.e., the body moves at a const. velocity of $20\,m/s$. so the dist. moved in 20 sec. is

$$s = vt = 20\,m/s \times 20\,s = 400\,m$$

3.

a, In either case the acc. force is $mg\sin\theta$

For W_1

$m_1 g \sin 30^\circ = 50\,kg \times 10\,m/s^2 \times 1/2$

$= 250\,N$

For W_2

$m_2 g \sin 53^\circ = 25\,kg \times 10\,m/s^2 \times 0.8$

$= 200\,N$

∴ the system will move to the left.

5,

$m_1 g \sin 30^\circ - T = m_1 a$, assuming the slope is frictionless

$T - m_2 g \sin 53^\circ = m_2 a$, assuming the cord does not stretch.

$$a = \frac{(m_1 \sin 30^\circ - m_2 \sin 53^\circ) g}{m_1 + m_2}$$

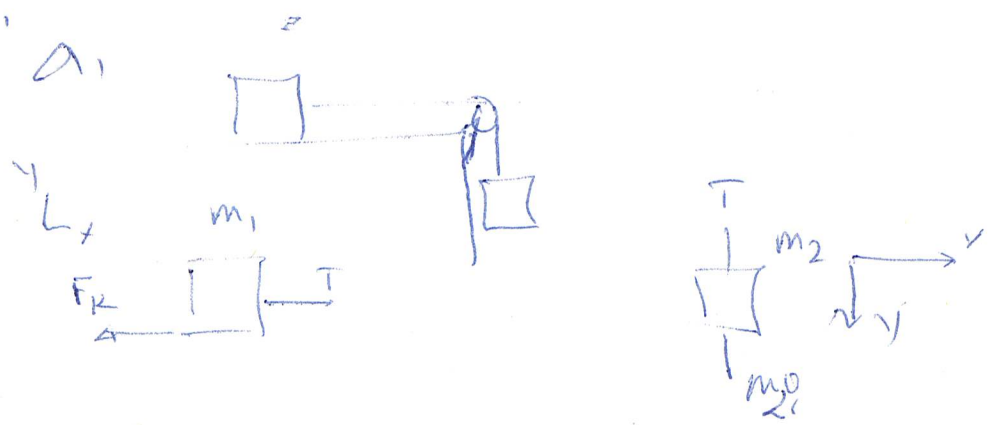
or $$a = \frac{(50 \times 1/2 - 25 \times 0.8)\,10}{50 + 25}\ m/s^2 = 2/3\ m/s^2$$

C,

$T = m_1 (g \sin 30^\circ - a)$

$= 50\,kg\ (10\,m/s^2 \times 1/2 - 2/3\,m/s^2) = \frac{13}{3} \times 50\ N$

4. a,



$T - F_K = m_1 a$ $\qquad$ $m_2 g - T = m_2 a$

Adding the two eqs.

$m_2 g - F_K = (m_1 + m_2) a$

or $$a = \frac{m_2 g - F_K}{m_1 + m_2} = \frac{m_2 g - \mu_K m_1 g}{m_1 + m_2} = \frac{(m_2 - \mu_K m_1) g}{m_1 + m_2}$$

$= (106/615 \times 10)\ m/s^2$

)

$$T = m_2 (g - a)$$

$$= m_2 \left(g - \frac{m_2 g + \mu_K m_1 g}{m_1 + m_2} \right)$$

$$= m_2 \left(\frac{m_1 g + m_2 g - m_2 g + \mu_K m_1 g}{m_1 + m_2} \right)$$

or

$$T = \frac{m_1 m_2 g (1 + \mu_K)}{m_1 + m_2}$$

5.

a, R, mg so the force due to the floor is now,

$$R = mg = 800\ N$$

b, a, R, mg

$$R - mg = ma$$

$$R = m(g + a) = 80(10 + 5)\ N$$

or

$$R = 1200\ N$$

c, $a = 0$, R, mg

$$R = mg = 800\ N$$

d, $a = 3 m/s^2$, R, mg

$$R - mg = -ma$$

$$R = m(g - a) = 80(10 - 3)N$$

or

$$R = 560\ N$$

6. a) since the block is frictionless the conservation of M.E. gives

$$mgh = \tfrac{1}{2} m v^2 \Rightarrow v = \sqrt{2gh}$$

or

$$v = \sqrt{2 \times 10 \times 40 \sin 30^\circ} = 20\ m/s$$

b) From the work energy theorem, and since the only acting force is F_K we have

$$+\tfrac{1}{2} m v_b^2 = F_K S = \mu_K mg S \Rightarrow v_b = 20\ m/s$$

i.e., $S = \frac{v_b^2}{2\mu_k g} = \frac{20^2}{2\times 0.1\times 10}\,m = 200\,m$

[illegible]

[illegible]

$$v_f^2 - v_b^2 = 2as \Rightarrow a = \frac{-v_b^2}{2s} = -\frac{400}{400}\,m/s^2 = -1\,m/s^2$$

Since the acc. is known,

$$v_f = v_b + at = 20\,m/s - 1\,m/s^2 \times 55 = 15\,m/s^2$$

7.

$$F_g = G\frac{m_e m_s}{r^2} = \frac{G m_e m_s}{(R_e + r_0)^2}, \quad r_0 = 40{,}000\,km$$

$$W_s = m_s g = m_s \cdot \frac{GM_e}{R_e^2} = G\frac{M_e m_s}{R_e^2}$$

$$\Rightarrow F_g / W_s = \frac{R_e^2}{(R_e + r_0)^2}$$

---

or

$$F_k = -ma \Rightarrow -\mu_k m g = a$$

or $$a = -\mu_k g$$

$$\therefore v_f = v_b + at = 20\,m/s - 0.1\times 10\,m/s^2 \times 55 = 15\,m/s$$

Department of Science

Phys.101, Work Sheet I

Vectors

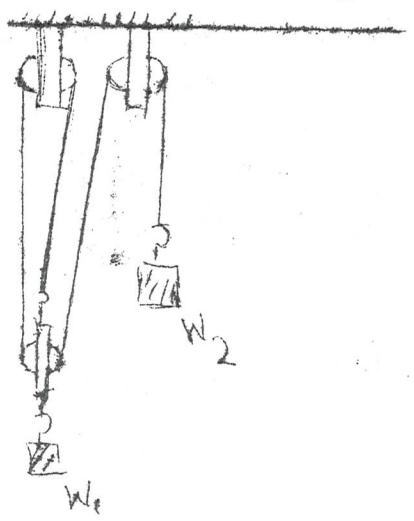
1. An air plane flies 60 km in a direction 40^{o} north of east. What is the eastward component of the displacement ? What is the north ward component ?

2. The following three forces act upon an object: 200 lb horizontally to the right, 150 lb downward and to the left at an angle of 30^{o} below the horizontal, and 500 lb upward and to the right at an angle of 53^{o} above the horizontal find the resultant force.

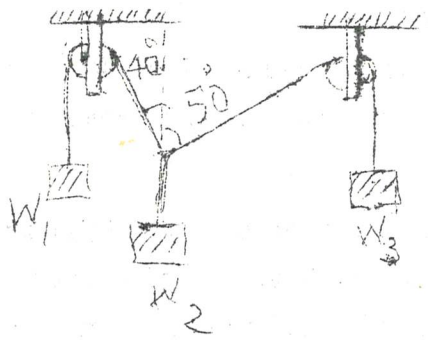
4. A 160 N force and a 360 N force are applied to the same object. If the angle between the two forces is 63^{o}, What is the magnitude of the resultant force ?

Equilibrium

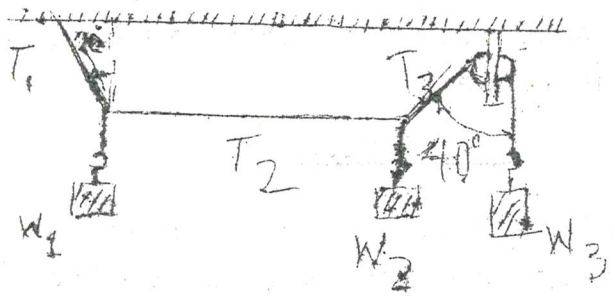
1. In the equilibrium situation shown in the figure below, the pulleys are nearly frictionless, and so the tension is the same in all parts of the cord, Assume that the weights of the pulley and cord are negligible and that the cords are essentially vertical. What is the tension in the cord in terms of W_2 ? What is the ratio of W_1 to W_2 ?



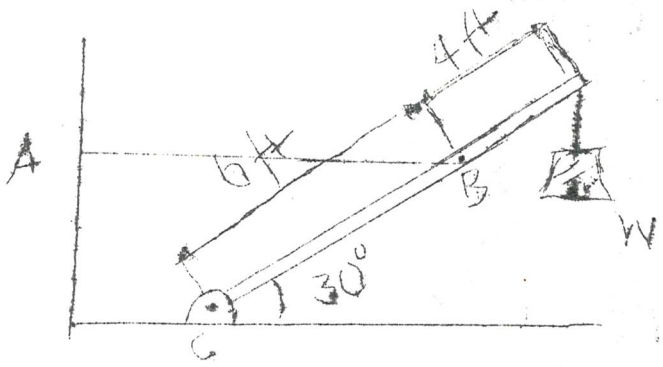
2. For the equilibrium situation shown below, find W_2 and W_3. Assume W_1 = 100 lb and consider the pulleys to be friction less.



3. For the equilibrium situation shown in the following figure, assume the pulley to be frictionless. Find W_1, W_2, T_1 and T_2.



4. If the weight of the boom in the figure below is neglected, and the weight W is 200 lb, a, what is the tension in the horizontal cable ? b) what are the horizontal and vertical components of the force exerted on the boom at its lower end ?



1. A can run at a top speed of 5 omls while B can run at a speed of only 3 omls. They are to race 200 m. To make the race more even, A is required to start t seconds later than B. How large should t be if they are to end in a tie ?

2. A truck initially travelling at 50 ft/s? decelevates at 4 ft/s find a) how long it takes to stop b) how far it moves in that time, and c) how far the truck moves in the third second and after the brakes are applied ?

3. A boy wants to throw a can straight up in the air and then hit it with a second can. He wants the collision to occur 5.0 m above the throwing point. In addition, he knows that the time he needs between throw is 3.05. Assuming he throws both cans with the same speed, what must be the initial speed of the cans ?

4. A stone is thrown straight up with aninitial speed of 80 ft/s. How high does the stone go and how long does it take to reach its highest point ?

5. A brick is thrown vertically down from the top of a bridge, and 45 later it strikes the water with a final velocity of 60 mls. a) what was the initial velocity of the brick ? b) how high is the bridge above the water.

Projectile Motion

1. A stone is thrown horizontially from the top of a building with an initial velocity of 1000 ft/s. At the same instant another stone is dropped from rest. a) compute the position and velocity of the latter after 35. b) How far has the first stone traveled horizontally during this same time ? c) How far has it traveled vertically ? d) What are the horizontal and vertical components of its velocity after 35 ?

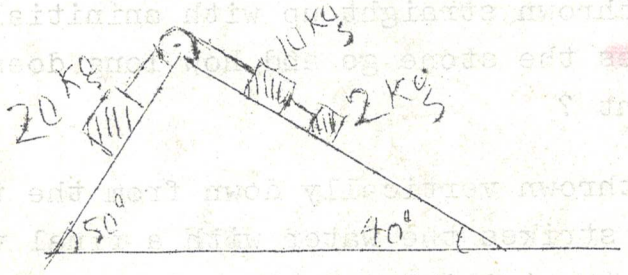
2. A bomber flying with a horizontal speed of 400 ft/s releases a bomb, which strikes the ocean below 65 later. At what altitude is the plane flying ? How far has the bomb traveled horizontally from the time if release ?

Newton's 2nd law of motion

1. A 64 lb box hangs at the end of a rope. Find the tension in the rope if the acceleration is a) upward 5 m/s and b) downward at 5m/s .

2. The mass m_1, m_2 and m_3 of the figure below are 10,8 and 6 kg. Neolecting friction, what are the acceleration of the system and the tension in each cord.

3. Find the tension in the various cords shown in the figure below and the acceleration of the blocks if for each block is 0.20.



Work, Energy, and Power

1. A 16 kg ball is held at a hight of 30 m above the ground
   a) What is the potential energy at this height ? b) If the ball is dropped, What is its down ward velocity. When it is 15 m above the ground ? c) What are the kinetic and potential energies at 15 m ? d) what is the total energy at this point ? e) what is its kinetic energy at the ground ? f, with what speed does it strike the ground ?

2. A car of mass m rests on a hill of height h and length L. Show that the speed of the car when it reaches the bottom of the hill is

$$\sqrt{2gh - \frac{2LF}{m}}$$

KOTEBE COLLEGE OF TEACHER EDUCATION

Department of Science

Physics Problems for Phy. III

Work Sheet I

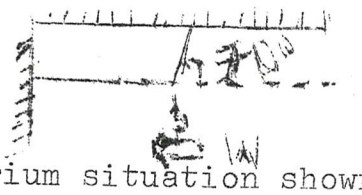
Vectors

1. If an airplane flier southwest for 100Km, how far west and how far south has it gone ?
2. A bug sits on a board which is floating south-ward down a stright stream with a speed of 5.0 cm/s. The bug now walks eastward at a speed of 2.0 cm/s what is the speed of the bug relative to the shore of the stream ?

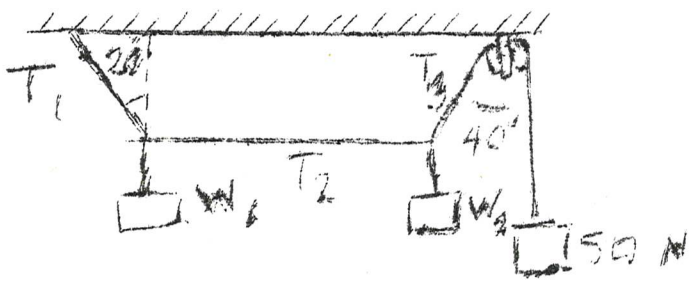
Equilibrium

1. Find the tension in the rope of the fig. below if the downward foree F is of magnitude 150 lb.

2. The weight W shown in the fig. below is 200 N. What is the tension in the horizontal rope ? Assume the repes to have negligible weight.



3. For the equilibrium situation shown below assume the pulley to be frictionless. Find W_1, W_2, T_1, & T_2.



Rectilnear Motion

Unless otherwise stated, assum uniform acceleration.
Use. g = 9.8 m/s = 32 ft /s^2. neglect air friction.

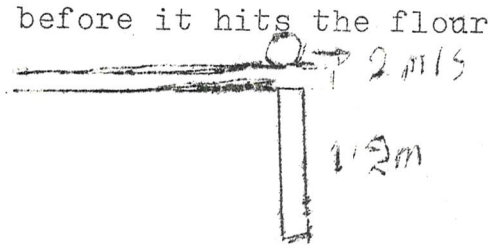
1. According to the theory of continental drift, south America was once attached to Africa. They have been drifting apart for about 200 million years. their present separation is about 400 mi. Assume these numbers to be exact, what was the average rate of separation of the continent in miles per year ?
2. Suppose two cars A & B are racing a 200M distance. Car A runs at a speed of 5.0 m/s while car B runs at a speed of 3.0 m/s. To make the race more even A is required to start to seconds 1 later then B. How large should t be if they are to end in a tie ?
3. A bullet moving with a speed of 150 m/s strikes a tree and penetrates 3.5 cm before stopping. Find its decelaration and the time taken to stop.
4. A truck initially traveling at 50 ft/s decelarates at 4ft/s^2
   Find a) how long it takes to stop
   b) how farit moves in that time
   c) how far the truck moves in the third second after the brakes are applied.
5. A boy throws a ball straight down from the top of a 50 ft buliding with a speed of 20 ft/s How long does it take for the ball to reach the ground, and how fast is it moving just before it hits ?

Laws of Dynamics

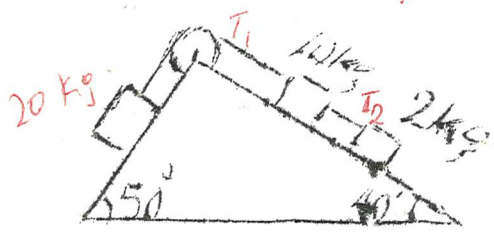
1. A 3600 lb car is to be pushed up a hill which is inclined 8^o to the horizontal. It friction forces are negligible, what is the minimum pushing force needed ?
2. A car is moving up a 15^o incline at a speed of 20 m/s when the driver disenoages the motor. How far up along the incline will the car coast if friction forces are negligible ?

3. If, in prob. 2, the 1400 kg car coasts 30 m before stopping, howlarge is the average (assumed constant) friction force which stopped it ?

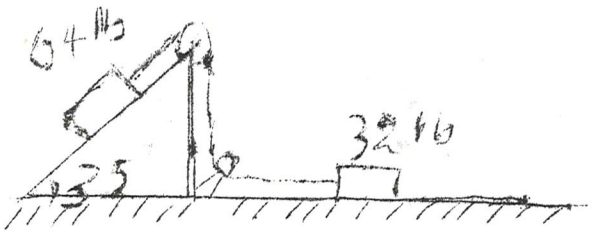
4. A marble roles off the edge of a horizontal table 1-2m high, as shown. Find the horizontal distance that can be moved by the marble if it has been rolling at a speed of 2m/s. What is its velocity just before it hits the flour ?



5. Find the tension in the various cords shown in the fig. below and the acceleration of the blocks if μ for each block is 0.20



6. Each of the two blocks in the fig. below experiences a friction force of 5 lb. How long will it take, starting from rest, to pull the 32 lb block through a distance of 19 ft ?



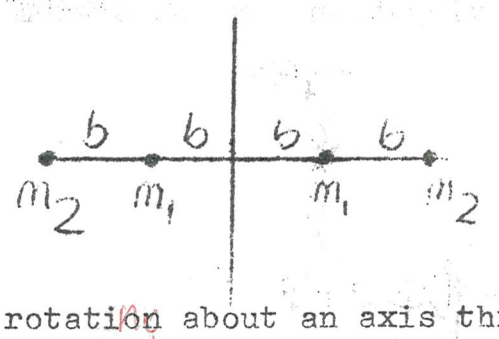
Phys.101 Work Sheet II

Retational Dynamics

1. Find the moment of inertia of a solid disk having a radius of 10 cm and a mass of 200 g (radius of gyration, K = b$\sqrt{2/5}$)

2. How large a torque is required to give an angular acceleration of 4.0 rad/s^2 to a wheel having a moement of inertia 0.20kg.m^2?

3. Refer to the gigure below. Prove that the moment of inertia for rotation about an axis parallel to the one shown but a distanced away from it is given by $I = md^2 + I_0$)
where II B is the total mass and I_0 is the moment of inertia about an axis through the center of mass. Neglect the mass of the bar (This relation is a general one and is called the parallel axis theorem)



4. A solid wheel rotation about an axis through its center has a moment of inertia I_0. Two identical masses are placed at opposite points on the rim of the wheel at opposite ends of a dimmeter. (alling the wheel radius a, and each mass M, show that the wheel now has moment of inertia given by $I = I_0 + 2Ma^2$

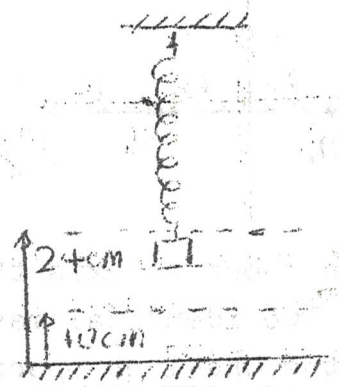
5. A childrens merry - go - round in a park consists of essentially uniform 150 kg solid. disk rotating about a vertical axis. the radius of the disk is 6.0m, and a 90kg. teacher is standing on it at its outeredge when it is rotating at a speed of 0.2rev/s.

How fast will the disk be rotating it the teacher walks 4.0m toward the center along the radius.
Hint: momentum must be conserved.

6. Suppose that the merry - go - round described in prob. 25 has no one on it but is moving at a speed of 0.20 rev/s. If a g 90 kg teacher quickly sits down on the edge of it, what will its new speed be ?

Harmonic Motion

1. As seen below, a mass vibrates between two limits, 10.0 and 24.0 cm above the floor. It veaches the lowest point 20 times a minute. What area a)frequency b) the period and c) the amplitude of vibration for the system ?



2. A body vibrates with SHM of amplitude 12cm and frequency $4H_Z$. What is the period ? What is the displacement of the body after 3.25sec ?

3. A body describing SHM has a maximum acceleration of $24m/s^2$ and a maximum speed of 2m/s. What are its period and amplitude of bibration ?

4. The prong of a tuning fork vibrates with a frequency of 330 H_Z and an amplitude of 2mm. What is its velocity when the displacement is 1.5mm?

5. A pendulum has a length of 2m and executes 21 complete vibr in 1min. What is the acceleration due to gravity ?

6. Show that the maximum speed of a pendulum bob is given by

$$v = \sqrt{2gL\,(1-\cos\theta)}$$

$\frac{1}{2}mV^2 = mgh$

$\frac{1}{2}V^2 = gh$

$V^2 = 2gh$

$V = \sqrt{2gh}$

$V = \sqrt{2gl(1-\cos\theta)}$

7. If a 2.0 kg mass vibrates at the end of a spring according to the relation $y = 0.20 \cos(9.42t)$ centimeters. Fina a) the amplitude of vibration b) the frequency of vibration c) the spring constant, and d) the maximum speed of the mass.

Fluid Mechanics

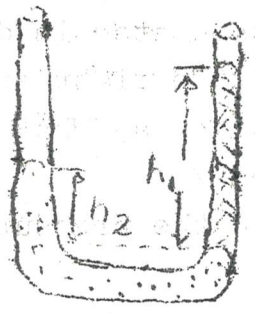
I. Fluids of Rest

1. Find the pressure in Kilopascals due to a column of mercury 60 cm high. What is this pressure in pounds per square inch ? In atmospheres ?

2. An open U - shaped tube containing mercury is $2cm^2$ in cross section. What volume of water must be poured into the left tube in order to cause the mercury in the right tube to rise 1cm above its original position ?

3. Two liquids which do not react chemically are placed in a bent tube as shown show that the heights of the liquids above their surface of separation are inversely propertional to their densities:

$$h_1/h_2 = \rho_2/\rho_1$$



4. Assume the two liquids in the U - shaped tube above are water and oil. Compute the density of the oil it the water stands 19 cm above the interface and the oil stands 24 cm above the interface.

5. A 64 lb block has a volume of 0.2 ft^3. The block is suspended from a cord and submerged in oil ($D_o = 48 lb/ft^3$) Find the bouyant force and the tension in the cord.

6. A stone of unknown composition weighs 82 lb in air. 2 lb apparent weight is 74 lb when submerged in water. What is the volume of the stone and what is its density ??

II Fluids in Motion

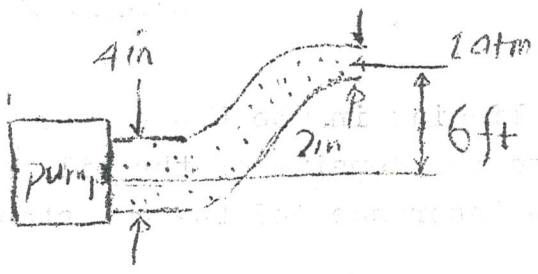
1. Gasoline flows through a 1in diameter hose at an average speed of 5 ft/s. What is the rate of flow in cubic feet per second ? How many minutes are required to fill a 20 gal tank ? ($1ft^3$ = 7.481 gal.)

2. Water flowing from a 2in pipe emerges horizontally at the rate of 8 gal/min. What is the horizontal range of the stream of water if the pipe is 4ft from the ground ?

3. An air craft wing 25 ft long and 5ft wide experiences a lifting force of 800 lb. What is the difference in pressure between the upper and lower surfaces of thw wing ?

4. Sea water (ρ = 64 lb/ft^3, is pumped through the system of pipes shown in the figure below at the rate of 4 ft^3/min. The pipe diameters at the lower and upper sections are 4 and 2in, respectively, the water is discharged into the atmosphere (Pa = 14.7) lb/in^2, at the right end of the upper section, 6 ft above the lower section. a) What are the speeds of flow in the upper and lower pipes ?

b) What are the pressures in the lower and upper sections



1. A particle moves according to the position function $x(t) = at^3 + bt^2 + ct$ with $a = 2 m/s^3$, $b = -5 m/s^2$, and $c = 2 m/s$. Find the acceleration at those points for which a, $x(t) = 0$ and b, $v(t) = 0$. 1 stencil

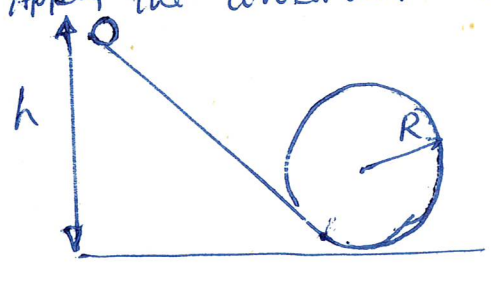
2. An athlete can throw a ball a maximum horizontal distance R. If air resistance effects are negligible, show that the athlete can throw the same ball to a maximum height $\frac{1}{2}R$. 1 stencil ½

3. Imagine a long stretch of smooth ice (such as a frozen river) over which a hockey puck can slide. If the initial velocity of the puck is v_0 and if the coefficient of kinetic friction between the puck and the ice has the constant value μ_K, derive an expression for the distance the puck will travel before coming to rest. Find the distance for the case in which $v_0 = 8 m/s$ and $\mu_K = 0.2$. ½ stencil ½

4. A particle with a mass of 0.5 kg is acted on by a force $\vec{F} = (ax + b)\hat{i}$, where $a = 2 N/M$ and $b = 4N$. If the initial velocity at $x = 0$ is $\vec{v}_0 = (1.5 m/s)\hat{i}$, what is the velocity of the particle when it arrives at $x = 6m$? (Hint: First give a general expression for $\vec{v}$ as a function of x.) ½ stencil ½

5. A hunter has a rifle that can fire 60 gm bullets with a muzzle velocity of 900 m/s. A 40 kg leopard springs at him with a speed of 20 m/s. How many bullets must the hunter fire into the leopard in order to stop him in his tracks? 1 stencil ½

6. A uniform disk of radius r rolls down THROUGH a loop-the-loop (fig. below). What minimum height h will keep the disk on the track at the top of the loop? Assume that r is small (compared to h and R. (Hint: Apply the conservation of M.E. and for the disk $I_c = \frac{1}{2}mr^2$) ½ stencil



Problems

1. A wheel 1m in diam. is rotat. about a fixed axis with an initial angular velocity of 2 rev. s^{-1}. The angular acc. is 3 rev. s^{-2}

a, Compute the angular velocity after 6s

b, Through what angle has the wheel turned in this time interval

c, What is the tangential velocity of a point on the rim of the wheel at $t = 6s$.

soln.

a, $\omega = \omega_0 + \alpha t$, b, $\theta = \omega_0 t + \frac{1}{2}\alpha t^2$

c, $v_T = r\omega$

2. A fly wheel requires 3s to rotate through 234 radians. Its angular velocity at the end of this time is 108 rad/s. Find its const. angular acc.

3. A fly wheel whose angular acc. is constand equal to 2 rad/s^2 rotates through an angle of 100 radians in 5s. How long had it been in motion at the beginning of the 5s interval if it started from rest?

Solutions to problems for worksheet III

1. a, Average & inst. velocity can be equal if the displacement increases by the same amount in equal time intervals i.e., if the displacement is a linear function of time

b, yes. That is there is a non zero finite displacement and that means we can compute the instantaneous velocity by considering a very small time interval.

2. $v_f = 0$ and a acc. $= -g$, due to gravity; acc $= g$ before it strikes.

3. $\bar{v} = \frac{x_2 - x_1}{t_2 - t_1} = \frac{\Delta x}{\Delta t}$ $\quad \vec{v}_{av} = \frac{\bar{x}_2 - \bar{x}_1}{\Delta t} = \frac{\Delta \vec{x}}{\Delta t}$

But $|\Delta \bar{x}| = |\bar{x}_2 - \bar{x}_1| \neq x_2 - x_1 = |\bar{x}_2| - |\bar{x}_1|$

or $\bar{v} = |\vec{v}_{av}|$

e.g. $\bar{x}_1 = 2\bar{i} - 3\bar{j}$, $\bar{x}_2 = -\bar{i} + \bar{j}$

$|\bar{x}_2 - \bar{x}_1| = |-3\bar{i} + 4\bar{j}| = 5$, $|\bar{x}_2| - |\bar{x}_1| = \sqrt{13} - \sqrt{2}$

4. a, yes - (prob. 2) b yes. - direction, c) No. d, yes - deceleration.

e. yes — free fall.

Problems

1. a, (i) $\Delta x = 4$, $\Delta t = 1$; $\bar{v} = 1\ m/s$.

(ii) $x(0) = 0$, $x(4) = -2$, $\Delta x = -2$, $\Delta t = 4$; $\bar{v} = -\frac{1}{2}\ m/s$.

(iii) $x(1) = 4$, $x(5) = 0$, $\Delta x = -4$, $\Delta t = 4$; $\bar{v} = -1\ m/s$.

(iv) $x(10) = 0$, $x(5) = 0$; $\Delta t = 5$; $\bar{v} = 0\ m/s$

b, (i) $x(0) = 0$, Slope $= 4 = \frac{x - 0}{t - 0} \Rightarrow x = 4t$

$x(0.5) = 2$, $x(0.5 + \Delta t) = 2 + \Delta t$

$\Delta x = \Delta t$

$\lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = v_{inst} = 1\ m/s$

(ii)

Slope $= \frac{4 - 0}{1 - 2} = -4 = \frac{x - 0}{t - 2} \Rightarrow x = -4(t - 2)$

$x(2) = -4(2-2) = 0$, $x(2+\Delta t) = -4(2+\Delta t-2) = -4\Delta t$

$\Delta x = -4\Delta t$

i.e.) $v_{inst} = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{-4\Delta t}{\Delta t} = -4$ m/s

(iii) $\frac{-2-0}{4-5} = \frac{x-0}{t-5} \Rightarrow x = 2(t-5)$

$x(4.5) = -1$, $x(4.5+\Delta t) = -1+2\Delta t$

$\Delta x = 2\Delta t$

i.e.) $v_{inst.} = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = 2$

2. a) a in m and b in m/s.

b) $\bar{v} = \frac{v - v_0}{t - t_0} = \frac{b(t-t_0)}{t-t_0} = b$

$v_{inst} = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \lim_{\Delta t \to 0} b = b \equiv \bar{v}$

3. a) In (0, 1)

slope $= a = \frac{-3-0}{1-0} = -3$

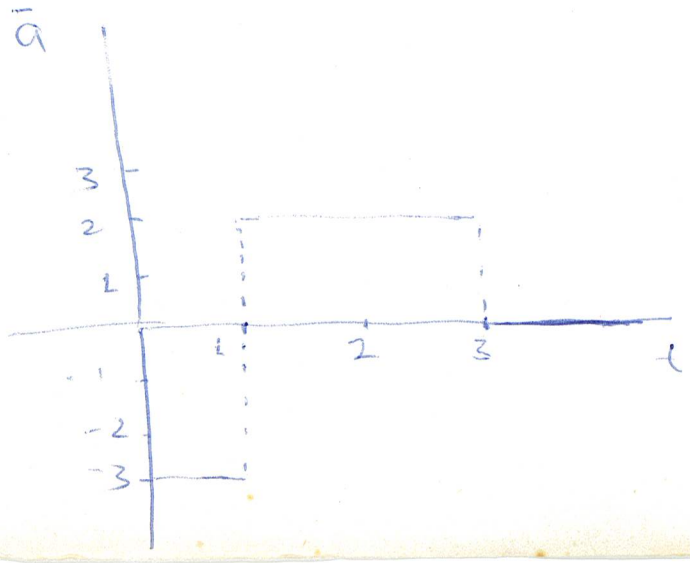
In (1, 3)

Slope $= \dot{a} = \frac{3-(-3)}{3-1} = \frac{6}{3} = 2$

In (3, t)

Slope $= a = \frac{3-3}{t-3} = 0$

Note $\bar{a} = a$ since v varies linearly with time.



b) From the graph, $\bar{a} = 2 m/s^2$

c) from the graph of v vs. t

$v(1 \le t \le 1)$ is in $(0, 1)$: $v(t) = -3t$

$v(1) = -3$, $v(1+\Delta t) = -3 - 3\Delta t \Rightarrow \Delta v = -3\Delta t$

$\therefore a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = -3.$

In $(1, 3)$ $v(t) = 2(t-1) - 3$

$v(1) = -3$, $v(1+\Delta t) = 2\Delta t - 3 \Rightarrow \Delta v = 2\Delta t$

$a = \lim_{\Delta t \to 0^+} \frac{\Delta v}{\Delta t} = 2$

Since the limit does not exist, the instantaneous acc. at $t = 1$ sec is not defined. This is also seen from the graph of $a(=\bar{a})$ vs t. At $t = 2$ sec, $a = \bar{a} = 2 m/s^2$.

4. a) $a = \frac{v^2 - v_0^2}{2s} = \frac{(8^2 - 5^2)10^2}{2 \times 2 \cdot 10^2} m/s^2 = 9.75 m/s^2$

b) $t = \frac{v - v_0}{a} = \frac{80 - 50}{9.75 m/s} s = 3.1$

5. a)

$x - x_0 \equiv s = 180 km$

$v_B = 45 km/hr$

$v_A = ?$

$t = 3 hr$

$s = \left(\frac{v_A + v_B}{2}\right) t$

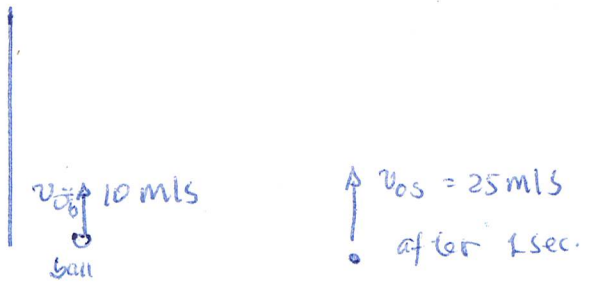
$v_A = \frac{2s}{t} - v_B$

or $v_A = \frac{2 \times 180 km}{3 hr} - 45 km/hr = 75 km/hr$

b) $a = \frac{v_B - v_A}{t} = -\left(\frac{75 - 45}{3}\right) km/hr^2 = -10 km/hr^2$

c) v_A, v_B, t are given, s, t, a are not known so that one cannot find s from the given information.

6. a.



Suppose the time taken by the ball to move a dist. y is t_b. Since the stone starts 1 sec. after the ball, the time taken by the stone to reach the same height y is

$$t_s = t_b + (-1s) = t_b - 1s$$

Thus,

$$y = v_{ob}(t_s + 1s) - \tfrac{1}{2} g (t_s + 1s)^2$$

$$= v_{os} t_s - \tfrac{1}{2} g t_s^2$$

or

$$v_{ob} t_s + v_{ob} - \tfrac{1}{2} g (t_s^2 + 2t_s + 1)$$

$$= v_{os} t_s - \tfrac{1}{2} g t_s^2$$

$$\Rightarrow (v_{os} - v_{ob} + g) t_s = v_{ob} - \tfrac{1}{2} g$$

$$\text{or} \quad t_s = \frac{v_{ob} - \frac{1}{2} g}{v_{os} - v_{ob} + g} = \frac{10 - 1}{25 - 10 + 2} s = \frac{9}{7} \text{ sec.}$$

b. $v_b = v_{ob} - g t_b$, $t_b = \left(\frac{9}{7} + 1\right)$ sec. $= \frac{16}{7}$ sec.

$$= 10 \text{ m/s} - 10 \times \frac{16}{7} \text{ m/s}$$

$$v_s = v_{os} - g t_s$$

$$= 25 \text{ m/s} - 10 \times \frac{9}{7} \text{ m/s}$$

c, That means the time taken to reach y_{max}.

$$v_b = 0 = v_{ob} - g t_b$$

$$\Rightarrow t_b = \frac{v_{ob}}{g} = \frac{10 \text{ m/s}}{10 \text{ m/s}^2} = 1 \text{ s}$$

Also, $t_s = \frac{v_{os}}{g} = \frac{25 \text{ m/s}}{10 \text{ m/s}^2} = 2.5 \text{ sec.}$

7. 1000m with const. velocity of 80 m/s. So the time taken to move this dist. is

$y_1 = v_1 t \Rightarrow t_1 = \frac{y_1}{v_0} = \frac{1000}{80} s = 12.5$ sec.

At the point where the engine of the rocket fails, its upward initial velocity is 80 m/s and it will move upward until this velocity goes to zero and the time taken for this motion is

$v_{y_f} = 0 \Rightarrow t_2 = \frac{v_0}{g} = \frac{80 m/s}{10 m/s^2} = 8$ sec.

The corresponding vertical dist. moved is

$$y_2 = v_0 t_2 - \tfrac{1}{2} g t_2^2$$
$$= (80 m/s)(8 sec.) - \tfrac{1}{2} \times 10 m/s^2 \times 64 sec^2$$
$$= 320 m$$

So the total upward dist. moved is

$$y = 1000 m + 320 m = 1320 m$$

Let us now find the time for the downward movement

$$t = \sqrt{\frac{2y}{g}} = \sqrt{\frac{2 \times 1320 m}{10} s} = \sqrt{264} \text{ sec.} = 16 \text{ sec.}$$

So the rocket was in motion for a total time of

$$t = 12.5 s + 8 sec. + 16 sec. = 36.5 sec.$$

b) The max altitude as calculated in a is 1320 m

a) When it starts its way down $v_{0y} = 0$

$$\therefore v_{yf} = -g t = (-10 m/s^2)(16 sec.) = -160 m/sec.$$

8. - For motion with an acc. of $2 m/s^2$

$$S_1 = \tfrac{1}{2} a_1 t_1^2 = \tfrac{1}{2} \times 2 m/s^2 \times 100 sec^2 = 100 m$$

- The final velocity in the first motion

$$v_{f1} = a_1 t_1 = 2 m/s^2 \times 10 sec. = 20 m/s$$

So the train has moved with this velocity speed for 30 sec. Thus the dist moved here is

$$s_2 = \bar{v}\, t = (20\ m/s)(30\ s) = 600\ m$$

- The train is decelerating from a speed of 20 m/s to zero at $-4\ m/s^2$. Now

$$v_f^2 - v_0^2 = 2as$$

or $$s_3 = \frac{-v_0^2}{2a} = -\frac{(20\ m/s)^2}{-2 \times 4\ m/s^2} = 50\ m$$

∴ the total dist moved by the train is

$$s = s_1 + s_2 + s_3$$

$$= 100\ m + 600\ m + 50\ m = 750\ m$$

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Physics problems for phys. 112

Worksheet I

Wave motion

1. A steel piano wire 50 cm long has a mass of 5 g and is under a tension of 400 N. What are the frequencies of its fundamental mode of vibration and first two overtones?

2. If 1.2 kg rope is stretched over a dist. of 5.2 m and placed under a tension of 1200 N. compute the speed of transverse wave in the rope.

3. A wooden float at the end of a fishing line makes eight complete oscillations in 20 s. If it takes 3.6 s for a single wave to travel 11 m, what is the wavelength of the water waves?

4. One end of a long horizontal rope oscillates with a frequency of 2 Hz and an amplitude of 50 mm. A 2 m length of the string has a mass of 0.3 kg. If the string is under a tension of ~48 N, how much energy per second must be supplied to the rope?

5. The second harmonic of a vibrating string is 200 Hz. If the length of the string is 3 m and its tension is 200 N, Compute the linear density.

6. A 4.3 m string has a tension of 300 N, and a mass of 0.50 g. If it is fixed at each end and vibrates in three segments, what is the frequency of the standing waves?

Sound

1. Find the fundamental frequency and the first three overtones for a 20 cm pipe at 20°C
   a, if the pipe is open at both ends?
   b, if the pipe is closed at one end?

2. What length of closed pipe is required to produce a fundamental frequency of 256 Hz? What length of open pipe will have this fundamental frequency? Assume the temp. to be zero?

Solutions to problems for worksheet I
(Phys-112)

Wave Motion

1. This is the case of a string fixed at two of its ends. Thus, the characteristic frequencies are given by,

$$f_n = \frac{n}{2l} v \quad , \quad n = 1, 2, 3, \cdots$$

But for here v is given to be

$$v = \sqrt{F/\mu}$$

where F is the force applied to the string and μ is the mass per unit length.

$$\Rightarrow \quad f_n = \frac{n}{2l}\sqrt{F/\mu}$$

The fundamental is found by setting $n = 1$ in the above eq.

$$\therefore \quad f_1 = \frac{1}{2l}\sqrt{F/\mu}$$

for the given values,

$$f_1 = \frac{1}{2 \times (0.5\,m)} \left(\frac{400\,N}{0.05\,kg/0.5\,m} \right)^{1/2}$$

$$= 200\,Hz$$

The first and second overtones are

$$f_2 = 2f_1 = 400\,Hz$$

$$f_3 = 3f_1 = 600\,Hz$$

2. In this case the length of the rope is $5.2\,m$

Then, $v = \sqrt{F/\mu}$

or $v = \sqrt{\frac{120\,N}{1.2\,kg/5.2\,m}}$

$= \sqrt{\frac{100}{5.2}}\ m/s$? Ans. $\sqrt{520}\ m/s$

3.

$$f = \frac{8}{10}\,cps = \frac{8}{10}\,Hz = 0.8\,Hz$$

$$v = \frac{11\,m}{3.6\,s} = \frac{11}{3.6}\,m/s$$

Hence the wavelength is given by

$$\lambda = v/f = \left(\frac{11}{3.6}\,m/s\right)\left(\frac{1}{0.8\,Hz}\right)$$

$$= \frac{11}{2.8\,8}\,m$$

4. The total energy of a particle on the rope that is oscillating is given by

$$E = E_p + E_k = (E_k)_{max}$$
$$= \tfrac{1}{2} m v_{max}^2$$

But from the consideration of harmonic motion, we found that the maximum velocity of a particle oscillating with frequency f and amplitude A is given by

$$v_{max} = 2\pi f A$$

$$\therefore E = \tfrac{1}{2} m (2\pi f A)^2$$
$$= 2\pi^2 f^2 A^2 m$$

Suppose that a wave travels down a length l of a given string with a speed v. The time t required for the wave to travel this length is

$$t = l/v$$

Now the power P of the wave is given by

$$P = E/t = E/(l/v) = \frac{E}{l} v$$

$$\text{or } P = \frac{2\pi^2 f^2 A^2 m}{l} v \quad \text{but } \frac{m}{l} = \mu \text{ (linear density)}$$
$$= 2\pi^2 f^2 A^2 \mu v$$

Therefore, coming to our problem,

$$P = 2(3.14)^2 (2\,Hz)^2 (0.05\,m)^2 \left(\frac{0.3\,kg}{2\,m}\right)\left(\sqrt{\frac{45\,N}{0.3\,kg/2\,m}}\right)$$

$$= 0.53\,W$$

5. From no. (1) the n^{th} frequencies of a vibrating string are given by,

$$f_n = \frac{n}{2l}\sqrt{F/\mu}, \quad n = 1, 2, 3, \ldots$$

for the 2nd harmonic

$$f_2 = \frac{2}{2l}\sqrt{F/\mu}$$

$$\Rightarrow (f_2)^2 = \frac{4}{4l^2}\frac{F}{\mu}$$

$$(f_3)^2 = \left(\frac{g}{4\ell^2}\right)\left(\frac{F}{m/\ell}\right)$$

$$(f_3)^2 = \left(\frac{gF\ell}{4\ell^2}\right)$$

$$\Rightarrow \mu = \frac{gF}{4\ell^2(f_3)^2}$$

$$\text{or } \mu = \frac{9 \times 200N}{4 \times 3m^2 \cdot (200s^{-1})^2} = \frac{9}{9 \times 200} kg/m = 0.056 \times 10^{-2} kg/m$$

$$\mu = 5.6 \times 10^{-4} kg/m$$

6. The vibration is illustrated diagramatically as shown.

From here, $\frac{3}{2}\lambda = \ell$.

$$\Rightarrow \lambda = \frac{2}{3}\ell = \frac{2}{3} \times 4.3m = \frac{8.6}{3}$$

But $f_n = \frac{n}{2\ell}\sqrt{F/\mu}$

3rd Since this is the case of the vibration of the string in its harmonic i.e., $n = 3$, we find

$$f_3 = \frac{3}{2\ell}\sqrt{F/\mu}$$

$$\text{or } f_3 = \frac{3}{2 \times 4.3}\sqrt{\frac{300N}{0.5 \times 10^{-3}}}$$

$$= \frac{3}{8.6}\sqrt{600 \times 4.3 \times 10^3} \; Hz$$

$$= \frac{3}{8.6}\sqrt{258 \times 10^6} \; Hz$$

$$= 560 Hz$$

Sound

1. The velocity of sound at a temp. T is given by

$$v = 331 m/s + 0.6 \frac{m/s}{C^\circ} T$$

Because the velocity increases by 0.6 m/s for each degree centigrade rise in temp. Therefore, for the given case the velocity

is

$$v = 331\ m/s + 0.6\frac{m/s}{C^\circ} \cdot 20^\circ C$$

$$= 331\ m/s + 12\ m/s$$

$$= 343\ m/s$$

a, If the pipe is open at both ends

$$f_n = \frac{n}{2l} v$$

the fundamental is, $f_1 = \frac{1}{2 \times 0.2m}\ 343\ m/s$

$= \frac{343.0}{4} m/s = 857.5\ Hz$, which is the fundamental frequency.

The first overtone is,

$$f_2 = \frac{2}{2 \times 0.2m} \cdot 343\ m/s = 1715\ Hz$$

The second overtone is,

$$f_3 = \frac{3}{2 \times 0.2m}\ 343\ m/s = 2572.5\ Hz$$

The third overtone is,

$$f_4 = \frac{4}{2 \times 0.2m}\ 343\ m/s = 3430\ Hz$$

b, If the pipe is opened at one end with the allowed frequencies are given by

$$f_n = \frac{nv}{4l}, \quad n = 1, 3, 5, \ldots$$

The fundamental is

$$f_1 = \frac{1}{4 \times 0.2m}\ 343\ m/s = 428.75\ Hz$$

The first overtone is,

$$f_3 = \frac{3}{4 \times 0.2m} \cdot 343\ m/s = 1286.25\ Hz$$

The second overtone is,

$$f_5 = \frac{5}{4 \times 0.2m} \cdot 343\ m/s = 2143.75\ Hz$$

The third overtone is,

$$f_7 = \frac{7}{4 \times 0.2m} \times 343\ m/s = 3001.25\ Hz$$

2. If the temp. is zero, the velocity of sound wave is 331 m/s

- closed pipe

$$f_n = \frac{n}{4l} v \quad , \quad n = 1, 3, 5, \ldots$$

$$\Rightarrow f_1 = \frac{1}{4l} v$$

$$\Rightarrow l = \frac{1}{f_1} v$$

$$\text{or } l = \frac{1}{4 \times 256\ \text{Hz}} \cdot 331\ \text{m/s}$$

$$= 0.32\ \text{m}$$

- open pipe

$$f_n = \frac{n}{2l} v \quad , \quad n = 1, 2, 3, \ldots$$

$$\Rightarrow f_1 = \frac{1}{2l} v$$

$$\Rightarrow l = \frac{1}{2f_1} v$$

$$\text{or } l = \frac{1}{2 \times 256\ \text{Hz}} \times 331\ \text{m/s}$$

$$= 2.54\ \text{m}$$

Solution to problems for worksheet II

1. - Stable equil. – After a small displacement the body comes to its initial stable state, e.g. an inverted cone.
   - Unstable equil. – After a small displacement the body continues to move e.g. a cone standing by its apex.
   - Neutral equil. – After a small displacement the body takes another equil. state, e.g. a cone lying by its side.

2. A crane is a system of pulleys and rigid bodies whereas a boom is a rigid body like a beam. So since a force applied on an extended body will try to bring rotation, the equil. cond will require that the sum of torques be zero i.e., torque is involved.

3. - Move up – pot energy increases so that the object will come back – unstable
   - Move down – pot. energy decreases so that the system may overcome the effect of this pot. i.e., it may go indefinitely – unstable
   - Same level – pot energy is the same and hence takes another equil. state – neutral.

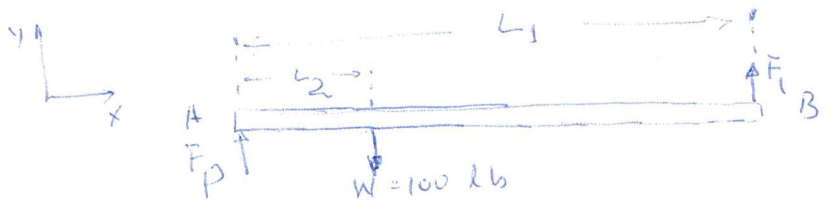
4. Let us apply the 2nd cond. of equil. which will suffice to find the unknown force F in this prob.

   Take an axis of rotation which passes through the center of the small wheel.

$$\Sigma C = 0 \quad : \quad F \cdot R_2 - W R_1 = 0$$

$$\Rightarrow F = \frac{R_1}{R_2} W = \frac{2}{6} \cdot 300 \text{ lb} = 100 \text{ lb}$$

5. a,



To find cond. of equil.

$$\Sigma F_y = 0 \quad : \quad F_p + F_1 - W = 0$$

Taking torque about A and using the second cond. of equil. we have

$$\Sigma C_A = 0 : \quad W \cdot L_2 - L_1 \cdot F_1 = 0$$

In this eq. it is only F_1 that is unknown and thus we can solve for it as

$$F_1 = \frac{L_2}{L_1} W = \frac{2}{8} \cdot 100 \text{ lb} = 25 \text{ lb}$$

b,

From the first condition of equil.

$$\Sigma F_y = 0 \quad : \quad F_p + F_1 - W = 0$$

or,

$$F_p = W - F_1 = 100 \text{ lb} - 25 \text{ lb} = 75 \text{ lb}$$

c, A similar steps is to be followed.

6.

a, (rough scale)

b, First cond of equil.

$$\Sigma F_y = 0: \quad F_1 + F_2 - (40+24+60)\text{lb} = 0$$

Axis of rotation through the c.g. and the 2nd cond.

$$\Sigma \tau_{cg} = 0: \quad 40\text{lb}\cdot 3 + F_2 \cdot 3 - F_1 \cdot 4 - 60\text{lb}\cdot 4 = 0$$

So taking the two eqs. together

$$4\begin{cases} F_1 + F_2 = 124\text{lb} \\ 3F_2 - 4F_1 = 120\text{lb} \end{cases}$$

+, $7F_2 = 616\text{lb}$

or $F_2 = \frac{616}{7}\text{lb} = 88\text{lb}$

From the first eq.,

$$F_1 = 124\text{lb} - F_1$$
$$= 124\text{lb} - 88\text{lb}$$

or $F_1 = 36\text{lb}$

c Neglecting the weight of the beam

1st cond. $F_1 + F_2 - (40+60)\text{lb} = 0$

2nd cond. $3\cdot 40\text{lb} + 3F_2 - 4F_1 - 4\cdot 60\text{lb} = 0$

or $4\begin{cases} F_1 + F_2 = 100\text{lb} \\ 3F_2 - 4F_1 = 120\text{lb} \end{cases}$

$7F_2 = 520\text{lb}$

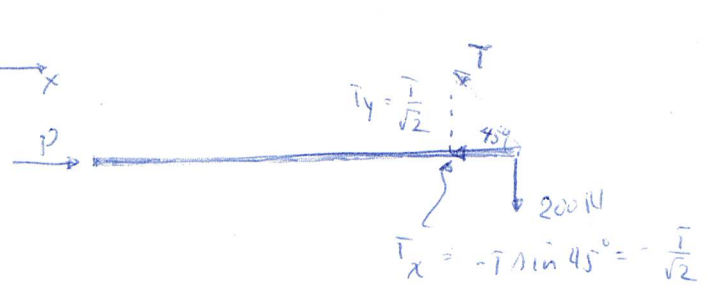
or $F_2 = \frac{520}{7}\text{lb} = 74.3\text{lb}$

And

$$F_1 = 100\text{lb} - F_2$$
$$= 100\text{lb} - 74.3\text{lb}$$

or $F_1 = 25.7\text{lb}$

7.



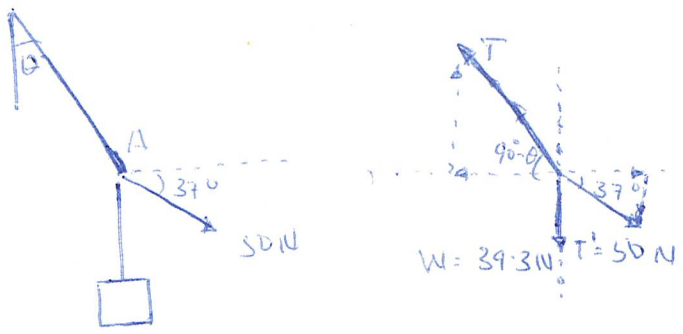
$\Sigma F_x = 0: \quad P - \frac{T}{\sqrt{2}} = 0$

$\Sigma F_y = 0: \quad \frac{T}{\sqrt{2}} - 200\,N = 0$

1st The 2nd eq. gives $T = 200\sqrt{2}\,N = 282.8\,N$ and from the eq. $P = T/\sqrt{2} = 200\,N$.

8.

a.



Comps.

$T'_x = 50\,N \cos 37^\circ = 40\,N$

$T'_y = -50\,N \sin 37^\circ = -30\,N$

$W_x = 0$

$W_y = -39.3\,N$

b) $T_x = -T \sin\theta \;, \quad T_y = T \cos\theta$

c) $\Sigma F_x = 0: \quad -T \sin\theta + 40\,N = 0$

$\Sigma F_y = 0: \quad T \cos\theta - 30\,N - 39.3\,N = 0$

i.e.)

$T \sin\theta = 40\,N$

$T \cos\theta = 69.3\,N$

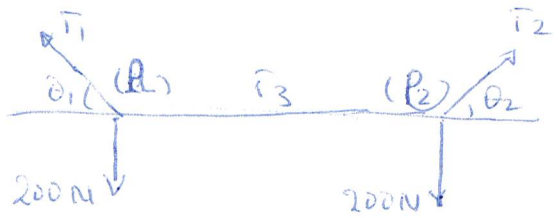
$\tan\theta = \frac{40}{69.3} = 0.5772$

or $\theta = 30^\circ$

$$T = \frac{40}{\sin 30^\circ} = \frac{40\,N}{1/2} = 80\,N.$$

9.

a,



P_1 is in equil.

$\Sigma F_x = 0$: $T_3 - T_1 \cos\theta_1 = 0$

$\Sigma F_y = 0$: $T_1 \sin\theta_1 - 200\,N = 0$

P_2 is in equil

$\Sigma F_x = 0$: $T_2 \cos\theta_2 - T_3 = 0$

$\Sigma F_y = 0$: $T_2 \sin\theta_2 - 200\,N = 0$

If $\theta_1 = \theta_2 = \theta$, say, then the eqs. for the x-forces give

$$T_3 = T_1 \cos\theta = T_2 \cos\theta$$

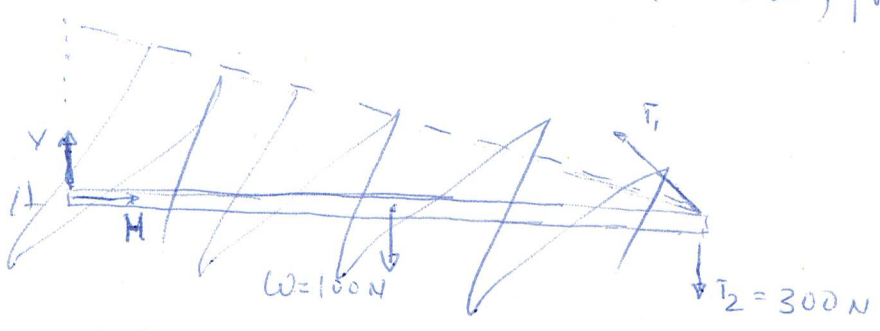
i.e.,

$$T_1 = T_2$$

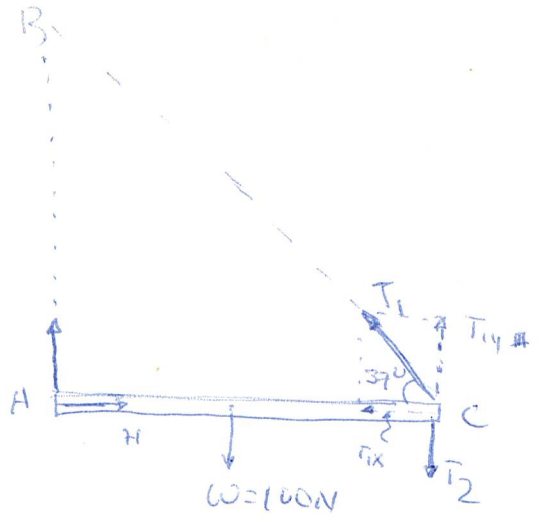
b, For $\theta = 8^\circ$, $T_1 = T_2 = \frac{200\,N}{\sin 8^\circ} = 1437.1\,N$ from the y-eq.

$$T_3 = (1437.1\,N)\cos 8^\circ = 1423.1\,N,$$ from the x-eq.

10.



10.



$T_{1x} = -0.8\,T_1$

$T_{1y} = 0.6\,T_1$

Point A

H, $\tau_H = 0$, line of action passes through A.

V, $\tau_V = 0$, " " " " " " ".

W, $\tau_W = -2\,m \cdot 100N = -200\,N\cdot m$

~~T_1~~ & T_{1x}; $\tau = 0$; line of action passes through A.

T_1, $\tau_{T_{1y}} = 4 \times 0.6\,T_1 = 2.4\,T_1$

T_2; $\tau_{T_2} = -4\,T_2 = -4 \times 300N = -1200\,N\cdot m$

~~point~~ B For $H = 466\frac{2}{3}\,W$, $V = 50N$, and $T_1 = 583\frac{1}{3}N$

$\tau_H = \tau_V = 0$; $\tau_W = -200\,NM$; $\tau_{T_{1x}} = 0$; $\tau_{T_{1y}} = 1400\,N\cdot M$

$\tau_{T_2} = -1200\,NM$

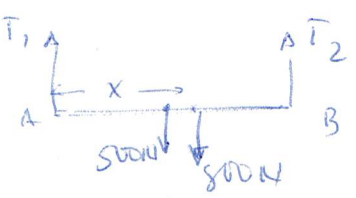
Point B

$\tau_H = 3m \times 466.67N = 1400\,N\cdot m$; $\tau_x = 0$; $\tau_W = 2M \times 100N = 200\,N\cdot m$

$\tau_{T_{1x}} = +3m \times 466.67N = -1400\,N\cdot m$; $\tau_{T_{1y}} = 4m \times 2.4 \times 583\frac{1}{3} = 5600\,N\cdot m$

$\tau_{T_2} = 4m \times 300N = 1200\,N\cdot m$

FL. (a)



$\Sigma F_y = 0$: $T_1 + T_2 - 1300 = 0$

$\Sigma \tau_A = 0$: $5 \cdot T_2 - \frac{5}{2} \times 500 - x \cdot 800 = 0$

$\Rightarrow T_2 = 250 + 160\,x$

T_2 will be max. if only $x = 5$

$T_{2,M} = 1050\ N\cdot m$

$\Sigma \tau_B = 0$: $2\cdot 500 + (5-x)\,800 - 5T_1 = 0$

$\Rightarrow T_1 = 200 + (1 - \frac{x}{5})\,800$

T is max. when $x = 0$, (i.e.)

$T_{1,M} = 1000\ N\cdot m$

b, Using the eq. of T_1 in terms of x (from a)

$410 = 200 + (1 - x/5)\,800$

$-590 = -x/5 \cdot 800$

(i.e.) $x = 3.69\ m$

Again for T_2,

$890 = 250 + (200x)$

(i.e.) $x = 4\ m$

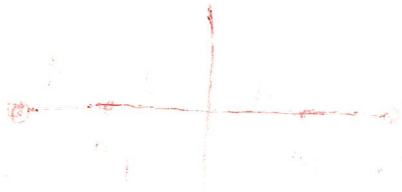
Phys. 101 Work sheet II

Rotational Dynamics

1. Find the moment of inertia of a solid disk having a radius of 20 cm and a mass of 200 g. (Radius of gyration, $k = b\sqrt{2/5}$)

2. How large a torque is required to give an angular acceleration of 4.0 rad/s^2 to a wheel having a moment of inertia 0.20 kg-m^2?

3. Refer to the figure below. Prove that the moment of inertia for rotation about an axis parallel to the one shown but a distance d away from it is given by $I = Md^2 + I_0$, where M is the total mass and I_0 is the moment of inertia about an axis through the center of mass. Neglect the mass of the bar. (This relation is a general one and is called the parallel-axis theorem)



4. A solid wheel rotating about an axis through its center has a moment of inertia I_0. Two identical masses are placed at opposite points on the rim of the wheel at opposite ends of a diameter (adding the wheel radius only), and each mass M. Show that the wheel now has moment of inertia given by $I = I_0 + 2Mr^2$

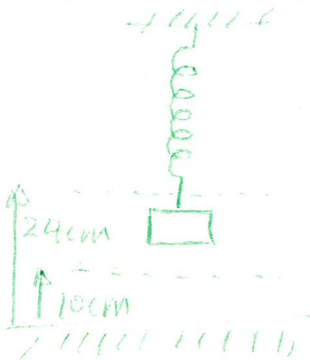
5. A children's merry-go-round in a park consists of essentially uniform 150 kg solid disk rotating about a vertical axis. The radius

of the disk is 6.0m, and a 90kg teacher is standing on it at its outer edge when it is rotating at a speed of 0.2rev/s. How fast will the disk be rotating if the teacher walks 4.0m toward the center along the radius. Hint: Momentum must be conserved.

6. Suppose that the merry-go-round described in prob. 25 has no one on it but is moving at a speed of 0.20 rev/s. If a 90kg teacher quickly sits down on the edge of it, what will its new speed be?

Harmonic Motion

1. As seen below, a mass vibrates between two limits, 16.0 and 24.0 cm above the floor. It reaches the lowest point 20 times a minute. What are a) the frequency b) the period and c) the amplitude of vibration for the system?



2. A body vibrates with SHM of amplitude 12 cm and frequency 4 Hz. What is the period? What is the displacement of the body after 3.25?

3. A body describing SHM has a maximum acceleration of $24 m/s^2$ and a maximum speed of 2 m/s. What are its period and amplitude of vibration?

4. The prong of a tuning fork vibrates with a frequency of 330 Hz and an amplitude of 2mm. What is its velocity when the displacement is 1.5m?

Ans.

5. A pendulum has a length of 2m and executes 21 complete vibrations in 1 min. What is the acceleration due to gravity?

6. Show that the maximum speed of a pendulum bob is given by

$$v = \sqrt{2gL(1-\cos\theta)}$$

7. If a 2.0kg mass vibrates at the end of a spring according to the relation $y = 0.20\cos(9.42t)$ Centimeters, find a, the amplitude of vibration b, the frequency of vibration c, the spring constant, and d, the maximum speed of the mass.

Ans.

Fluid Mechanics

I. Fluids at Rest

1. Find the pressure in kilopascals due to a column of mercury 60 cm high. What is this pressure in pounds per square inch? In atmospheres?

2. An open U-shaped tube containing mercury of 1 cm^2 in cross section. What volume of water must be poured into the left tube in order to cause the mercury in the right tube to rise 1cm above its original position?

3. Two liquids which do not react chemically are placed in a bent tube as shown. Show

that the heights of the liquids above their surface of separation are inversely proportional to their densities:

$$h_1/h_2 = \rho_2/\rho_1$$

Assume the two liquids in the U-shaped tube are water and oil. Compute the density of the oil if

4. Assume the two liquids in the U-shaped tube above are water and oil. Compute the density of the oil if the water stands 19 cm above the interface and the oil stands 24 cm above the interface.

5. A 64 lb block has a volume of 0.2 ft^3. The block is suspended from a cord and submerged in oil ($\rho = 48$ lb/ft^3). Find the bouyant force and the tension in the cord.

6. A stone of unknown composition weighs 82 lb in air. Its apparent weight is 74 lb when submerged in water. What is the volume of the stone and what is its density?

II Fluids in Motion

1. Gasoline flows through a 2 in diameter hose at an average speed of 5 ft/s. What is the rate of flow in cubic feet per second? How many minutes are required to fill a 20 gal tank? (1 ft^3 = 7.481 gal)

2. Water flowing from a 2in pipe emerges horizontally at the rate of 8 gal/min. What is the horizontal range of the stream of water if the pipe is 4ft from

the ground?

3. An aircraft wing 25 ft long and 5 ft wide experiences a lifting force of 900 lb. What is the difference in pressure between the upper and lower surfaces of the wing?

4. Sea water (ρ = 64 lb/ft^3) is pumped through the system of pipes shown in the figure below at the rate of 4 ft^3/min. The pipe diameters at the lower and upper sections are 4 and 2 in, respectively. The water is discharged into the atmosphere (p_a = 14.7 lb/in^2) at the right end of the upper section, 6 ft above the lower section. a) What are the speeds of flow in the upper and lower pipes? b) What are the pressures in the lower and upper sections?

Solutions to problems of Worksheet 11

1. $I = mk^2$

or $I = (0.200\,kg)(0.1\,m\sqrt{2/5})^2$

$= 0.200\,kg \times 0.\frac{02}{5}\,m^2$

$= (0.04\,kg)(0.02\,m^2)$

$= 0.0008\,kg \cdot m^2$

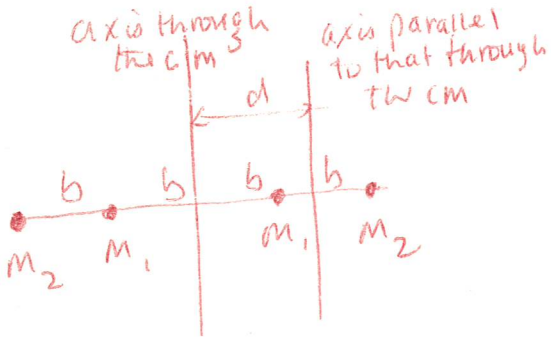
2. $\tau = I\alpha$

or $\tau = (0.20\,kg \cdot m^2)(4\,rad/s^2)$

$= 0.8\,kg \cdot m^2/s^2$

$= 0.8\,N\,m$

3.



About the axis through the cm

$I_0 = m_2(2b)^2 + m_1 b^2 + m_1 b^2 + m_2(2b)^2$

$= 2[m_1 b^2 + m_2(2b)^2]$

About the new axis

$I = m_2(2b+d)^2 + m_1(b+d)^2 + m_1(d-b)^2 + m_2(2b-d)^2$

$= m_2[(2b+d)^2 + (2b-d)^2] + m_1[(b+d)^2 + (d-b)^2]$

$= m_1[(b+d)^2 + (d-b)^2] + m_2[(2b+d)^2 + (2b-d)^2]$

$= m_1(b^2 + 2bd + d^2 + d^2 - 2bd + b^2)$

$+ m_2(4b^2 + 4bd + d^2 + 4b^2 - 4bd + d^2)$

$= m_1(2b^2 + 2d^2)$

$+ m_2(8b^2 + 2d^2)$

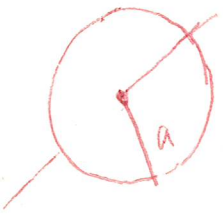
$$m_1 \cdot 2b^2 + m_1 2d^2 + m_2 \cdot 8b^2 + m_2 2d^2$$

$$= (m_1 \cdot 2b^2 + m_2 8b^2) + m_1 2d^2 + m_2 \cdot 2d^2$$

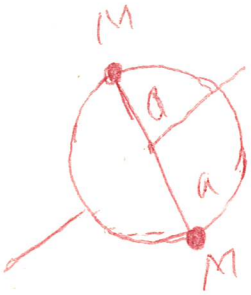
$$= 2\underbrace{(m_1 b^2 + m_2 (2b)^2)}_{I} + 2\underbrace{(m_1 + m_2)}_{m} d^2$$

$$\therefore I = I_0 + md^2$$

4.



About an axis through the center the moment of inertia of the wheel is I_0.

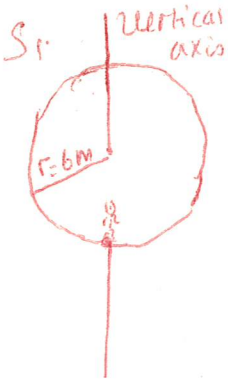


Now we have two particles of each mass M placed on the rim of the wheel at opposite ends of the diameter

Now, $I = I_0 + Ma^2 + Ma^2$

$= I_0 + 2Ma^2$

5.

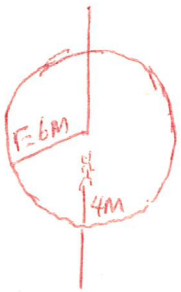


Initially the teacher stands on the outer edge of the disk.

$$I_0 = \frac{1}{2} M_{disc} r^2 + M_{teacher} r^2$$

or $I_0 = \frac{1}{2}(150\,kg)(6m)^2 + (90\,kg)(6m)^2$

$= 165 \times 36\ kg \cdot m^2$



Now the teacher is 4m from the axis of rotation

$$I = \tfrac{1}{2} M_{disc} r^2 + M_{teacher} (r - 4m)^2$$

or $$I = \tfrac{1}{2}(150kg)(6m)^2 + 90kg(2m)^2$$

$$= 85 \times 36\ kg \cdot m^2$$

But in the whole of this situation angular mom. is conserved.

$$I_0 \omega_0 = I \omega$$

$$\Rightarrow \omega = \left(\frac{I_0}{I}\right)\omega$$

or $$\omega = \left(\frac{165 \times 36\ kg \cdot m^2}{85 \times 36\ kg \cdot m^2}\right)(0.2\ rev/s)$$

$$= 0.388\ rev/s$$

6. In this,

$$I_0 = \tfrac{1}{2} M_{disc} \cdot r^2$$

$$= \tfrac{1}{2}(150\ kg)(6m)^2$$

$$= 75 \times 36\ kg \cdot m^2$$

$$I = \tfrac{1}{2} M_{disc} r^2 + M_{teacher} r^2$$

or $$I = 75 \times 36\ kg \cdot m^2 + 90 \times 36\ kg \cdot m^2$$

$$= 165 \times 36\ kg \cdot m^2$$

Once again angular mom. is conserved when ce,

$$\omega = \left(\frac{75 \times 36\ kg \cdot m^2}{165 \times 36\ kg \cdot m^2}\right)(0.2\ rev/s)$$

$$= 0.091\ rev/s$$

Harmonic Motion

1- a) 20 vibrations in 1 minute

$\Rightarrow f = \frac{20 \text{ vibration}}{60 \text{ sec}} = \frac{1}{3} \text{ vps} = \frac{1}{3} \text{ Hz}$

b) from $|a|$, $|T| = |3|$

$T = \frac{60 \text{ sec}}{20 \text{ vib}} = 3 \text{ sec/vibrations}$

i.e.) the period is 3 sec.

c) The range of vibration is

$2y_0 = 24 \text{ cm} - 10 \text{ cm} = 14 \text{ cm}$

or $y_0 = 7 \text{ cm}$

2. $T = 1/f$

or $T = 1/4 \text{ Hz} = 0.25 \text{ sec}$

$x = x_0 \cos(2\pi f) t =$

or $x = 12 \text{ cm} \cos(2\pi \times 4 \text{ rad/s})(3.2 \text{ sec})$

$= 3.71 \text{ cm}$

3. $a_{max} = \omega^2 x_0 = \frac{v_s^2}{x_0}$

$\Rightarrow x_0 = \frac{v_s^2}{a_{max}}$

or $x_0 = \frac{(2 \text{ m/s})^2}{24 \text{ m/s}^2} = \frac{1}{6} \text{ m}$

$A = \frac{v_{max}^2}{a_{max}}$

An.

$T = \frac{2\pi x_0}{v_0}$ ✓ $\quad T = 2\pi\sqrt{m/k}$, $\frac{kx}{m} = a \Rightarrow \frac{kA}{m} = a_{max}$

$m/k = \frac{A}{a_{max}}$

or $T = \frac{2\pi \times \frac{1}{6} m}{2 m/s} = \frac{\pi}{6}$ sec. $\quad m/k = \frac{v_{max}^2}{a_{max}} = \frac{v_{max}^2}{a_{max}^2}$

4. $v = \omega(x_0^2 - x^2)^{1/2}$

$= 2\pi f (x_0^2 - x^2)^{1/2}$

$T = 2\pi\sqrt{\left(\frac{v_{max}}{a_{max}}\right)^2}$

$\boxed{T = \frac{2\pi v_{max}}{a_{max}}}$ ✓

or $v = 2\pi \times 330 (4m^2 - 2.25m^2)^{1/2} 10^{-3} = 2.74 m/s$

5. $T = \frac{60 sec}{21} = 2.86 sec$

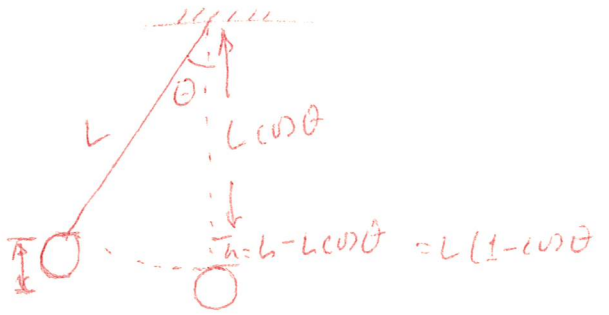
$T = 2\pi\sqrt{l/g}$

$\Rightarrow g = 4\pi^2 l/T^2$

or $g = 4\pi^2 \frac{2m}{(2.86 sec)^2}$

$= 9.65 m/s^2$

6.



max $P.E. = mgh = mgl(1-\cos\theta)$

max $K.E. = \frac{1}{2} m v^2$

But max $K.E. =$ max $P.E.$

$\frac{1}{2} m v^2 = mgL(1-\cos\theta)$

$\Rightarrow v = \sqrt{2gL(1-\cos\theta)}$

7. This is the case of sinusoidal vibration.

$$y = Y_0 \cos \omega t$$

But $y = 0.20 \cos(9.42t)$ cm

a) $Y_0 = 0.20$ m, amplitude

b) $f = \frac{\omega}{2\pi}$

or $f = \frac{9.42 \text{ rad/s}}{2\pi \text{ rad}} = 1.50 \text{ Hz}$

c) $T = 2\pi\sqrt{m/k}$

$\Rightarrow k = 4\pi^2 \frac{m}{T^2}$

or $k = 4\pi^2 \frac{2 \text{ kg}}{(0.667 \text{ sec})^2} = 177.4 \text{ N/m}$

d)

$v_{max} = \omega Y_0$

or $v_{max} = (9.42 \text{ rad/s})(0.20 \text{ m})$

$= 1.88 \text{ m/s}$

Fluid Mechanics

I. Fluids at rest

1.

$p = \rho g h$

or $p = (13,600 \text{ kg/m}^3)(9.8 \text{ m/s}^2)(0.6 \text{ m}) = 79968 \text{ N/m}^2$

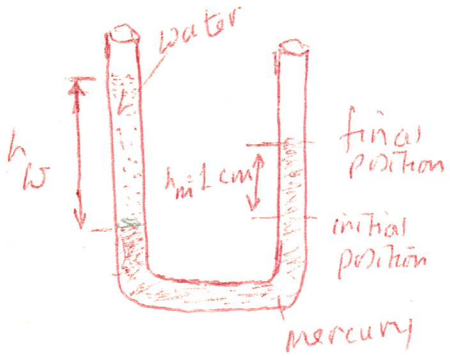
$1 \text{ N/m}^2 = 1 \text{ Pascal}$

$\Rightarrow p = 79.968 \text{ kPa}$

$1 \text{ kPa} = 0.145 \text{ lb/in}^2$

$\Rightarrow p = 11.60 \text{ lb/in}^2$

2.



The mercury rises 1 cm above its initial position until equilibrium is reached i.e., the pressure due to the 1 cm column of mercury is equal to pressure due to a column h_w of water.

Then, $\rho_w g h_w = \rho_m g h_m$

$\Rightarrow h_w = \frac{\rho_m}{\rho_w} \cdot h_m$

or $h_w = \left(\frac{13.6 \text{ g/cc}}{1 \text{ g/cc}}\right) 1 \text{ cm}$

$= 13.6 \text{ cm}$

The volume of water required is thus given as

$V_w = A h_w$

or $V_w = (2 \text{ cm}^2)(13.6 \text{ cm})$

$= 27.2 \text{ cm}^3$

3. Here since equilibrium is reached,

$\rho_1 g h_1 = \rho_2 g h_2$

$\Rightarrow h_1/h_2 = \rho_2/\rho_1$

4.

$\frac{h_1}{h_2} = \rho_2/\rho_1$

if we refer to quantities with subscript 1 to water we have

$h_1 = 1 \text{ g}$, $\rho_1 = 1 \text{ g/cc}$

for oil, $h_2 = 24 \text{ cm}$

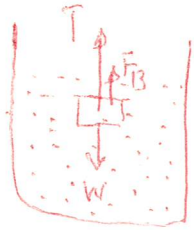
Then,

$$\rho_2 = \frac{h_1}{h_2}\rho_1$$

or $$\rho_2 = \left(\frac{19}{24}\right)(1\,gm/cc)$$

$$= 0.792\,gm/cc$$

5.



The object is in equilibrium at the indicated position.

We apply then the first condition of equilibrium.

$$\Sigma F_y = 0 \quad : \quad T + F_B - W = 0$$

But $$F_B = V_o w_o$$

where V_o is the volume of the oil & w_o the weight density.

$$\therefore F_B = (0.2\,ft^3)(48\,lb/ft^3)$$

$$= 9.6\,lb$$

From the equation for the condition of equilibrium we have that

$$T = W - F_B$$

or $$T = 64\,lb - 9.6\,lb = 54.4\,lb$$

6. The force of buoyancy is given by the difference between the weight in air and the apparent weight.

$$\therefore F_B = 82\,lb - 74\,lb = 8\,lb$$

But the force of buoyancy equals the weight of the displaced water (i.e.,)

$$F_B = V_w w_w$$

The subscript w refers to water

b) $V_w = \frac{F_B}{D_w}$

or $V_w = \frac{8\,lb}{62.4\,lb/ft^3} = 0.128\ ft^3$

This is actually equal to the volume V_s of the stone.

$$V_s = 0.128\ ft^3$$

The weight density is thus

$$D_s = \frac{8\,lb}{0.128\ ft^3} = 639.6\ lb$$

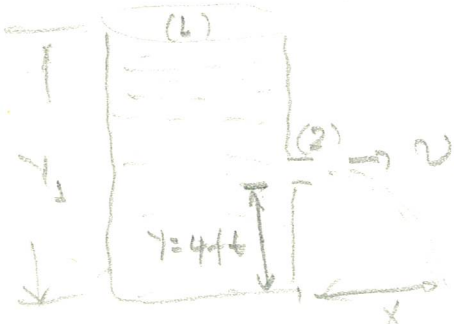
II Fluids in Motion

1. The rate of flow is given by

$R = Av = \pi R^2 v$

But also, $R = \frac{V}{t} \Rightarrow t = \frac{V}{R}$

2.



$\Rightarrow v_0 =$

$R = A v_0 \Rightarrow v_0 = \frac{R}{A}$

or $v_0 = R/\pi r^2$

3. $A = 25 \times 5\ ft^2 = 125\ ft^2$

$F_2 - F_1 = F_B = 800\ lb$

or $A(P_2 - P_1) = F_B \Rightarrow P_2 - P_1 = \frac{F_B}{A} = \frac{800\,lb}{125\ ft^2}$

4 a, $R = A_1 v_1 = A_2 v_2 \Rightarrow v_1 = R/A_1 \; ; \; v_2 = R/A_2$

b, Bernoulli eq.

$$P_1 + \rho g h_1 + \tfrac{1}{2}\rho v_1^2 = P_2 + \rho g h_2 + \tfrac{1}{2}\rho v_2^2$$

$$\Rightarrow P_1 - P_2 = \rho g h_2 + \tfrac{1}{2}\rho (v_2^2 - v_1^2)$$

$$= \rho h_2 + \tfrac{1}{2}\rho (v_2^2 - v_1^2)$$

Kotebe College of Teacher Education
Department of Physics

12 June 1990

Name _____

I.D. No. _____

Time Allowed 2hrs.

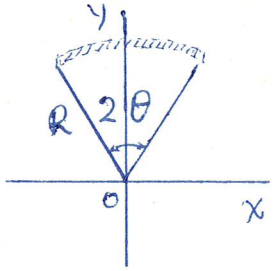
Instructions:

Solve any ten of the following twelve problems on the spaces provided after each problem. Neatness is considered.

In case you need you may use:

$K = 9 \times 10^9 \, N m^2/c^2$, $\mu_0 = 4\pi \times 10^{-7} \, Wb/mA$.

1. A uniformly charged rod with charge per unit length λ is bent into the shape of a circular arc of radius R as shown below. The arc subtends an angle 2θ at the center of the circle. Show that the electric field at the center of the circle is in the y direction with a magnitude given by $E_y = 2\lambda \sin\theta / 2\pi\epsilon_0 R$.

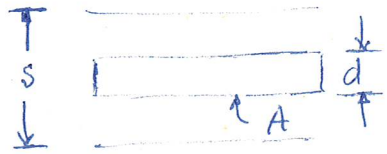


2. Each of two small spheres is charged positively, the combined charge being 5×10^{-5} C. If each sphere is repelled from the other by a force of 1 N when the spheres are 2m apart, how is the total charge distributed between the spheres?

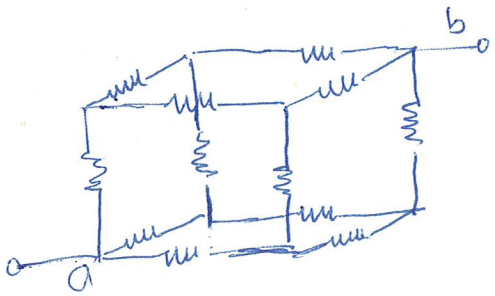
3. A nonconducting plane slab of thickness d has a uniform charge density ρ. Find the magnitude of the electric field at all points in space both a, inside and b, outside the slab.

4. Three point charges are located as follows: $-q$ at $x = -\ell$, $+2q$ at $x = 0$, and $-q$ at $x = +\ell$. Calculate the potential along the x-axis in the distant region ($|x| >> \ell$).

5. A conducting slab with a thickness d and area A is inserted into the space between the plates of a parallel-plate capacitor with spacing S and area A, as shown in the diagram. What is the value of the capacitance of the system?

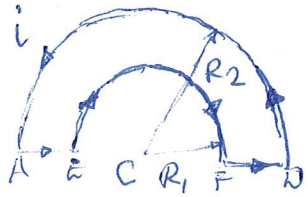


6. Determine the equivalent resistance of twelve resistors each having a resistance R are connected to form a cube as shown below. Find the equivalent resistance between the terminals a and b.

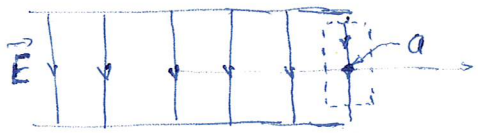


7. A long straight wire carries a current i (63 A). A proton is projected parallel to the wire and in the direction of the current a distance d (10cm) and speed v (10^6 m/s). What force acts on the proton.

8. In the figure below suppose that the positive x axis points outward from the plane of the figure. a, Use the Biot-Savart law to calculate the magnetic field $\vec{B}$ at C, the common center of the semicircular arcs AD and EF, of radii R_2 and R_1 respectively, forming part of the circuit AEFDA and carrying current i as shown. b, Calculate magnetic dipole moment $\vec{m}$ of this circuit.



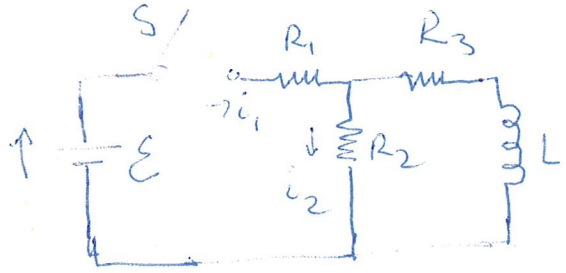
9. Prove that the electric field $\vec{E}$ in a charged parallel-plate capacitor cannot drop abruptly to zero as one moves at right angles to it, as suggested by the arrow in the figure below. (See point a). In actual capacitors fringing of the lines of force always occurs, which means that $\vec{E}$ approaches zero in a continuous and gradual way. (Hint: Apply Faraday's law to the rectangular path shown by the dashed lines.)



10. Prove that if the flux through of a magnetic field $\vec{B}$ through a coil of N turns changes in any way from ϕ_1 to ϕ_2, then the charge q that flows through the circuit of the coil of total resistance R is given by

$$q = \frac{N(\phi_2 - \phi_1)}{R}.$$

11. An emf $\mathcal{E}$ is connected in the circuit shown below where $\mathcal{E} = 100$ volts, $R_1 = 10\,\Omega$, $R_2 = 20\,\Omega$, $R_3 = 30\,\Omega$, and $L = 2$ H, find the values of i_1 and i_2 a, immediately after S is closed b, a long time later c, immediately after S is opened d, a long time later.



12. Show that the self-inductance for a length ℓ of a long wire associated with the flux inside the wire is $\mu_0 \ell / 8\pi$, independent of the diameter. Assume a uniform current density.

Work Sheet - II

Wave Motion

1. A 1.2 kg rope is stretched over a distance of 5.2 m and placed under a tension of 120 N. Compute the speed of transverse wave in the rope

2. A string 4m long has a mass of 100 g and a tension of 64 N. What is the frequency of its fundamental mode of vibration? What are the frequencies of the first and second overtones?

3. A 30 m cable weighing 400 N is stretched between two telephone poles with a tension of 1800 N. What length of time is required for a pulse to make a round trip if the cable is struck?

Sound

1. On a day when the air temperature is 27°c, a stone is dropped down a mine shaft 200 m deep. How much later will the impact be heard at the bottom be heard? (Allow for time to drop.) The speed of sound increases by 0.6 m/s for every for each celsius degree rise in temperature (or $v = 331\ m/s + (0.6\ \frac{m/s}{c^\circ})T$) where T is the celsius temp. of the air.

2. What length of closed pipe is required to produce a fundamental frequency of 256 Hz? What length of open pipe will have this fundamental frequency. Assume the temp. to be 0°c.

3. The fundamental frequency of a train whistle is 300 Hz, and the speed of the train is 60 km/h. On a day the temperature is 20°C, What frequencies will be heard by a stationary observer as the train passes with its whistle blowing?

4. To what three lowest frequencies will a pipe open at both ends resonate? The pipe is 0.90m long. Draw the wave within the tube at each frequency.

Solution to Assignment II

Newtons Laws

2.

a,

The bullet undergoes rectilinear motion with const. acc. so that

$$x = \bar{v}\, t = \frac{v_0 + v}{2}\, t$$

or,

$$t = \frac{2x}{v_0 + v} = \frac{2 \times 0.1}{360}\, s = \frac{1}{18} \times 10^{-2}\, s = 5.55 \times 10^{-4}\, s.$$

b, To find this force let us first determine the acc. of the bullet

$$a = \frac{v^2 - v_0^2}{2x} = -\frac{360^2}{2 \times 0.1}\, m/s^2 = -6.48 \times 10^5\, m/s^2$$

Then,

$$F = ma = -1.8 \times 10^{-3} \times 6.48 \times 10^5\, N = -1.17 \times 10^3\, N$$

Which is the force of friction.

3. From the W-E theorem

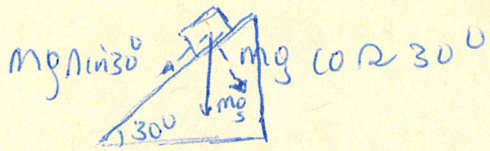
$$-F_k x = \tfrac{1}{2} m v^2 - \tfrac{1}{2} m v_0^2 = -\tfrac{1}{2} m v_0^2$$

or,

$$\mu_k m g x = \tfrac{1}{2} m v_0^2$$

$$\mu_k = \frac{v_0^2}{2gx} = \frac{10^2}{2 \times 10 \times 40} = 0.125$$

4.



a, If the plane is frictionless

$$mg\sin 30^\circ = ma$$

or, $$a = g\sin 30^\circ = g/2 = 4.9\ m/s^2.$$

b) If the reacts a force of friction

$$mg\sin 30^\circ - \mu mg\cos 30^\circ = ma$$

or, $$a = g\sin 30^\circ - \mu g\cos 30^\circ = (0.5 - 0.17)g = 0.33g$$

5. a,

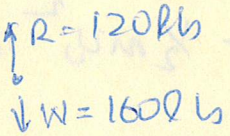
$$R - W = Ma$$

$$(200 - 160)\,lb = \frac{160}{32}\ slug\ a$$

or,

$$a = 8\ ft/s^2,\ upward\ since\ R > W.$$

b,

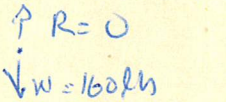


$$(120 - 160)\,lb = 5\ slug\ a$$

or,

or, $$a = -8\ ft/s^2,\ downward.$$

c)

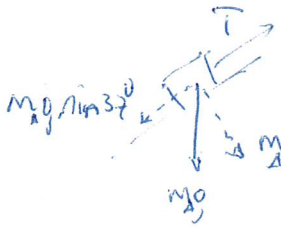


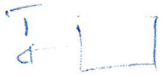
$$-160\,lb = +5\ slug\ a$$

or, $$a = -32\ ft/s^2,\ free\ fall.$$

6.

a,





$$m_A g \sin 37^0 - T = m_A a \,, \quad T = m_B a$$

or,

$$m_A g \sin 37^0 - m_B a = m_A a$$

or,

$$a = \frac{m_A g \sin 37^0}{m_B + m_A} = \frac{20 \times 10 \times 0.6}{40} \, m/s^2 = 3 \, m/s^2$$

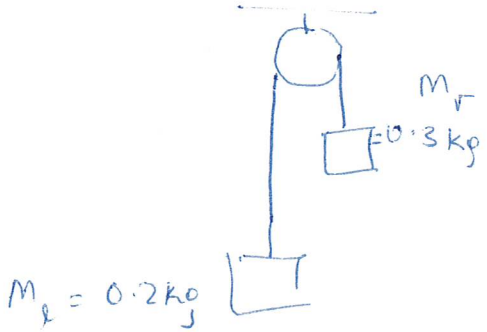
Hence,

$$1m = \frac{1}{2} \times 3 \, m/s^2 \times t^2$$

i.e.,

$$t = \sqrt{\tfrac{2}{3}} \text{ sec.}$$

7.



Left block

$$T - m_l g = m_l a \quad \text{--- (1)}$$

Right block

$$m_r g - T = m_r a \quad \text{--- (2)}$$

1 + 2 :

$$(m_r - m_l) g = (m_l + m_r) a$$

or,

$$a = \frac{m_r - m_l}{m_l + m_r} g = \frac{0.3 - 0.2}{0.2 + 0.3} g = g/5$$

After two seconds the velocity of each block will be

$$v = at = \frac{10}{5} \times 2 \, m/s = 4 \, m/s \,, \text{ start from rest.}$$

Now after the 0.1kg block is removed the system moves at this const. velocity so that the dist. moved by each block in the first second after each block is removed is

$$y = vt = 4\,m/s \times 1s = 4m.$$

b. Before the 0.1kg block is removed

$$T = m_1(a+g) + m_2(a+g) \quad \text{from (1)}$$

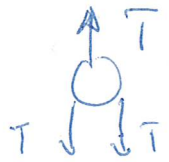
Hence,

$$T = 0.2(g/5 + g) = 0.2\,\frac{6}{5}g = 2.4\,N.$$

After it is removed a=0 and again from (1)

$$T = m_2 g = 0.2 \times g = 2\,N.$$

c, Neglecting the weight of the pulley the free body diagram of this body is as shown below.



Then, $T' = 2T = 2 \times 2.4\,N = 4.8N.$

Work and Energy

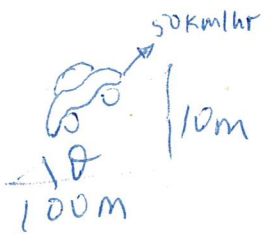
13. a, $50\,Km/hr = 50 \times 10^3 m / 3600\,s = \frac{500}{36}\,m/s = \frac{125}{9}\,m/s$

$P_r = F_r v$ or $20 \times 746\,W = F_r \frac{125}{9}\,\frac{m}{s}$

i.e.,

$$F_r = 1074.24\,N.$$

b,



Now in addition to the work done against retarding forces there is work done against gravity,

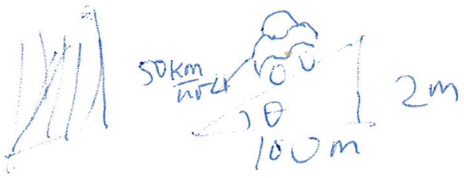
Thus if P_g represents the rate of work done against gravity we have

$$P_g = mg \sin\theta \cdot v = 1200 \times 10 \times \frac{10}{100} \times \frac{125}{9} W$$
$$= 22.2 hp$$

So the total power need will be

$$P_T = P_r + P_g = 20hp + 22.2hp = 42.2hp.$$

c,



In this case gravity helps since the car is acted upon by this force down along the incline,

$$P_g = mg \sin\theta v = 1200 \times 10 \times \frac{2}{100} \times \frac{125}{9} W = \frac{10}{3} W = 4.46hp$$

Now the total power required is

$$P_T = P_r - P_g = 20hp - 4.46hp = 15.54 hp.$$

d,

Now the whole power developed due to gravity equals the power developed by the engine.

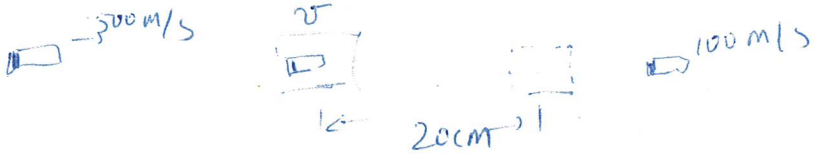
Thus,

$$P_g = mg \sin\theta \, v = 2hp$$

$$\sin\theta = \frac{P_g}{mgv} = \frac{20 \times 746}{1200 \times 10 \times \frac{125}{9}} = 0.09 = 9\%$$

Impulse and Momentum

4. a)



immediately after the impact the mom. is the same as that if its original value so that

$$2\times10^{-3}\times5\times10^{2} = (2+1)\times10^{-3}\times v \quad 2\times10^{-3}\times10^{2} + 1\cdot v$$

or,

$$v = \frac{1}{1+0\times10} \quad 8\times10^{-3}\times10^{2}\ m/s = 0\cdot8\ m/s.$$

Now, the W-E theorem gives

$$\frac{1}{2}\times1\times0\cdot8^{2} \quad \mu\, m g \times = \frac{1}{2} m v_0^{2}$$

or,

$$\mu = \frac{m v_0^{2}}{2 g \times} = \frac{0\cdot8^{2}}{2\times10\times0\cdot2} = \frac{0\cdot64}{4} = 0\cdot16$$

b,

$$\Delta K = \frac{1}{2}\times2\times10^{-3}\,(25\times10^{4} - 10^{4})\,J = 10^{-3}\times24\times10^{4}\,J = 240\,J.$$

c,

$$K_{block} = \frac{1}{2}\times1\times0\cdot8^{2}\,J = 0\cdot32\,J$$

5. Mom. is conserved.

$$m_{bullet}\, v_{bullet} = (m_{bullet} + m_{block})\, v$$

$$\frac{1}{2}(m_{bullet} + m_{block})\, v^{2} = \mu(m_{bullet} + m_{block})\, g\, x \quad \text{– W-E theorem}$$

$$v = \sqrt{2\mu g x}$$

(i.e.)

$$v_{bullet} = \frac{m_{bullet} + m_{block}}{m_{bullet}} \sqrt{2\mu g x}$$

$$= \frac{5\times10^{-3} + 3}{5\times10^{-3}} \sqrt{2\times10\times\frac{1}{4}\times0\cdot2}\ m/s$$

$$= 601\ m/s$$

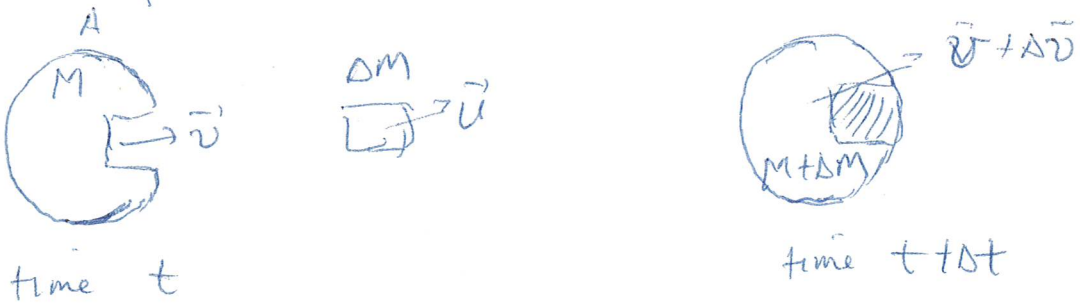
Motion with a Variable Mass

There are two ways in which a body's mass can change, either it acquires mass from or loses mass to its surroundings.

- Mass Accretion:

This is a situation where the mass of a body increases. For example, when a raindrop falls through a region of saturated vapor in a cloud, the drop's mass will increase by vapor condensation on its surface.

Consider an object A that, at some time t, has a mass M and velocity $\vec{v}$ in a certain inertial frame. The object encounters a mass Δm that has a velocity $\vec{u}$ in the same frame. Assume these two objects form a combined body in a time Δt. At the time $t+\Delta t$, the combined mass has a velocity $\vec{v}+\Delta\vec{v}$ in the inertial frame.



Let $\vec{F}_e$ be the external force that acts on the entire system.

From Newton's 2nd law

$$\vec{F}_e = \frac{d\vec{P}}{dt} \simeq \frac{\Delta P}{\Delta t} = \frac{P_f - P_i}{\Delta t}$$

where

$$\vec{P}_i = M\vec{v} + \Delta m\,\vec{u}$$

$$\vec{P}_f = (M+\Delta m)(\vec{v}+\Delta\vec{v})$$

Neglecting terms of 2nd order smallness we get

and remembering that $\Delta m = \Delta M$, we get

$$\vec{F}_e = M\frac{\Delta\vec{v}}{\Delta t} + \vec{v}\frac{\Delta M}{\Delta t} - \vec{u}\frac{\Delta M}{\Delta t}$$

In the limit $\Delta t \to 0$, this eq. becomes

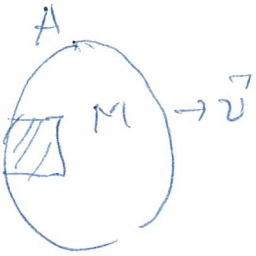
$$M\frac{d\vec{v}}{dt} = \vec{F}_e + (\vec{u}-\vec{v})\frac{dM}{dt}$$
$$= \vec{F}_e + \vec{v}_r\frac{dM}{dt},$$

$\vec{v}_r = \vec{u} - \vec{v}$ is the relative velocity of ΔM wrt A.

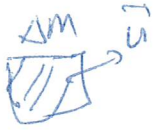
We can also write

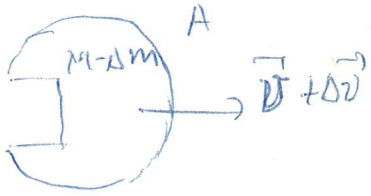
$$\frac{d}{dt}(M\vec{v}) = \vec{F}_e + \vec{u}\frac{dM}{dt}.$$

- Loss of Mass



time t





time $t+\Delta t$

Now, $\Delta M = -\Delta m$.

If $\vec{F}_e$ is the external force on the entire system, then

$$\vec{F}_e = \frac{d\vec{p}}{dt} = \frac{\Delta\vec{p}}{\Delta t} = \frac{\vec{p}_f - \vec{p}_i}{\Delta t}$$

where

$$\vec{p}_i = M\vec{v}$$
$$\vec{p}_f = (M-\Delta m)(\vec{v}+\Delta\vec{v}) + \Delta m\,\vec{u}$$

$$\Rightarrow \vec{F}_e = M\vec{v} + M\Delta\vec{v} - \Delta m\,\vec{v} - \Delta m\,\Delta\vec{v} + \Delta m\,\vec{u} - M\vec{v}$$
$$= (M\Delta\vec{v} + \Delta M\vec{v} - \Delta M\vec{u} + \Delta M\,\Delta\vec{v})/\Delta t$$

Neglecting the last term we find

or,

$$\vec{F}_e = m\frac{\Delta\vec{v}}{\Delta t} + \vec{v}\frac{\Delta m}{\Delta t} - \vec{v}\frac{\Delta m}{\Delta t} \quad (\vec{u} \neq \vec{v})$$

Rearranging and taking the limit $\Delta t \to 0$ we find

$$m\frac{d\vec{v}}{dt} = \vec{F}_e + (\vec{u} - \vec{v})\frac{dm}{dt}$$

Since $\vec{u} - \vec{v} = \vec{v}_r$ (the relative velocity of Δm wrt object A), we write

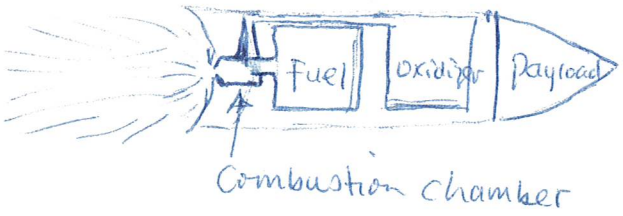
$$m\frac{d\vec{v}}{dt} = \vec{F}_e + \vec{v}_r\frac{dm}{dt}.$$

Thus we have exactly the same eq. for the case of mass loss as for the case of mass accretion. However, we must remember that in the last eq. above, $dm/dt < 0$.

For the description of the motion of a rocket this eq. may be interpreted as follows.

$$m\frac{d\vec{v}}{dt} = \vec{F}_e + \vec{v}_r\frac{dm}{dt} = \vec{F}_e - \vec{v}_r\frac{dm}{dt}.$$

This eq. is often called the rocket eq.



The mass Δm corresponds to the combustion ejected gases due to the combustion.

The term $\vec{\Omega} = \vec{v}_r\frac{dm}{dt} = -\vec{v}_r\frac{dm}{dt}$ is referred to as the thrust of the rocket engine. Note that $\vec{v}_r$, the velocity of the ejected gases relative to the rocket, is to the left in the above figure; because $dm/dt < 0$, the thrust gives a contribution to $d\vec{v}/dt$ that is to the right, thereby producing a +ve acc.

Example: A spacecraft is stationary in deep space when its rocket engine is fired for a 100 s "burn". Hot gases are ejected at a const. rate of 150 kg/s with a velocity relative to the spacecraft of 3000 m/s. The initial mass of the spacecraft (plus fuel and oxidizer) is 25,000 kg. Determine the thrust of the rocket engine and the initial acc. in units of g_0. What is the final velocity of the spacecraft?

Take the direction of motion of the spacecraft to be positive. Then, the thrust is

$$\Omega = -v_r \frac{dm}{dt} = -(-3000\,\text{m/s})(150\,\text{kg/s})$$
$$= 450{,}000\,\text{N}.$$

Since the rocket is in deep space $\vec{F}_e = 0$, and its initial acc. is given by

$$a = \frac{dv}{dt} = -\frac{v_r}{M}\frac{dm}{dt} = \frac{\Omega}{M} = \frac{450{,}000\,\text{N}}{25{,}000\,\text{kg}} \frac{g_0}{9.8\,\text{m/s}^2}$$
$$= 1.84\,g_0.$$

Also,

$$M\frac{dv}{dt} = v_r \frac{dM}{dt}$$

or,

$$\frac{dM}{M} = \frac{dv}{v_r}$$

Integration gives

$$\int_{M_i}^{M_f} \frac{dM}{M} = \frac{1}{v_r}\int_0^v dv$$

$$-\ln\frac{M_i}{M_f} = \frac{v}{v_r} \quad \text{or} \quad v = -v_r \ln\frac{M_i}{M_f}.$$

For the 100-second burn, $M_i = 25{,}000\,\text{kg}$, $M_f = M_i - 150\frac{\text{kg}}{\text{s}} \times 100\,\text{s}$ $= 10{,}000\,\text{kg}$, and $v_r = -3000\,\text{m/s}$, so that

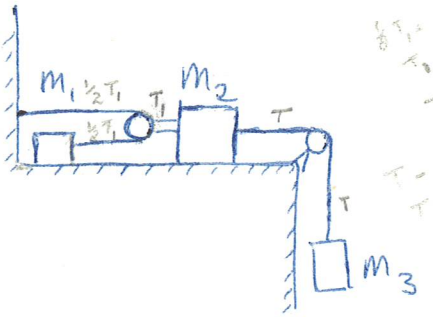
$$v = -(-3000\,\tfrac{\text{m}}{\text{s}})\ln\frac{25{,}000}{10{,}000} = 2749\,\text{m/s}$$
$$= 6150\,\text{mi/h}$$

Problems for Exercise

* 1. A car is traveling at 60 mi/h along a road parallel to a railroad track. How long does it take the car to pass a ½-mi-long train traveling at 40 mi/h in the same direction? In opposite direction?

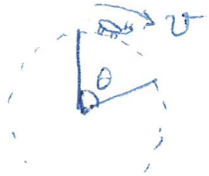
2. The driver of a car notes the car's speed to be 30 m/s in a 20 m/s zone. Just then, the driver notices a police car sitting behind a tree. Hoping that the police are not using radar, the driver at once releases the accelerator. The police are actually timing cars between two check points 150 m apart. The first check point coincides with the point where the accelerator was released. What must be the deceleration of the car if the police are to find its speed as 20 m/s?

** 1. Find the tensions in the two cords and the accelerations of the blocks shown in the figure below if friction is negligible. Assume the pulleys to be massless and frictionless. ($m_1 = 2000$; $m_2 = 5000$; $m_3 = 3000$)



2. If the friction force on each of the blocks in the above figure is 2 lb, find the acceleration of each block and the tensions in the strings. (The weights of masses 1, 2, and 3 are 8, 32, and 16 lb, respectively.

*** 1. The boy shown in the figure below has just lost its footing while near the top of the bowling ball. It slides down the ball without appreciable friction. Show that it will leave the surface of the ball at the angle shown, where θ is given by $\cos\theta = 2/3$.

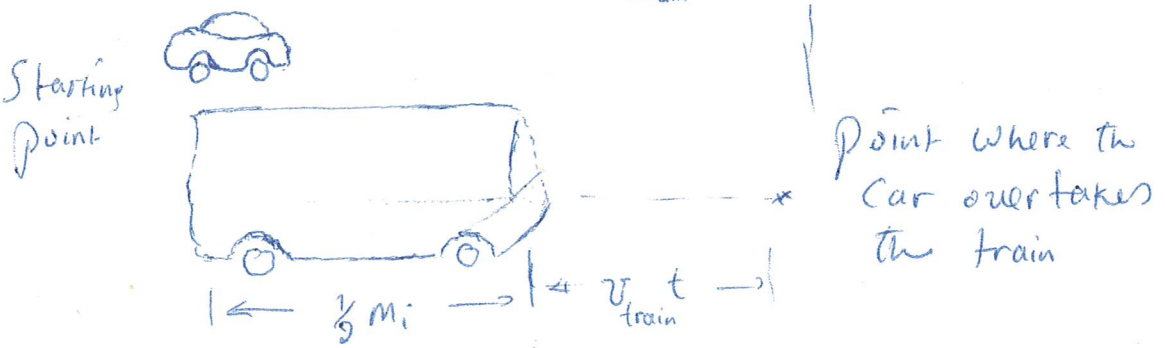


Solutions

* 1. For the first case let the car (Hadithi/Emmanuel) be opposite the rear of the train as shown below.

- In the same direction

$S_{car} = \frac{1}{2}mi + v_{train}\, t = v_{car}\, t$



As can be seen from the figure

$$S_{car} = \tfrac{1}{2}mi\; v_{car}\, t = \tfrac{1}{2}mi + v_{train}\, t$$

$$\Rightarrow (60\,mi/h)\,t = \tfrac{1}{2}mi + (40\,mi/h)\,t$$

$$\Rightarrow (60\,mi/h - 40\,mi/h)\,t = \tfrac{1}{2}mi$$

$$(20\,mi/h)\,t = \tfrac{1}{2}mi$$

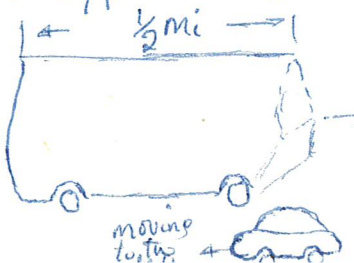
$$\Rightarrow t = \frac{(1/2)\,mi}{20\,mi/h}$$

$$= \frac{1}{40}\,hr$$

$$\text{or } t = \tfrac{1}{40}hr = \frac{1}{40}hr \times 60\,min/hr \times 60\,sec/min$$

$$= 90\,sec, \text{ in the same direction}$$

- In opposite direction



- to can be seen when they move in opposite directions the distance moved is ½ mi and this is shared between the car and the train.

$$\tfrac{1}{2}\,\text{mi} = v_{car}\,t + v_{train}\,t$$

$$\Rightarrow \frac{1}{t} = \frac{v_{car} + v_{train}}{\tfrac{1}{2}\,\text{mi}}$$

$$\text{or } t = \frac{60\,\text{mi/h} + 40\,\text{mi/h}}{\tfrac{1}{2}\,\text{mi}}$$

$$= 100 \times 2\ \text{/h}$$

$$t = \frac{1}{100 \times 2}\,\text{h}$$

$$= 5 \times 10^{-3}\,\text{h}$$

$$= 18\ \text{sec}.$$

2. Actually in the 150 m zone the allowed speed is 20 m/s.

At the first check point the driver released the accelerator and there the speed of the car had to be 30 m/s since it is not possible to reduce the speed as soon as the accelerator is released. So $v_0 = 30$ m/s. Also, the allowed average speed of the car must be 20 m/s to cover the zone in the allowed time.

$$\frac{v_0 + v_f}{2} = \bar{v}$$

$$\Rightarrow v_f = 2\bar{v} - v_0$$

$$\text{or } v_f = 2 \times 20\,\text{m/s} - 30\,\text{m/s}$$

$$= 10\,\text{m/s}$$

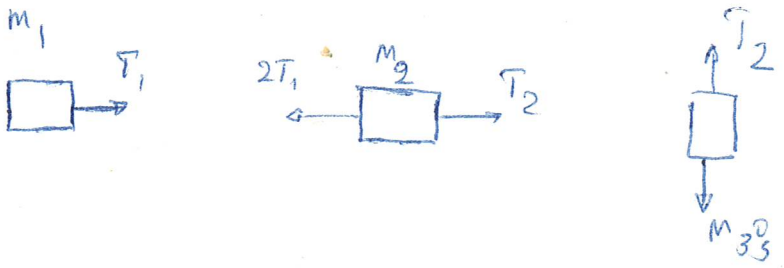
Then to find the acceleration

$$v_f^2 - v_0^2 = 2as$$

$$\Rightarrow a = \frac{v_f^2 - v_0^2}{2s}$$

$$\text{or } a = \frac{(10\,\text{m/s})^2 - (30\,\text{m/s})^2}{2 \times 150\,\text{m}} = -2.67\,\text{m/s}^2$$

** 1.



Apply Newtons 2nd law of motion

$$T_1 = m_1 a_1 \quad \text{---} \quad (1)$$

$$T_2 - 2T_1 = m_2 a_2 \quad \text{---} \quad (2)$$

$$m_3 g - T_2 = m_3 a_2 \quad \text{---} \quad (3)$$

But $a_1 = 2a_2$ because the mass m_1 is acted upon by T_1 to the right and in the same cord we have a tension T_1 to the left which accelerates it.

$$\begin{cases} T_1 = m_1 \, 2a_2 \\ T_2 - 2T_1 = m_2 a_2 \\ m_3 g - T_2 = m_3 a_2 \end{cases}$$

Multiplying the first eq. by 2 and adding the resulting eq. to the second eq. gives

$$T_2 = (4m_1 + m_2) a_2$$

Adding this eq. to the last eq. above we obtain

$$m_3 g = (4m_1 + m_2 + m_3) a_2$$

or $$a_2 = \left(\frac{m_3}{4m_1 + m_2 + m_3} \right) g$$

or $$a_2 = 1.84 \text{ m/s}$$

From which $a_1 = 2a_2 = 3.68 \text{ m/s}^2$

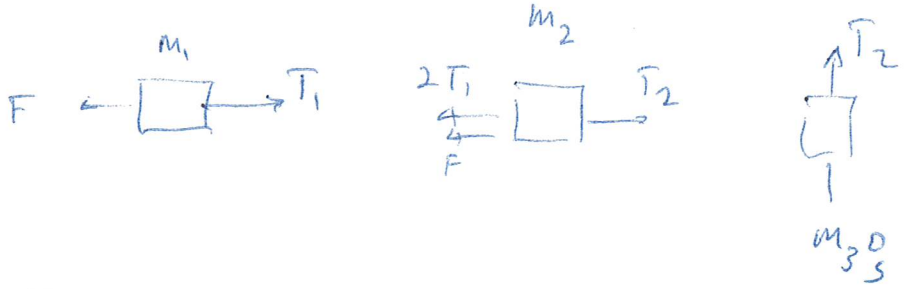
The tension in the first cord

$$T_1 = m_1 a_1 = (0.2 \text{ kg})(3.68 \text{ m/s}^2)$$
$$= 0.735 \text{ N}$$

Also,

$$T_2 = m_3(g - a_2)$$
$$= 0.3 \text{ kg}\,(9.8 \text{ m/s}^2 - 1.84 \text{ m/s}^2)$$
$$= 2.39 \text{ N}$$

2. The same free body diagram as in (1) except that here we have to include the force of friction



$$T_1 - F = m_1 a_1 = m_1 2a_2 \quad \text{--- 1}$$

$$T_2 - 2T_1 - F = m_2 a_2 \quad \text{--- 2}$$

$$m_3 g - T_2 = m_3 a \quad \text{--- 3}$$

Multiply (1) by 2 and add it to the second eq.

$$T_2 - 3F = (4m_1 + m_2) a_2$$

Adding this with the 3rd eq

$$m_3 g - 3F = (4m_1 + m_2 + m_3) a_2$$

$$\Rightarrow \quad a_2 = \frac{m_3 g - 3F}{4m_1 + m_2 + m_3}$$

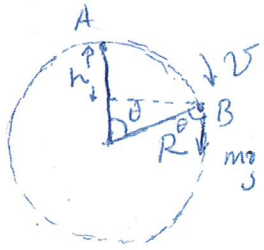
or $a_2 = 4 \text{ ft/s}^2$

$$a_1 = 2a_2 = 8 ft/s^2$$

$$T_1 = F + m_1 2a_2 = 4 lb$$

once again, $T_2 = m_3(g - a) = 14 lb$

*** 1.



Let the radius of the circle be R

Since friction is neglected and there is no other dissipating force energy is conserved i.e.) the P.E. at point A equals the K.E. at point B.

$$m_s g h = \tfrac{1}{2} m v^2$$

But $h = R - R\cos\theta = R(1 - \cos\theta)$

$$\Rightarrow m_s g R(1 - \cos\theta) = \tfrac{1}{2} m v^2$$

$$\Rightarrow \frac{m v^2}{R} = 2 m_s g (1 - \cos\theta)$$

Since the left side of the eq. represents the centripetal force at ~~point~~ B the bug experiences ~~at~~ point B, we have

$$F_c = 2 m_s g (1 - \cos\theta)$$

Also, at B an inward force F is acting on the bug.

$$F = m_s g \cos\theta$$

On the other hand, since friction is neglected the reaction force $\vec{R}$ from the ball on to the bug will be just perpendicular to the surface of the ball. This reaction

force is given by

$$\vec{R} = \vec{F} - \vec{F}_c$$

$$R = F - F_c$$

i.e., the centripetal force is provided by part of $mg\cos\theta$.

$$R = mg\cos\theta - 2mg(1-\cos\theta)$$

$$= 3mg\cos\theta - 2mg$$

But the bug leaves the surface of the ball means the reaction force ceases i.e., $R = 0$.

$$3mg\cos\theta - 2mg = 0$$

$$3\cos\theta - 2 = 0$$

or $\cos\theta = 2/3$

simple Alternative Method

Since friction is neglected, M.E. is conserved. At the top, the bug has only P.E. and taking the center of the ball as a reference point.

$$P.E.|_A = mgR$$

At point B, the bug has both P.E. and K.E.

$$P.E.|_B = mgR\cos\theta$$

$$K.E.|_B = \tfrac{1}{2}mv^2$$

Since the motion is along a circular path, the centripetal force at point B is

$$F_c = \frac{mv^2}{R} = mg\cos\theta$$

$$\Rightarrow v^2 = mgR\cos\theta$$

Now by conservation of M.E.

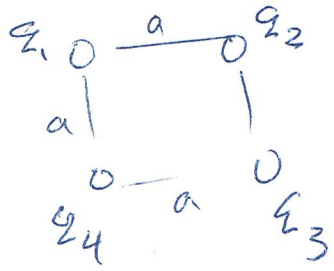
$$mgR = mgR\cos\theta + \tfrac{1}{2}mgR\cos\theta = \tfrac{3}{2}mgR\cos\theta$$

(i.e.) $\cos\theta = 2/3$

Electrostatics

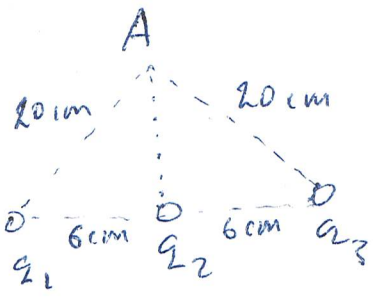
1. An alpha particle consists of two protons and two neutrons. What is the electrostatic force between two alpha particles separated by a distance of 2 nm? Is it attractive or repulsive?

2. As shown below, four point charges are placed on the four corners of a square. If $q_1 = q_2 = q_3 = q_4 = 5\mu C$ and $a = 30$ cm, find the magnitude and direction of the force on q_3.



3. If point charge of $+2\mu C$ is placed on a straight line defined by a $+2\gamma$- and a $+3$-μC charge which are 100 cm apart. a, where must the $2\mu C$ charge be placed if the resultant force on it is to be zero? b, Is it in a stable or unstable equilibrium?

4. In the fig. below if $q_1 = q_2 = +6\mu C$, how large must q_3 be if the field at A is to be directed along a line parallel to the line of the three charges? b, How large is the field in that case



5. In the fig. below is shown a particle between two parallel oppositely charged plates. Assume the particle to be an electron ($q = -e$, $m = 9.1 \times 10^{-31}$ kg).

a, How large must the electric field be between the plates if the particle is to remain motionless under the combined effects of gravity and electric field?

b, Should the top plate be positive or negative?

c, If the distance between the plates is 2.00 mm, what is the required p.d. between the plates?

A ——— B

• 2

Circuit Elements

1. An electron ($m = 9.1 \times 10^{-31}$ kg, $q = -e$) initially at rest falls from the negative filament to the positive plate in a vacuum tube. The terminals of a 12.0 V battery are connected to the filament and plate.

a, What is the speed of the electron just before it hits the plate?

b, What is its K.E. in electron volts (eV) at that time?

2. An ordinary 60 W tungsten filament light bulb has a resistance of 240 Ω when lit. The temp. of the filament is about 2000 °C under these conditions. If the α value $4.5 \times 10^{-3}\ °C^{-1}$ at 20 °C is valid even for temperatures this high, what is the resistance of the bulb at 20 °C?

3. a) What is the p.d. across a 4 Ω resistor with a current of 20 A through it?

b, What is the resistance of a rheostat if the drop in potential is 48 V and the current through it is 6 A?

c, Determine the current through a 5-Ω resistor that has a p.d. of 30 V across it.

4. A water turbine delivers 2000 hp to an electric generator. The generator is only 80% efficient and has an output terminal voltage of 1200 V. What current is delivered?

5. The plates of a certain capacitor are 3 mm apart and have an area of $0.04\ m^2$. For a dielectric of air, find,

a, the capacitance

b, the electric field intensity between the plates

and c, the charge on each plate if 200 V is applied to the capacitor.

6. Answer the questions in prob. 5 if mica ($K = 5.0$) replaces air as the dielectric.

7. A capacitor has a capacitance of 12 μF when the dielectric is air ($K = 1$). If is charged to 800V by means of a battery.

a) What is the charge on each plate?

The capacitor is now disconnected and immersed in transformer oil ($K = 4.0$).

b) What is the new capacitance?

c) How much charge is on each plate?

d) what is the new p.d.

8. Three capacitors (A, B & C) have respective capacitances of 2, 4, and 6 μF. Compute the equivalent capacitance if they are connected in series with an 800 V source of p.d. What is the charge on each capacitor? What is the voltage on each capacitor? Answer all the questions if they are connected in parallel.

9. How much work is required to place 1600 μC on a 30 μF capacitor?

10. A capacitor is formed from 30 parallel plates 20 by 20 cm. If each plate is separated by 2mm of dry air, what is the total capacitance?

Electrostatics

1. The effective charge of an alpha particle is $+2e$, i.e., the charge of the two protons. Therefore, the electrostatic force between two alpha particles that are separated by 2nm is given from coulomb's law as

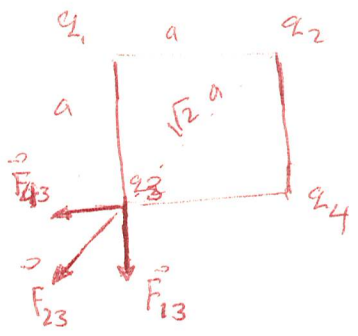
$$F = \frac{(9\times10^{9}\ N\cdot m^2/c^2)(2e)(2e)}{(2\times10^{-9}\ m)^2}$$

$$= 9\times10^{9}\ N/c^2\ \frac{2\times1.6\times10^{-19}\ c)(2\times1.6\times10^{-19}\ c)}{4\times10^{-18}}$$

$$= 24.04\times10^{-11}\ N$$

$$= 2.404\times10^{-10}\ N$$, 2 + ve repulsive.

2.



$q_1 = q_2 = q_3 = q_4 = q = 5\,\mu c$

$a = 30\ cm$

Since all the charges are similar (like charges) the forces on q_3 exerted by q_1, q_2 & q_4 are repulsive as shown in the figure. From Ohm's law

$$F_{13} = \frac{kq^2}{a^2},\quad F_{23} = \frac{kq^2}{2a^2},\quad F_{43} = \frac{kq^2}{a^2}$$

Force	x-comp	y-comp
$\vec{F}_{13}$	0	$-\frac{kq^2}{a^2}$
$\vec{F}_{23}$	$-\frac{kq^2}{2\sqrt{2}a^2}$	$-\frac{kq^2}{2\sqrt{2}a^2}$
$\vec{F}_{43}$	$-\frac{kq^2}{a^2}$	0
resultant, $\vec{F}_R$	$-\frac{kq^2}{a^2}\left(\frac{1}{2\sqrt{2}}+1\right)$	$-\frac{kq^2}{a^2}\left(1+\frac{1}{2\sqrt{2}}\right)$

(i.e.) $F_{Rx} = -\frac{kq^2}{a^2}\left(\frac{1}{2\sqrt{2}}+1\right) = -1.354\frac{kq^2}{a^2}$

$F_{Ry} = -\frac{kq^2}{a^2}\left(1+\frac{1}{2\sqrt{2}}\right) = -1.35\frac{kq^2}{a^2}$

Then,

$$F_R = \sqrt{F_{Rx}^2 + F_{Ry}^2} = \sqrt{2(1.35)^2}\,\frac{kq^2}{a^2} = 1.91\,\frac{kq^2}{a^2}$$

or using numerical values

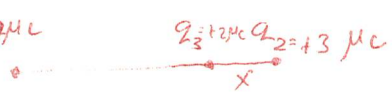
$$F_R = \frac{(1.91 \times 9\times10^{9}\ N\cdot m^2/c^2)(5\times10^{-6}\ c)^2}{(0.3m)^2}$$

$$= 5.32\times10^{-1}\ N$$

$$= 0.532\ N$$

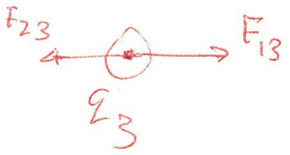
And it makes an angle $\theta = \tan^{-1}(1) = 45$, with the negative x-axis.

3.



The forces exerted on the $+2\,\mu c$ charge by the $+27\,\mu c$ & the $+3\,\mu c$ charges are repulsive so that they act in opposite directions. So if the resultant force on the $+2\,\mu c$ charge is to be zero, it has to be placed nearer to the $3\,\mu c$ charge. Let then it be placed at a distance x from the $3\mu c$ charge. Its distance from the $+27\,\mu c$ charge will be $100-x$. Therefore, from coulomb's law

$$F_{13} = k\frac{q_1 q_3}{(100-x)^2}, \quad F_{23} = \frac{kq_2q_3}{x^2}$$



a) If the resultant force is to be zero

$$F_{13} - F_{23} = 0$$

$$\Rightarrow \frac{kq_1q_3}{(100-x)^2} - \frac{kq_2q_3}{x^2} = 0$$

$$\frac{q_1}{(100-x)^2} - \frac{q_2}{x^2} = 0$$

$$\frac{27\mu c}{(100-x)^2} - \frac{3\mu c}{x^2} = 0$$

$$\frac{9}{(100-x)^2} - \frac{1}{x^2} = 0$$

$$\Rightarrow 9x^2 = (100-x)^2$$

$$9x^2 = 100^2 - 200x + x^2$$

$$8x^2 + 200x - 100^2 = 0$$

$$x^2 + 25x - 1250 = 0$$

$$x = \frac{-25 \pm \sqrt{625 + 5000}}{2} \text{ cm}$$

$$= \frac{-25 \pm \sqrt{5625}}{2} \text{ cm}$$

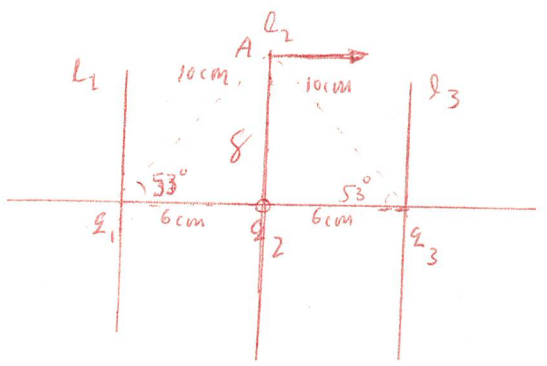
$$= \frac{-25 \pm 75}{2} \text{ cm}$$

$x = -50$ cm, or $x = 25$ cm

Since x is distance we leave out the negative value. And, Therefore, the $+2\,\mu c$ charge has to be placed 25cm from the $+3\,\mu c$ charge.

(b) It is in an unstable equilibrium.

4.



L, the line of the three charges

Then if the field is to be directed parallel to the line of the three charges, its vertical comp. should be of zero magnitude (Note Q_3 has to be a negative charge.)

$$E_1 = \frac{kQ_1}{r_{1A}^2}, \quad E_2 = \frac{kQ_2}{r_{2A}^2}, \quad E_3 = \frac{kQ_3}{r_{3A}^2}$$

Electric field vector	x- comp	y- comp
$\vec{E}_1$	$\frac{kQ_1}{r_{1A}^2} \times 0.6$	$\frac{kQ_1}{r_{1A}^2} \times 0.8$
$\vec{E}_2$	0	$\frac{kQ_2}{r_{2A}^2}$
$\vec{E}_3$	$\frac{kQ_3}{r_{3A}^2} \times 0.6$	$-\frac{kQ_3}{r_{3A}^2} \times 0.8$
resultant, $\vec{E}$	$0.6k\left(\frac{Q_1}{r_{1A}^2} + \frac{Q_3}{r_{3A}^2}\right)$	$0.8k\left(\frac{0.8Q_1}{r_{1A}^2} + \frac{Q_2}{r_{2A}^2} - \frac{0.8Q_3}{r_{3A}^2}\right)$

i.e.) $E_x = 0.6k\left(\frac{Q_1}{r_{1A}^2} + \frac{Q_3}{r_{3A}^2}\right)$, $E_y = k\left(\frac{0.8Q_1}{r_{1A}^2} + \frac{Q_2}{r_{2A}^2} - \frac{0.8Q_3}{r_{3A}^2}\right)$

But for the given condition, $E_y = 0$.

$$K\left(\frac{0.8q_1}{r_{1A}^2} + \frac{q_2}{r_{2A}^2} - \frac{0.8q_3}{r_{3A}^2}\right)$$

$$\Rightarrow \frac{0.8q_1}{r_{1A}^2} + \frac{q_2}{r_{2A}^2} - \frac{0.8q_3}{r_{3A}^2} = 0$$

$$\Rightarrow q_3 = \frac{r_{3A}^2}{0.8}\left(\frac{0.8q_1}{r_{1A}^2} + \frac{q_2}{r_{2A}^2}\right)$$

$$\text{or } q_3 = \frac{(0.1\text{m})^2}{0.8}\left(\frac{0.8 \times 6.4\times10^{-6}\text{C}}{(0.1\text{m})^2} + \frac{6.4\times10^{-6}\text{C}}{(0.08\text{m})^2}\right)$$

$$= 6.4\times10^{-6}\text{C} + 1.25\times10^{-5}\text{C}$$

$$= 6.4\times10^{-6}\text{C} + 0.125\times10^{-4}\text{C}$$

$$= 18.900\times10^{-6}\text{C}$$

This is the magnitude of q_3.

$$E_x = 0.6 \times 9\times10^{9}\,\text{N}\cdot\text{m}^2/\text{C}^2\left(\frac{6.4\times10^{-6}\text{C}}{(0.1\text{m})^2} + \frac{18.9\times10^{-6}\text{C}}{(0.1\text{m})^2}\right)$$

$$= 0.6 \times 9\times10^{9}\,\text{N}\cdot\text{m}^2/\text{C}^2\,(2530\times10^{-4}\,\text{C/m}^2)$$

$$= 13602\times10^{5}\,\text{N/C}$$

$$1.37\times10^{7}\,\text{N/C}$$

$$E_y = 9\times10^{9}\,\frac{\text{N}\cdot\text{m}^2}{\text{C}^2}\left(\frac{0.8\times6.4\times10^{-6}\text{C}}{(0.2\text{m})^2} + \frac{6.4\times10^{-6}\text{C}}{(0.08\text{m})^2} - \frac{0.8\times18.9\times10^{-6}\text{C}}{(0.1\text{m})^2}\right)$$

$$= 9\times10^{9}\,\frac{\text{N}\cdot\text{m}^2}{\text{C}^2}\left(\frac{6.4\times10^{-6}\text{C}}{(0.08\text{m})^2} - \frac{0.8\times0.125\times10^{-4}\text{C}}{(0.1\text{m})^2}\right)$$

$$= 9\times10^{9}\,\frac{\text{N}\cdot\text{m}^2}{\text{C}^2}\left(10^{-3}\,\frac{\text{C}}{\text{m}^2} - 10^{-3}\,\text{C/m}^2\right)$$

$$= 0$$

∴ the electric field has a magnitude

$$E = E_x = 1.37\times10^{7}\,\text{N/C}$$

5. a, If the particle (electron) remains motionless it means the force of gravity and electricity are balanced, i.e.,

$$m_e g = qE$$

$$\Rightarrow E = m_e g/q$$

$$\therefore E = \frac{(9.1\times10^{-31}\,kg)(9.8\,m/s^2)}{1.6\times10^{-19}\,C} = 5.57\times10^{-11}\,N/C$$

b) The top plate should be positive, otherwise the electron will be pushed down.

c) The p.d. between oppositely charged parallel plates is given by

$$V = Ed$$

$$\therefore \quad V = (5.57\times10^{-11}\,\tfrac{N}{C})(2\times10^{-3}\,m)$$

$$= 1.114\times10^{-13}\,J/C$$

$$\text{or } V \simeq 1.11\times10^{-13}\,V$$

Circuit Elements

1 - a) The electron here is accelerated then through a p.d. of 12.0 V. so its kinetic energy before it hits the plate is

$$E_k = \tfrac{1}{2}mv^2 = qV \quad \text{where } V \text{ is the p.d}$$

$$\Rightarrow v = \sqrt{\frac{2qV}{m}}$$

$$\text{or } v = \sqrt{\frac{2(1.6\times10^{-19}\,C)(12V)}{9.1\times10^{-31}\,kg}} = 2.05\times10^{6}\,m/s$$

b) From a, the K.E. of the electron is given to be

$$E_K = qV$$

But taking the magnitude of $q (= e)$ we have that

$$E_k = eV$$

$$\text{or } E_K = e\cdot 12\,V$$

$$= 12\,eV$$

i.e.) the K.E. of the electron is 12 electron volts (eV).
Reactivation By def. 1 eV is the energy that is gained by a charge

of $q = 1.6 \times 10^{-19}$ C, when it is accelerated through a p.d. of 1 volt. or 1 eV is the energy that is gained by an electron which is accelerated through a p.d. of 1 volt.

2. For a conductor over a limited temperature range the resistivity may be represented by an equation of the form

$$\frac{\rho - \rho_{20}}{\rho_{20}} = \alpha_{20}(t - 20^\circ)$$

or $$\frac{R - R_{20}}{R_{20}} = \alpha_{20}(t - 20^\circ)$$

where R is the resistance at a temp. t which is actually the celsius temp. Then,

$$\frac{R - R_{20}}{R_{20}} = \alpha_{20}(t - 20^\circ)$$

$$\Rightarrow R - R_{20} = R_{20}\alpha_{20}(t - 20^\circ)$$

$$\Rightarrow R = R_{20}(1 + \alpha_{20}(t - 20^\circ))$$

$$\Rightarrow R_{20} = \frac{R}{1 + \alpha_{20}(t - 20^\circ)}$$

or $$R_{20} = \frac{240\,\Omega}{1 + 4.5 \times 10^{-3}\,^\circ C^{-1}(200^\circ C - 20^\circ C)}$$

$$= \frac{240\,\Omega}{1 + 4.5 \times 10^{-3}(180)}$$

$$= 24\,\Omega$$

3. A, since this is the voltage drop across the resistor it is given by the product of the current and the resistance. Thus

$$\text{p.d.} = V = (10A)(4\Omega)$$
$$= 40 \text{ Volts}$$

B,

$$V = \sum R$$

$$\Rightarrow R = \frac{V}{\sum}$$

or $$R = \frac{48\,V}{6A} = 8\Omega$$

c, $I = \frac{V}{R}$

or $I = \frac{30V}{5\Omega} = 6A$

4. Since the generator is not 100% efficient the whole of the 2000 hp is not used for doing useful work. Thus the actual power that is supplied by the generator is

$$P = 2000hp \times 80\%$$
$$= 1600\ hp$$

Since $1\ hp = 746\ W$, then $1600\ hp = 1600hp \times 746\frac{W}{hp}$

i.e.) $P = 1193600W$.

But $P = IV$

$$\Rightarrow I = \frac{P}{V}$$

or $I = \frac{1193600W}{12\,00V} = 994.67A$

∴ the current delivered by the generator is 994.67A.

5.

a, In general for a material of dielectric constant k the capacitance C is given by

$$C = kC_0$$

where C_0 is the capacitance of a capacitor with no material between its plates during vacuum.

But for a parallel plate capacitor we have that

$$C = \frac{k\epsilon_0 A}{d}$$

For our case i.e.) air $k \approx 1$.

$$\Rightarrow C = \frac{\epsilon_0 A}{d}$$

or $C = \frac{8.85 \times 10^{-12} F/m \cdot \times 0.04m^2}{3 \times 10^{-3} m}$

$$= 1.19 \times 10^{-10} F$$
$$= 119\ pF \quad (p \text{ stands for pico meant } 10^{-12})$$

b) For a parallel plate capacitor

$$V = Ed$$

$$\Rightarrow E = \frac{V}{d}$$

Then $$E = \frac{200\ V}{3 \times 10^{-3}\ m} = 66.7\ k\frac{N}{C}$$

c) If the charge is Q

$$Q = CV$$

or $$Q = (119\ pF)(200\ V)$$

$$= 23.8\ nC$$

6) a) $$C = k C_0$$

Since for air we have $k \approx 1$, $C_{air} \approx C_0$.

$$C_{mica} = k_{mica} C_0 = 119\ pF$$

$$\Rightarrow C_{mica} = 5 \times (119\ pF)$$

$$= 595\ pF$$

b) Still, here, $V = Ed$

$$\Rightarrow E_{mica} = \frac{V_{mica}}{d}$$

But the insertion of a dielectric in a capacitor cannot result in an increase or decrease of charges so that

$$C = k C_0$$

$$\Rightarrow \frac{Q}{V} = k \frac{Q}{V_0}$$

From which

$$k = \frac{V_0}{V}$$

Also, $V_0 = E_0 d$, $V = Ed$

From which

$$K = \frac{V_0}{V} = \frac{E_0}{E}$$

Then $$E_{mica} = k E_0$$

or $$E_{mica} = 5\left(66.7\ k\frac{N}{C}\right)$$

$$= 333.5\ k\frac{N}{C}$$

c) the charge is the same 23.8 nc.

7. a) $Q_{air} = C_{air} V_{air}$

or $Q_{air} = 12\mu F \times 800V =$ 9600 μc

b) $C_{t\cdot oil} = K C_o = K C_{air}$

$\Rightarrow C_{t\cdot oil} = 4.0 \times 12\,\mu F = 48\,\mu F$

c) The charge is not affected so that here as well

$Q_{t\cdot oil} =$ 9600 μc

d) $V_{t\cdot oil} = \frac{V_o}{K}$ or $V_{t\cdot oil} = \frac{800V}{4} = 200V$

$V_{t\cdot oil} = \frac{Q_{t\cdot oil}}{C_{t\cdot oil}}$

or $V_{t\cdot oil} = \frac{9600\,\mu c}{48\mu F} = 200\,V$

8. Series connection

$\frac{1}{C_e} = \frac{1}{2\mu f} + \frac{1}{4\mu f} + \frac{1}{6\mu F} = \frac{11}{12\mu F}$

or $C_e = \frac{12}{11}\mu F = 1.09\mu F$

$Q_A = Q_B = Q_C = Q_T = C_e \cdot V_T$

or $Q_A = Q_B = Q_C = (1.09\mu F)(800V) = 872\,\mu c$

$V_A = \frac{Q_A}{C_A}$

or $V_A = \frac{872\mu c}{2\,\mu F} = 436\,V$

$V_B = \frac{Q_B}{C_B}$

or $V_B = \frac{872\mu c}{4\mu F} = 218\,V$

$V_C = \frac{Q_C}{C_C}$

or $V_C = \frac{872\,\mu c}{6\,\mu F} = 145.3\,V$

Parallel connection

- $C_e = 2\mu F + 4\mu F + 6\mu F = 12\mu F$
- $V_A = V_B = V_C = V_T = 800V$
- $Q_A = C_A V_A = 2\mu F \times 800V = 1600mV$

  $Q_B = C_B V_B = 4\mu F \times 800V = 3200mV$

  $Q_C = C_C V_C = 6\mu F \times 800V = 4800mV$

9. The work done to charge the capacitor is equal to the energy stored in the capacitor. Then

$$\text{Work} = \frac{1}{2}CV^2 = \frac{1}{2}\frac{Q^2}{C}$$

$$\therefore \text{Work} = \frac{1}{2}\left[\frac{(1.6\times10^{-3}C)^2}{30\times10^{-6}F}\right]$$

$$= \frac{1}{2}\cdot\frac{2.56\times10^{-6}}{3\times10^{-5}}J$$

$$= \frac{2.56}{6}\times10^{-1}J$$

$$= 0.427J$$

10. If we have a multiple-plate capacitor of N plates separated by air, the capacitance is given by

$$C_0 = \frac{(N-1)\epsilon_0 A}{d}$$

where A is the area of each plate
d is the separation of the plates.

Thus for our case,

$N = 30$, $A = 0.2m \times 0.2m = 0.04m^2$, $d = 2\times10^{-3}m$

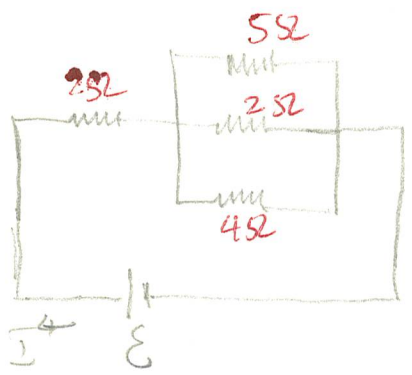
$$\Rightarrow C_0 = \frac{(29\times8.85\times10^{-12}F/m)(0.04m^2)}{2\times10^{-3}m}$$

$$= 5.13nF$$

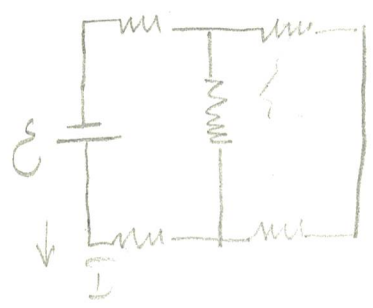
Work Sheet II

DC - circuits

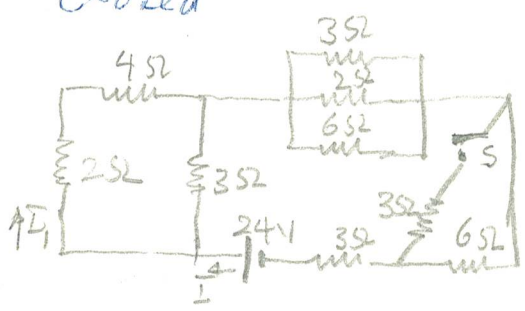
1. a, find the equivalent resistance for the circuit of the figure below. b, Find I if the value of $\mathcal{E}$ is 12.0 V.



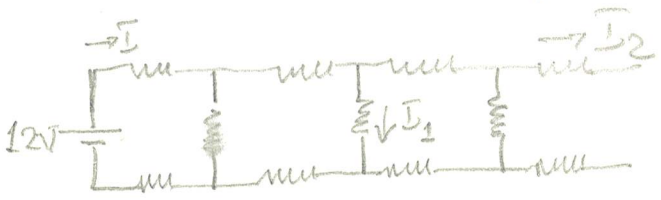
2. In the fig. below, the value of $\mathcal{E}$ is 12.0 V, and each resistor is 2.0 Ω. Find the equivalent resistance and I.



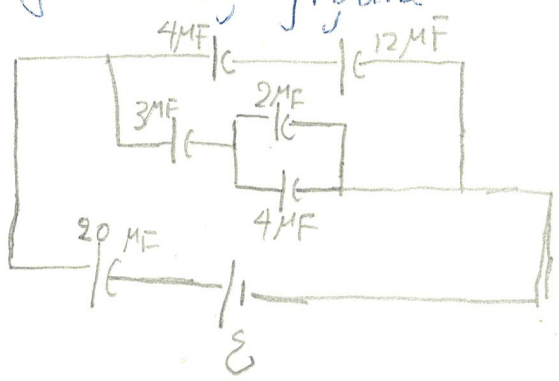
3. Find the equivalent resistance for the circuit shown below when a, the switch S is kept open and b, S is closed



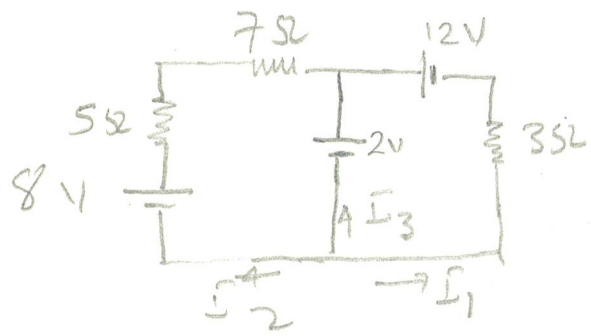
4. Find the equivalent capacitance resistance for the circuit shown in the figure below if all the resistors are 4 Ω.



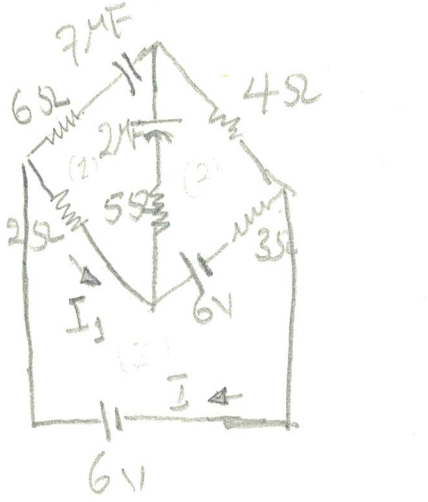
6- 5. Find the equivalent capacitance for the combination shown in the following figure.



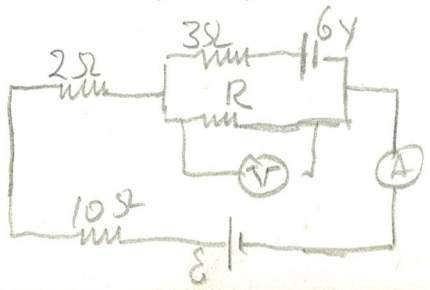
6. Find I_1, I_2 & I_3 in the circuit of the figure below.



9- 7. In the circuit of the figure below, find I, I_1 and the charge on the 7 μF capacitor. Hint: Note that the currents in many of the wires are zero.



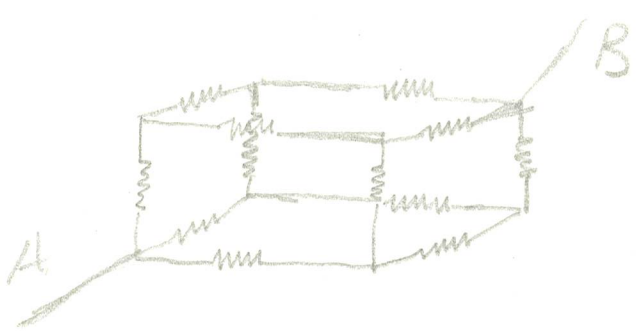
11- 9. Refer to the circuit of the figure below. The voltmeter reads 5.0V, and the ammeter reads 2.0A with the current flowing in the direction indicated. Find a) the value of R and b) the value of ε. (Notice in writing the circuit eq. for a) that the voltage drop across R is 5 V).



10. In the above circuit, how large must $\mathcal{E}$ be if the current through the 6V battery is to be zero and R is 12Ω?

11. In the same circuit shown in the figure above, what would a, the ammeter and b, the voltmeter read if $\mathcal{E}$ was 20 V and R was 6Ω?

12. Each of the resistors shown in the figure below has a value of 2Ω. Find the equivalent resistance between A & B. Hint: From symmetry, many of the wires contain identical currents.



Magnetostatics

1. A power line parallel to the earth's surface carries a current of 20 A straight west. At that point, the earth's magnetic field on a 15 m length of the wire is 0.80 G parallel to the earth's surface and directed straight north. a, Find the force due to the field on a 15 m length of the wire is, what is the direction.

2. In Miami, Florida, the earth's magnetic field is approximately straight north but is directed at an angle of 57° below the horizontal. (The dip angle is 57°). The magnitude is about 0.51 G. Find a, the magnitude and b, the direction of the force it causes on a 15 m length of wire carrying a current of 20 A straight upward.

3. A long straight wire carries a current of 12 A straight west along the floor in a house. a, How large is B 2m above the wire? b, What is the direction of the force?

5. Two parallel long straight wires 20 cm apart carry currents of 20 and 10 A in opposite directions. a, How large is B resulting from the 10 A wire at the position of the other wire b, Find the magnitude and direction of the force on a 1 m length of the 20 A wire. c, Repeat part b, for the other wire.

6. A constant horizontal field of flux density 0.5 T pierces a rectangular loop 10 cm by 5 cm. Determine the magnetic flux through the loop when its plane makes the following angles with the B field: a, 0°, b, 30°, c, 60°, d, 90°.

7. A coil of wire 20 cm in diameter is situated so that its plane is perpendicular to a field of flux density 0.3 T. Determine the magnetic flux through the coil.

8. A proton is projected from right to left into a B field of 0.4 T directed vertically upward. If the velocity of the proton is 2×10^6 m/s, what is the magnetic force on the proton.

9. If 40 cm of a straight wire is at an angle of 53° with a B field of 0.25 T, what current is required to give a force of 2 N on this length of wire?

10. Two parallel wires carrying currents I_1 and I_2 are separated by a distance d. Show that the force per unit length F/l each wire exerts on the other is given by

$$F/l = \frac{\mu I_1 I_2}{2\pi d}$$

11. Two wires lying in a horizontal plane carry parallel currents of 15 A and are 20 cm apart in air. If both currents are directed to the right, what are the magnitude and direction of the flux density at a point midway between the wires? What is the force per unit length that each wire exerts on the other? Is it attraction or repulsion?

Induction

1. The magnetic flux linking a loop of wire changes from 5 to 2 mWb in 0.1 s. What is the average induced emf?

2. A bar magnet can be moved in or out of a 200 turn coil into which it fits snugly. It is found that an average emf of 0.30 V is induced in the coil when the magnet is suddenly brought up and inserted in it in 0.20 s. If the cross-sectional area of the magnet is 2.0 cm^2, find the value of B in it.

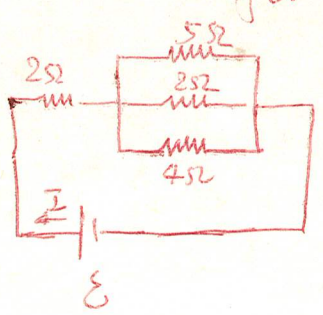
3. The flux through a coil having 200 turns changes from 0.06 to 0.03 Wb in 1 s. The coil is connected to an electric light, and the combined resistance is 2 Ω. a) Calculate the average induced emf. b) What is the average current delivered to the light filament.

4. A coil of area 0.2 m^2 has 80 turns of wire and is suspended with its plane perpendicular to a uniform magnetic field. What must the flux density of the field be for an average emf of -2 V to be induced when the coil is flipped parallel to the field in 0.5 s.

Alternating current

Solution to worksheet 8
for phys. 112

1. a)



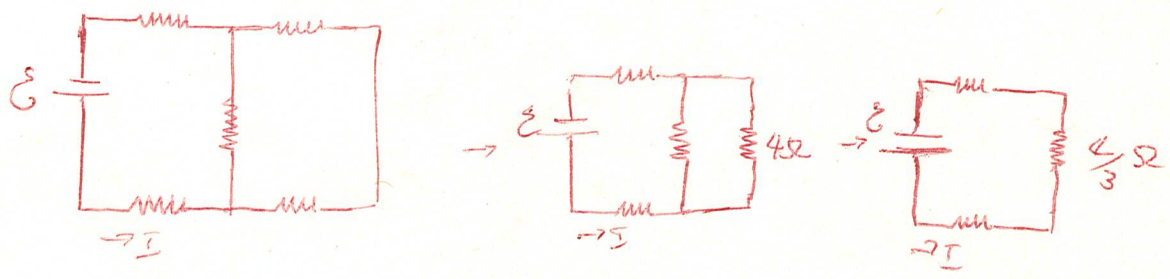
$$\frac{1}{R_{5,2,4}} = \frac{1}{5\Omega} + \frac{1}{2\Omega} + \frac{1}{4\Omega} = \frac{4+10+5}{20\Omega} = \frac{19}{20\Omega}$$

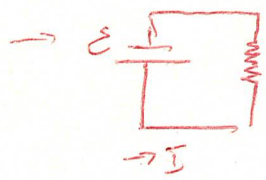
$$\Rightarrow R_{5,2,4} = \frac{20}{19}\Omega$$

$$\therefore R_{eq} = 2\Omega + \frac{20}{19}\Omega = \frac{58}{19}\Omega \approx 3\Omega$$

b) $$I = \mathcal{E}/R_{eq} = \frac{12.0\,V}{3\Omega} = 4A$$

2.

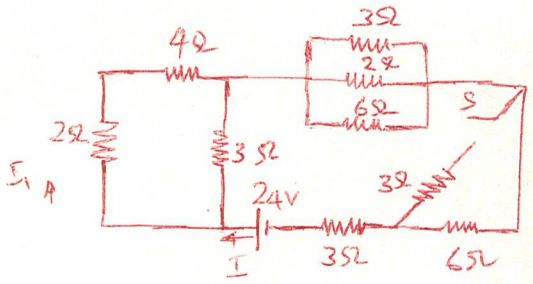




, (i.e.) $R_e = \frac{16}{3}\Omega \approx 5.3\Omega$

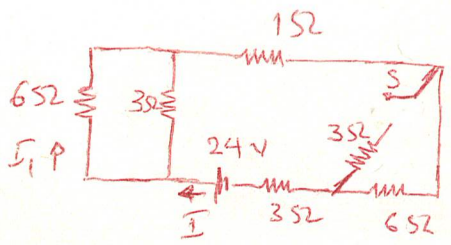
$$I = \mathcal{E}/R_e = \frac{12.0\,V}{(16/3)\Omega} = \frac{9}{4}A = 2.25\,A$$

3.



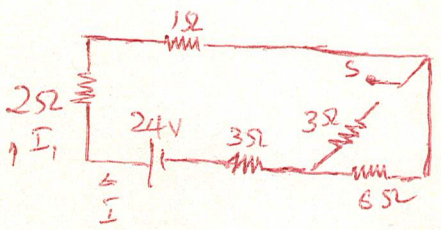
$$\frac{1}{R_{3,2,6}} = \frac{1}{3\Omega} + \frac{1}{2\Omega} + \frac{1}{6\Omega} = \frac{6}{6\Omega} = \frac{1}{1\Omega}$$

$$\Rightarrow R_{3,2,6} = 1\Omega$$



$$\frac{1}{R_{6,3}} = \frac{1}{6\Omega} + \frac{1}{3\Omega} = \frac{3}{6\Omega} = \frac{1}{2\Omega}$$

$$\Rightarrow R_{6,3} = 2\Omega$$



a) When the switch is open no current passes through the branch that consists of S. So we neglect the resistance in this branch.

$$\therefore R_e = 2\Omega + 1\Omega + 6\Omega + 3\Omega = 12\Omega$$

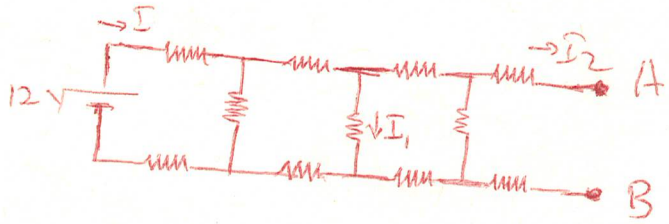
6) When the switch is closed we consider the above resistance and now it is parallel to the 6Ω resistance

$$\frac{1}{R_{3,6}} = \frac{1}{3\Omega} + \frac{1}{6\Omega} = \frac{3}{6\Omega} = \frac{1}{2\Omega}$$

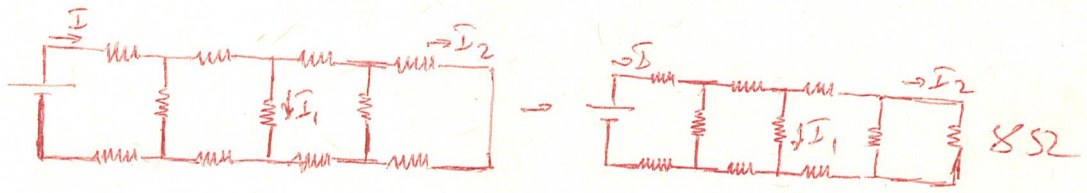
$$\Rightarrow R_{3,6} = 2\Omega$$

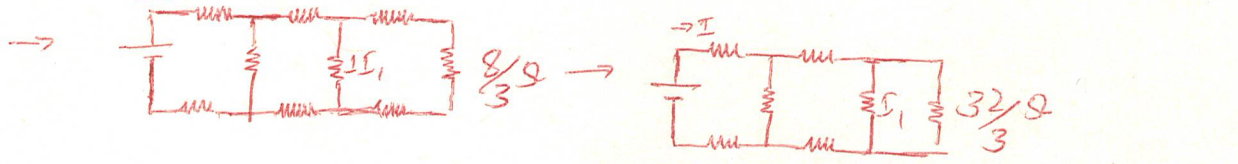
$$\therefore R_e = 2\Omega + 1\Omega + 2\Omega + 3\Omega = 8\Omega$$

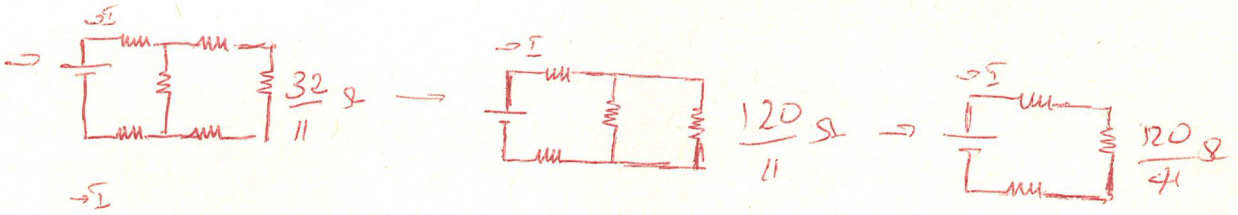
4.

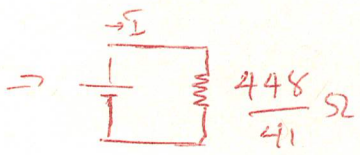


Let the potential at A be V_A and that at B V_B. These two points are actually at the same potential since we have equivalent potential drops in the resistors before. across the points. Thus we can connect the two points by a line i.e., by a resistanceless wire in the circuit. We redraw the circuit diagram then as shown below.

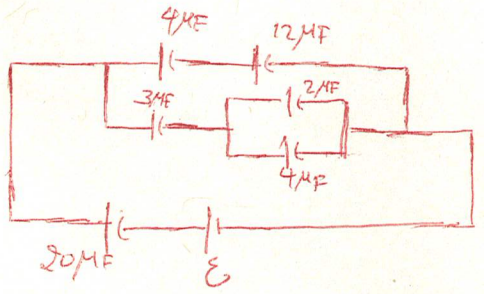








5.



$C_{2,4} = 2MF + 4MF = 6MF$

$$\frac{1}{C_{4,12}} = \frac{1}{4MF} + \frac{1}{12MF} = \frac{4}{12MF} = \frac{1}{3MF}$$

$$C_{4,12} = 3MF$$

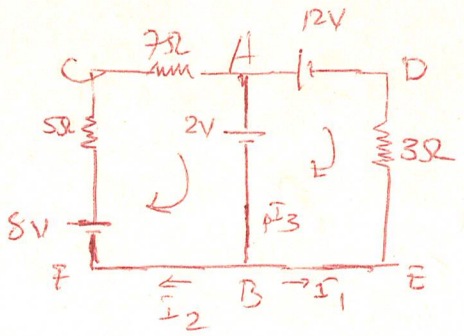
$$\frac{1}{C_{3,6}} = \frac{1}{3MF} + \frac{1}{6MF} = \frac{3}{6MF} = \frac{1}{2MF}$$

$$C_{3,6} = 2MF$$

$C_{4,12} \parallel C_{3,6} \Rightarrow 3MF + 2MF = 5MF$

Now, $\frac{1}{C_e} = \frac{1}{20MF} + \frac{1}{5MF} = \frac{5}{20MF} = \frac{1}{4MF}$ i.e., $C_e = 4MF$

6.



At junction B, $I_1 + I_2 + I_3 = 0$ ---(1)

Loop ABFCA, $-2V + 8V - 5\Omega I_2 - 7\Omega I_2 = 0$ ---(2)

Loop ABEBA, $-12V + 3I_1 + 2V = 0$ ---(3)

From eq. (2), $-2V + 8V - 5\Omega I_2 - 7\Omega I_2 = 0$

$6V - 12\Omega I_2 = 0$

(i.e.) $I_2 = 0.5\,A$

From eq. (3) $-12V + 3\Omega I_1 + 2V = 0$

$-10V + 3\Omega I_1 = 0$

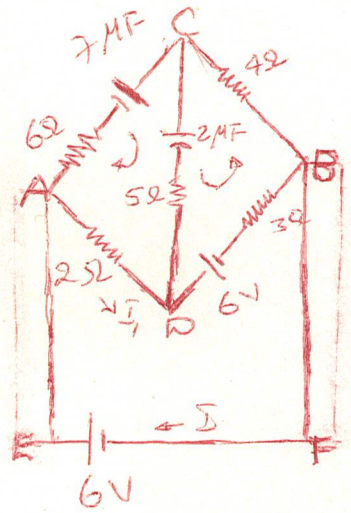
i.e.) $I_1 = 10/3\,A$

From (1), $I_1 + I_2 + I_3 = 0$

$\Rightarrow I_3 = -(I_1 + I_2)$

$\therefore I_3 = -(0.5A + 10/3\,A) = -\frac{23}{6}A \approx -4A$

7.



No current passes through the branches AC & CD since these branches consist of capacitors

(i.e.) $I = I_1$.

Loop ACDA : $-\frac{Q_7}{7\mu F} + \frac{Q_2}{2\mu F} + 2\Omega I_1 = 0$

Loop BCDB: $+\frac{Q_2}{2\mu F} - 6V - 3\Omega I_1 = 0$

Loop ADBFEA : $-2\Omega I_1 - 6V - 3\Omega I_1 + 6V = 0$

(i.e.) $I_1 = 0$

Then as calculated above, $I = I_1 = 0$.

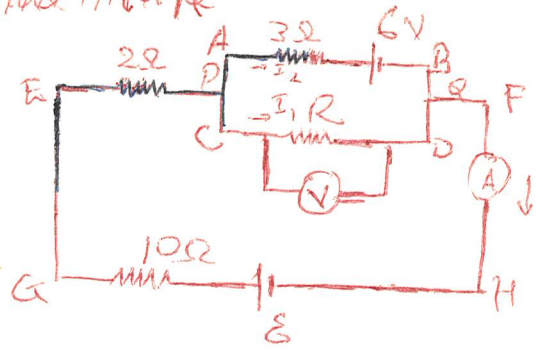
Now, $-\frac{Q_7}{7\mu F} + \frac{Q_2}{2\mu F} = 0$

$+\frac{Q_2}{2\mu F} = 6V$

Subtracting we get, $-\frac{Q_7}{7\mu F} = -6V$

(i.e.) $Q_7 = +42\,\mu C$

9. Although not labeled



Loop EPCQEFHG: $-2\Omega \cdot 2A - I_1R + \varepsilon - 10\Omega \cdot 2A = 0$

$-4V - I_1R + \varepsilon - 20V = 0$

$-I_1R + \varepsilon - 24V = 0$

Since $I_1R = 5V$

$-5V + \varepsilon - 24V = 0$

i.e., $\varepsilon = 29V$

Loop ABQDCPA: $-3\Omega I_2 - 6V + I_1R = 0$

$-3\Omega I_2 - 6V + 5V = 0$

i.e., $I_2 = -\frac{1}{3}A$

But at junction P, $I = I_1 + I_2$

i.e., $I_1 = I - I_2 = 2A - (-\frac{1}{3}A) = \frac{7}{3}A$

Now $5V = I_1R = \frac{7}{3}A \cdot R$

$\therefore R = \frac{5V}{\frac{7}{3}A} = \frac{15}{7}\Omega$

10. From $I = I_1 + I_2$, if $I_2 = 0$, then $I = I_1$.

From 9 we have that

Also, $\begin{cases} -I_1R + \varepsilon - 24V = 0 \\ -3\Omega I_2 - 6V + I_1R = 0 \end{cases}$

Adding gives $\varepsilon - 30V = 0$

i.e., $\varepsilon = 30V$

11. Once again, $\begin{cases} -2\Omega I - 6\Omega I_1 + 20V - 10\Omega I = 0 \\ -3\Omega I_2 - 6V + 6\Omega I_1 = 0 \end{cases}$

But $I_2 = I - I_1$

$$\Rightarrow \begin{cases} -2\Omega I - 6\Omega I_1 + 20V - 10\Omega I = 0 \\ -3\Omega(I - I_1) - 6V + 6\Omega I_1 = 0 \end{cases}$$

$$4 \begin{cases} -12\Omega I - 6\Omega I_1 + 20V = 0 \\ -3\Omega I_1 + 9\Omega I_1 - 6V = 0 \end{cases}$$

$$- \begin{cases} -12\Omega I - 6\Omega I_1 + 20V = 0 \\ -12\Omega I + 36\Omega I_1 - 24V = 0 \end{cases}$$

$$-42\Omega I_1 + 44V = 0$$

$$\Rightarrow I_1 = \frac{44}{42} A = 1.05\,A$$

$$-3\Omega I + 9\Omega \cdot \frac{44}{42} A - 6V = 0$$

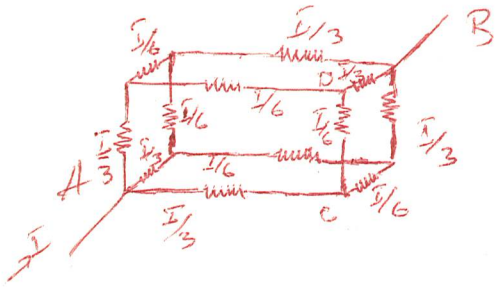
$$-3\Omega I + \frac{66}{7} V - 6V = 0$$

$$\text{i.e.)} \quad I = \frac{24}{7} \cdot \frac{1}{3} A = \frac{8}{7} A = 1.14A$$

a, the ammeter A would reads $\frac{8}{7}$ A

b, The voltmeter would read, $V = \frac{44}{42} A \cdot 6\Omega = \frac{44}{7} V = 6.28V$

12.



One can take any of the paths from A to B since in each case the total potential drop is the same.

Now take the path A C D B. The potential drop is

$$R_e \cdot I = \frac{I}{3} R + \frac{I}{6} R + \frac{I}{3} R \text{, where } R = 2.0\Omega$$

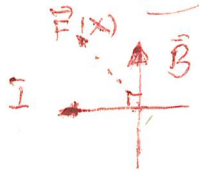
$$\Rightarrow R_e I = \frac{5}{6} IR$$

$$\text{i.e.)} \quad R_e = \frac{5}{6} R$$

$$\therefore R_e = \frac{5}{6} \cdot 2\Omega = \frac{5}{3}\Omega$$

Magnetostatics

1.



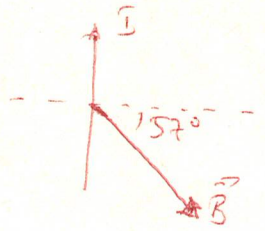
a, $F = I l B \sin\theta$

$\therefore F = 20A \times 15M \times 0.8 \times 10^{-4} \frac{N/C}{M^2} \sin 90^\circ, \theta = 90^\circ$

$= 2.4 \times 10^{-2} N$

b) From the right hand rule the force is directed to the center of (or into) the earth.

2.



a) $F = IlB\sin(90^\circ + 57^\circ)$

$= 20A \times 15m \times 0.5 \times 10^{-4} \frac{Wb}{m^2} \cdot \cos 57^\circ$

$= 0.0091\ N$

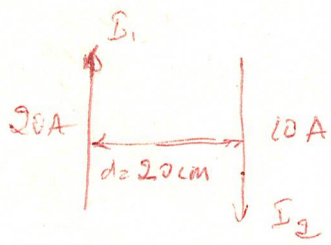
b) From the right hand rule it is directed to the west.

3.

a) $B = \frac{\mu_0 I}{2\pi r} = \frac{4\pi \times 10^{-7} \frac{Wb}{A\cdot m} \cdot 12A}{2\pi \cdot 2m} = 1.2 \times 10^{-6} \frac{Wb}{m^2}$

b) the $\vec{B}$-field circles the wire

4.



a) $B_2 = \frac{4\pi \times 10^{-7} \frac{Wb}{A\cdot m} \times 10A}{2\pi \times 0.2m} = 10^{-5} \frac{Wb}{m^2}$

b) $F_2 = I_1 l B_2 \sin 90^\circ$

$= 20A \times 1m \times 10^{-5} \frac{Wb}{m^2} = 2 \times 10^{-4}\ N$

(attractive force i.e.) to the right)

c) $B_1 = \frac{\mu_0 I_1}{2\pi d}$, $F_1 = I_2 l_2 B_1 \sin 90^\circ = I_2 l_2 \frac{\mu_0 I_1}{2\pi d} = \frac{\mu_0 I_1 I_2 l_2}{2\pi d}$

$F_1 = \frac{4\pi \times 10^{-7} \frac{Wb}{A\cdot m} \cdot 20A \cdot 10A \cdot 1m}{2\pi \times 0.2m} = 2 \times 10^{-4}\ N$, attractive (i.e.) to the left

5.

$\Phi = BA\sin\theta$

a) $\theta = 0 \Rightarrow \Phi = 0$

b) $\theta = 30^\circ \Rightarrow \phi = 0.5T \times 5 \cdot 10^{-3} m^2 \times \frac{1}{2} = 1.25 \times 10^{-3}\ Wb$

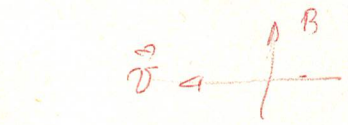
c) $\theta = 60^\circ \Rightarrow \phi = 0.5T \times 5 \times 10^{-3} m^2 \times 0.866 = 2.17 \times 10^{-3}\ Wb$

d) $\theta = 90^\circ \Rightarrow \phi = 0.5T \times 5 \times 10^{-3} m^2 \times 1 = 2.5 \times 10^{-3} \frac{Wb}{m^2}$

6. $A = \pi r^2 = \pi (0.1m)^2 = 0.01\pi\ m^2$

$\phi = B_1 A = 0.3T\ (0.01\pi\ m^2) = 0.003\pi\ Wb$

7.

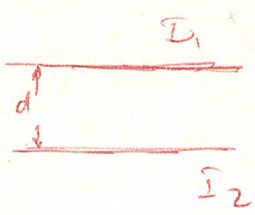


$F = qv_1 B = 1.6 \times 10^{-19} C \times 2 \cdot 10^6 m/s \times 0.4T$

$= 1.28 \times 10^{-13}\ N$, into the page

8. $F = IlB\sin\theta \Rightarrow I = \frac{F}{lB\sin\theta}$, i.e.) $I = \frac{2N}{0.4m \times 0.25T \times 0.8} = \frac{2}{0.08} A = 25A$

9.



B_1 (due to I_1 at the site of I_2) $= \frac{\mu_0 I_1}{2\pi d}$

$\Rightarrow F_1 = I_2 l B_1$ (for a length l of the wire carrying I_2)

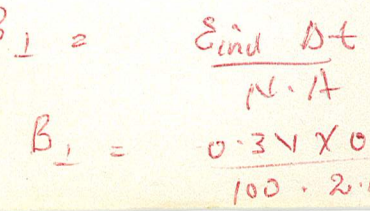
$= \frac{I_2 l \mu_0 I_1}{2\pi d} = \frac{\mu_0 I_1 I_2 l}{2\pi d}$

Similarly, $F_2 = \frac{\mu_0 I_1 I_2 l}{2\pi d}$

i.e.) $F_1 = F_2 = F = \frac{\mu_0 I_1 I_2 l}{2\pi d}$

$\therefore F/l = \frac{\mu_0 I_1 I_2}{2\pi d}$, for a medium of permeability μ_0

10.



$B_{1p} = \frac{\mu_0 I_1}{2\pi d/2} = \frac{\mu_0 I_1}{\pi d}$

$B_{2p} = \frac{\mu_0 I_2}{2\pi d/2} = \frac{\mu_0 I_2}{\pi d}$

But from the right hand rule the two fields are directed oppositely giving

$B = B_{1p} - B_{2p} = \frac{\mu_0 I_1}{\pi d} - \frac{\mu_0 I_2}{\pi d} = \frac{\mu_0}{\pi d}(I_1 - I_2) = \frac{\mu_0}{\pi d} \cdot 0 = 0$

AD calculated in (9)

$F/l = \frac{\mu_0 I_1 I_2}{2\pi d} = \frac{4\pi \times 10^{-7} \frac{Wb}{A \cdot m} \cdot 15A \cdot 15A}{2\pi \times 0.2m} = 2.25 \times 10^{-4} N$

Induction

1. $\mathcal{E}_{ind} = -N \frac{\Delta\phi}{\Delta t}$

Since $N = 1$, $\mathcal{E}_{ind} = -1 \frac{2mWb - 5mWb}{0.1 sec.} = 3 \times 10^{-9} V = 30 mV.$

2. $\mathcal{E}_{ind} = -N \frac{\Delta\phi}{\Delta t}$, $\phi = B_\perp A$

$\Delta\phi = \phi_f - \phi_i = 0 - B_\perp A \cos 180 = -B_\perp A$

$\Rightarrow \mathcal{E}_{ind} = -N(-B_\perp A) = \frac{N B_\perp A}{\Delta t}$

$\Rightarrow B_\perp = \frac{\mathcal{E}_{ind} \Delta t}{N \cdot A}$

$\therefore B_\perp = \frac{0.3V \times 0.1s}{100 \cdot 2 \cdot 10^{-4} m^2} = 1.5 \frac{Wb}{m^2}$

3. a) $\mathcal{E}_{ind} = -N\frac{\Delta\Phi}{\Delta t} = -200\left(\frac{0.03\,Wb - 0.06\,Wb}{1s}\right) = 6V$

b) $I = \frac{\mathcal{E}_{ind}}{R} = \frac{6V}{2\Omega} = 3A$

4.

$$\mathcal{E}_{ind} = -N\frac{\Delta\Phi}{\Delta t}$$

$$\Delta\Phi = \Phi_f - \Phi_i = B_1 A - 0 = B_1 A$$

$$\Rightarrow \mathcal{E}_{ind} = -N\frac{B_1 A}{\Delta t}$$

(-e.) $B_1 = -\frac{\mathcal{E}_{ind}\,\Delta t}{N \cdot A}$

$$\therefore B_1 = -\left(\frac{-2V \times 0.5s}{80 \times 0.2m^2}\right) = 62.5\,mT$$

Alternating currents

At junction W,

$$I_2 = I_1 + 5A$$

Loop ADCWA

$$-5A \cdot 2\Omega - I_2 \cdot 7\Omega + 16V = 0$$

$$-10V - 7I_2\Omega + 16V = 0$$

$$6V - 7I_2\Omega = 0$$

$$\Rightarrow I_2 = 6/7\ A$$

$$\therefore I_1 = \frac{6}{7}A - 5A = -\frac{29}{7}A$$

Loop BCDB

$$+\mathcal{E} - 7\Omega \cdot I_2 - 3\Omega I_1 = 0$$

$$+\mathcal{E} - 7\Omega \cdot \frac{6}{7}A - 3\Omega\left(-\frac{29}{7}A\right) = 0$$

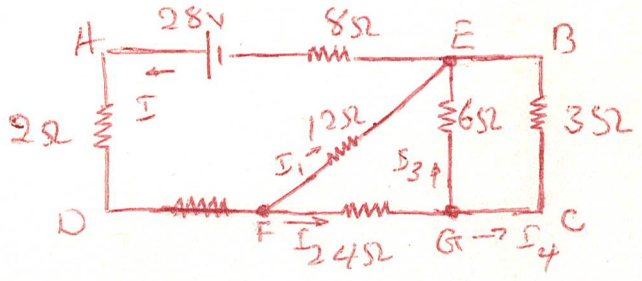
$$+\mathcal{E} - 6V + \frac{87}{7}V = 0$$

$$+\mathcal{E} + \frac{45}{7}V = 0$$

$$\mathcal{E} = -\frac{45}{7}V$$

The negative sign tells that I_1 & $\mathcal{E}$ be reversed in direction

12,



a, At junction F, $I = I_1 + I_2$

At junction G $I_2 = I_3 + I_4$

$\Rightarrow I = I_1 + I_3 + I_4$ --(1)

Loop AEFDA,

$$-28 + 8I + 12I_1 + 2I = 0 : -28 + 10I + 12I_1 = 0 \quad \text{--(2)}$$

Loop EFGE

$$12I_1 - 4I_2 - 6I_3 = 0 \quad \text{--------- (3)}$$

Loop AEGDA

$$-28 + 8I + 6I_3 + 4I_2 + 2I = 0 : -28 + 10I + 4I_2 + 6I_3 = 0 \quad \text{--(4)}$$

Loop BCGEB

$$3I_4 - 6I_3 = 0 \quad \text{- - - - - - - - - - (5)}$$

$\Rightarrow I_4 = \frac{3}{2} I_3$

From $I_2 = I_3 + I_4$ we obtain

$I_2 = I_3 + \frac{3}{2} I_3 = \frac{5}{2} I_3$

i.e., $I_3 = \frac{2}{5} I_2$ (2)

Take (2) & (3)

$$\begin{cases} -28 + 10I + 12I_1 = 0 \\ 12I_1 - 4I_2 - 6I_3 = 0 \end{cases}$$

$I = I_1 + I_2, \quad I_3 = \frac{2}{5} I_2$

$$\Rightarrow \begin{cases} -28 + 10(I_1 + I_2) + 12I_1 = 0 \\ 12I_1 - 4I_2 - \frac{12}{5} I_2 = 0 \end{cases}$$

$32/5$ $\quad -28 + 22I_1 + 10I_2 = 0$

10 $\quad 12I_1 - \frac{32}{5} I_2 = 0$

$$+ \begin{cases} -179.2 + 140.8 I_1 + 64 I_2 = 0 \\ 120 I_1 - 64 I_2 = 0 \end{cases}$$

$$-179.2 + 260.8 I_1 = 0$$

$$\therefore I_1 = \frac{179.2}{260.8} A = 0.69 A$$

i.e., the current through the 12Ω resistor is 0.69 A

b)

$$12 \times 0.69 A - \frac{32}{5} I_2 = 0$$

$$\Rightarrow I_2 = \frac{12 \times 0.69}{32/5} A = 1.29 A$$

$$I = I_1 + I_2 = 0.69 A + 1.29 A = 1.98 A$$

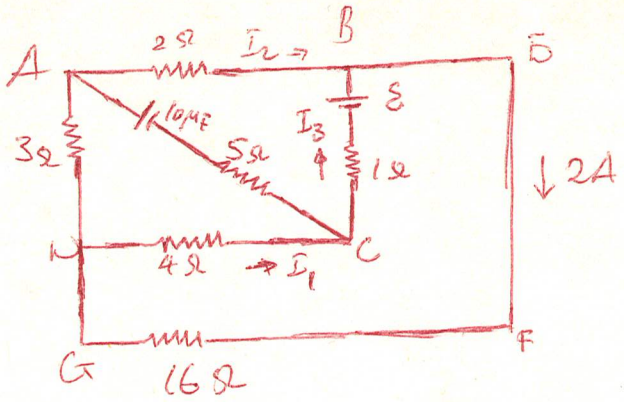
i.e., the current through the battery is 1.98 A

b) The current through the 12Ω resistor as calculated above is 0.69 A

c) the power loss in the 8Ω resistors

$$P = I^2 R = (1.98 A)^2 8\Omega = 31.36 W$$

13.



At junction B, $I_2 + I_3 = 2A$ ---- (1)

Loop ABCDA,

$$-2I_2 - \mathcal{E} + I_3 + 4I_1 - 3I_2 = 0 \quad \text{---- (2)}$$

Loop ABCA,

$$-2I_2 - \mathcal{E} + I_3 + \frac{Q}{10} = 0 \quad \text{---- (3)}$$

Loop ACDA

$$-\frac{Q}{10} + 4I_1 - 3I_2 = 0 \quad \text{---- (4)}$$

Loop AEFGA,

$$-2I_2 - 32 - 3I_2 = 0 \quad \text{---- (5)}$$

$$\Rightarrow -5I_2 - 32 = 0$$

$$\therefore I_2 = -\frac{32}{5}A = -6.4A$$

$$I_2 + I_3 = 2A$$

$$\therefore I_3 = 2A - I_2 = 2A - \left(-\frac{32}{5}A\right) = +\frac{42}{5}A = 18.4A$$

Loop DCBEFGD,

$$-4I_1 - I_3 + \mathcal{E} - 32 = 0 \quad \text{---- (6)}$$

Take (2) and (6)

$$-\begin{cases} -2I_2 - \mathcal{E} + I_3 + 4I_1 - 3I_2 = 0 \\ -4I_1 - I_3 + \mathcal{E} - 32 = 0 \end{cases}$$

$$-5I_2 + 8I_1 + 2I_3 + 2\mathcal{E} - 32 = 0$$

$$-5\left(-\frac{32}{5}\right) + 8I_1 + 2\frac{42}{5} - 2\mathcal{E} - 32 = 0$$

$$32 + 8I_1 + 16.8 - 2\mathcal{E} - 32 = 0$$

$$\Rightarrow \mathcal{E} = 4I_1 + 8.4$$

Take (3) and (4)

$$\begin{cases} -2I_2 - \mathcal{E} + I_3 + \frac{Q}{10} = 0 \\ -\frac{Q}{10} + 4I_1 - 3I_2 = 0 \end{cases}$$

$$-5I_2 - \mathcal{E} + 4I_1 + I_3 = 0$$

Loop A C B E F G A

$$-\frac{Q}{10} - I_3 + \mathcal{E} - 32 - 3I_2 = 0 \quad \cdots (7)$$

Take (3) and (7)

$$\begin{cases} -2I_2 - \mathcal{E} + I_3 + \frac{Q}{10} = 0 \\ -\frac{Q}{10} - I_3 + \mathcal{E} - 32 - 3I_2 = 0 \end{cases}$$

$$-5I_2 - 32 = 0$$

So no loop equation can give solution to I_1 or $\mathcal{E}$ or Q. Rather, since the current through the portion AC is zero, $I_1 = I_3$ (-e.)

$$I_1 = \frac{42}{5}A = 8.4A$$

from Eq. (4)

$$-\frac{Q}{10} + 4I_1 - 3I_2 = 0$$

$$\Rightarrow -\frac{Q}{10} + 4\left(\frac{42}{5}\right) - 3\left(\frac{-32}{5}\right) = 0$$

$$-\frac{Q}{10\mu F} + \frac{168}{5} + \frac{96}{5} = \frac{264}{5}V$$

$$\text{or} \quad Q = 10\mu F \cdot \frac{264}{5} = 528\mu C$$

Then from eq. (3)

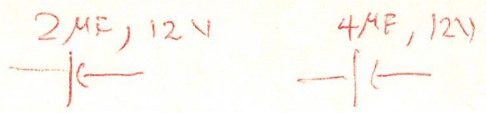
$$-2I_2 - \mathcal{E} + I_3 + \frac{Q}{10} = 0$$

$$\Rightarrow \mathcal{E} = -2I_2 + I_3 + \frac{Q}{10}$$

$$\therefore \mathcal{E} = \left[-2\left(\frac{-32}{5}\right) + \frac{42}{5} + 52.8\right]V$$

$$= \left(\frac{106}{5} + 52.8\right)V = 74V$$

14.



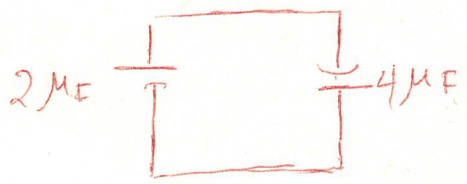
a) This can be treated as a parallel connection since here the positive plate of one is connected to the positive plate of the other and the same for the negative plates.

$\therefore \quad V_2 = V_4 = V_{2,4} = 12\,V$

b) $Q_2 = V_2 C_2 = 12V \cdot 2\mu F = 24\mu C$

$Q_4 = V_4 C_4 = 12V \cdot 4\mu F = 48\mu C$

15.



This is the case of series connection

$V_{2,4} = V_2 + V_4 = 24\,V$ i.e.) the sum of the initial potentials

$\frac{1}{C_{tot}} = \frac{1}{2\mu F} + \frac{1}{4\mu F} = \frac{3}{4\mu F}$

i.e.) $C_{tot} = 4/3\ \mu F$

$Q_{tot} = C_{tot} \cdot V_{2,4} = \frac{4}{3}\mu F \times 24V = 32\mu C$

$Q_2 = Q_4 = Q_{tot} = 32\mu C$

a) $V_2 = \frac{Q_2}{C_2} = \frac{32\mu C}{2\mu F} = 16\,V$

$V_4 = \frac{Q_4}{C_4} = \frac{32\mu C}{4\mu F} = 8\,V$

b) $Q_2 = Q_4 = 32\mu C$

Work Sheet IV

For phys. 102

Optics

1. How far will a beam of light travel in water in the time it takes to travel 1 cm through air?
2. At what angle must a fish look to see an insect sitting on the shore at the water's edge? Assume the fish to be below the surface of a still lake
3. a, Where must the object be placed if the image formed by a convex mirror is to be half as far from the mirror as the object? b, How large is the magnification?
4. A virtual image is formed by a convex mirror having a 10 cm focal length. a, Where must the object be placed if the image is to be half the size of the object? b, Is it possible to obtain a virtual image larger than the object for this type of object?
5. A concave mirror of radius 800 mm is placed from a plane mirror which faces it. A source of light placed midway between the mirrors is shielded so that light is reflected from the concave surface. What are the position and magnification of the image formed after reflection from the mirror?
6. An object is placed 40 cm in front of a converging lens, $f = 20$ cm, which in turn is 50 cm in front of a plane mirror. Find all the images formed by this system.
7. An 8 cm focal length diverging lens is placed 16 cm to the left of a concave spherical mirror having a radius of 20 cm. If an object is placed 8 cm to the left of the lens, find all the images formed by this system?

8. In a double-slit experiment the slit separation is 0.20 cm, and the slit-to-screen distance is 100 cm. Calling the position of the central bright fringe zero, locate the positions of the first three maxima on both sides of the central maximum. The wavelength is 5000 Å

9. When sodium yellow light ($\lambda = 589$ nm) is used in a double-slit experiment, the first order maximum is 0.03 cm from the central maximum. When the light is replaced by a source of an unknown wavelength, the second maximum occurs at 0.04 cm a) What is the wavelength of this latter light? b) What region of the spectrum is it in?

10. A certain diffraction grating has 5000 lines per centimeter. At what angle does the second-order spectrum of the sodium yellow line ($\lambda = 5890$ Å) occur?

11. A certain grating has 4000 lines per centimeter. What is the angular separation between the blue (4358 Å) and green (5461 Å) mercury lines in a) the first order spectrum? b) the second order spectrum?

12. A parallel beam of light illuminates a diffraction grating with 6000 lines/cm. The second-order image is located 32 cm from the central image on a screen 500 cm from the grating. Calculate the wavelength of the light.

Solutions to problems for Worksheet IV of phyp. 102

1. Let the time interval required in both cases be t. Now,

$$1\text{ cm} = ct \quad \text{– for air}$$
$$d = v_w t \quad \text{– for water}$$

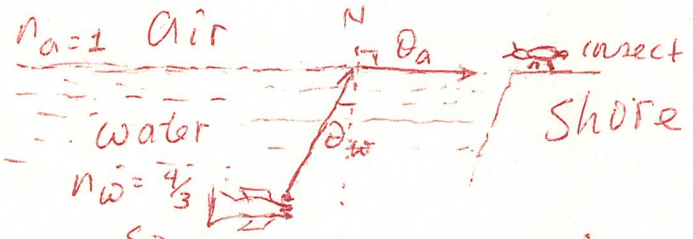
But $v_w = c/n_w$

$$\Rightarrow d = \frac{c}{n_w} t = \frac{ct}{n_w} = \frac{1\text{ cm}}{n_w}$$

whence (since $n_w = 4/3$)

$$d = \frac{1\text{ cm}}{4/3} = \frac{3}{4}\text{ cm} = 0.75\text{ cm}$$

2.



So the law of refraction is to be applied.

$$n_w \sin\theta_w = n_a \sin\theta_a$$
$$\Rightarrow \sin\theta_w = \frac{n_a}{n_w}\sin\theta_a$$
$$\text{i.e.,}\quad \theta_w = \sin^{-1}\left(\frac{n_a}{n_w}\sin\theta_a\right)$$
$$\therefore \theta_w = \sin^{-1}\left(\frac{1}{4/3}\sin 90^\circ\right) = \sin^{-1}(3/4) = 48.6^\circ$$

Therefore the fish must look to see the insect at the critical angle of water.

3. a) $q = -p/2$ (for a convex mirror image distance is negative)
From the mirror equation

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{f}$$
$$\Rightarrow \frac{1}{p} + \frac{1}{-p/2} = \frac{-1}{f}$$
$$-\frac{1}{p} = -\frac{1}{f}$$
$$\therefore p = f \quad \text{i.e.,}$$

The object must be placed in front of the convex mirror at a distance equal to the focal length of the lens.

b) $M = -q/p = -\left(\frac{-p/2}{p}\right) = 1/2 = 0.5$

4. a, $f = -10\,cm$, $M = -q/p = 0.5 \Rightarrow q = -0.5p$

Then,

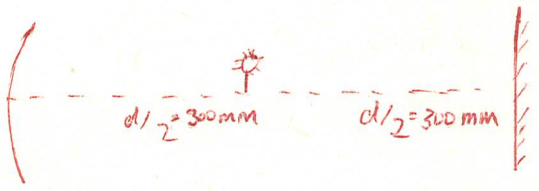
$$\frac{1}{p} + \frac{1}{-0.5p} = -\frac{1}{10\,cm}$$

$$-\frac{1}{p} = -\frac{1}{10\,cm}$$

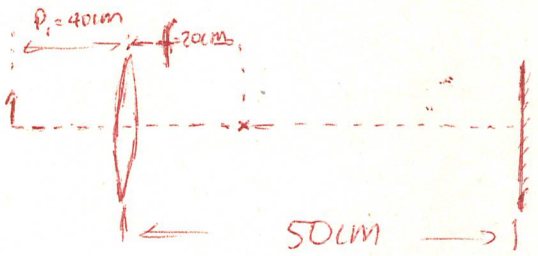
$\therefore p = 10\,cm$

b, NO

5.



6.



$1/p_1 + 1/q_1 = 1/f$

$\frac{1}{40\,cm} + \frac{1}{q_1} = \frac{1}{20\,cm} \Rightarrow \frac{1}{q_1} = \frac{1}{20\,cm} - \frac{1}{40\,cm} = \frac{2-1}{40\,cm} = \frac{1}{40\,cm}$

i.e.) $q_1 = 40\,cm$, on the other side of the lens.

This intermediate image acts as an object for the mirror.

Thus, $p_2 = 50\,cm - q_1 = 50\,cm - 40\,cm = 10\,cm$

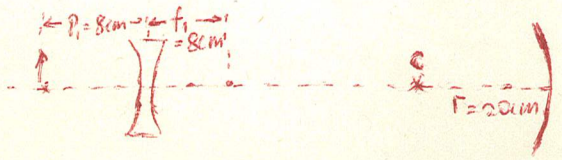
for the image formed by the mirror (plane),

$q_2 = p_2 = 10\,cm$, behind the mirror

$M_1 = -q_1/p_1 = \frac{-40\,cm}{40\,cm} = -1$, $M_2 = -q_2/p_2 = \frac{-(10\,cm)}{10\,cm} = +1$

$M = M_1 \cdot M_2 = (-1)(+1) = -1$

7.



$\frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{f_1}$ — for the diverging lens.

$\frac{1}{8cm} + \frac{1}{q_1} = -\frac{1}{8cm} \Rightarrow \frac{1}{q_1} = -\frac{1}{8cm} - \frac{1}{8cm} = -\frac{2}{8cm} = -\frac{1}{4cm}$

VIRT (i.e.) $q_1 = -4cm$. Thus the image is located to the left of the lens.

For the mirror, $p_2 = 16cm + 4cm = 20cm$

$\frac{1}{p_2} + \frac{1}{q_2} = \frac{1}{f_2}$

$\frac{1}{20cm} + \frac{1}{q_2} = \frac{1}{10cm} \Rightarrow \frac{1}{q_2} = \frac{1}{10cm} - \frac{1}{20cm} = +\frac{1}{20cm}$

i.e.) $q_2 = 20cm$, in front of the mirror or $(20-16)cm = 4cm$ to the left of the lens.

8. The conditions for maxima are

$$d\frac{y}{x} = n\lambda, \quad n = 1, 2$$

$$\Rightarrow y(\text{cm}) = n\frac{\lambda x}{d}$$

For the first maximum, $n=1$, for the second maximum $n=2$ and for the third maximum $n=3$.

$n=1$: $y = 1 \cdot 5000\times10^{-8}\,cm \frac{100cm}{0.2cm} = 2.5\times10^{-2}\,cm$

$n=2$: $y = 2 \cdot 5000\times10^{-8}\,cm \frac{100cm}{0.2cm} = 5\times10^{-2}\,cm$

$n=3$: $y = 3 \cdot 5000\times10^{-8}\,cm \frac{100cm}{0.2cm} = 7.5\times10^{-2}\,cm$

9. To find d: $d = n\lambda\frac{x}{y}$

Then, from the first condition:

$d = 1 \cdot 5890\times10^{-7}\,cm \frac{x}{0.03cm}$ ... two unknowns

In the second case λ is unknown. So one cannot find three unknowns from two equations.

10. $d = \frac{1}{5000}\,cm = 2\times10^{-4}\,cm$

$d \sin\theta_2 = 2\cdot\lambda$

$\therefore \theta_2 = \sin^{-1}(2\frac{\lambda}{d}) = \sin^{-1}\left(2\cdot\frac{5890\times10^{-8}\,cm}{2\times10^{-4}\,cm}\right) = \sin^{-1}(0.2995)$

11. $d = \frac{1}{4000/cm} = 2.5 \times 10^{-4}\ cm$

a) $d \sin\theta_{b_1} = 1\lambda_b$; $d\sin\theta_{g_1} = 1\lambda_g$

$\theta_{b_1} = \sin^{-1}\left(\frac{\lambda_b}{d}\right)$; $\theta_{g_1} = \sin^{-1}\left(\frac{\lambda_g}{d}\right)$

$\theta_{b_1} - \theta_{g_2} = \sin^{-1}\left(\frac{\lambda_b}{d}\right) - \sin^{-1}\left(\frac{\lambda_g}{d}\right)$

$\therefore \Delta\theta_1 = \sin^{-1}\left(\frac{4358 \times 10^{-8}\ cm}{2.5 \times 10^{-4}\ cm}\right) - \sin^{-1}\left(\frac{5461 \times 10^{-8}\ cm}{2.5 \times 10^{-4}\ cm}\right)$

=

b) $d \sin\theta_{b_2} = 2\lambda_b$; $d\sin\theta_{g_2} = 2\lambda_g$

$\theta_{b_2} = \sin^{-1}\left(2\frac{\lambda_b}{d}\right)$; $\theta_{g_2} = \sin^{-1}\left(2\frac{\lambda_g}{d}\right)$

$\therefore \theta_{b_2} - \theta_{g_2} = \sin^{-1}\left(2\frac{4358 \times 10^{-8}\ cm}{2.5 \times 10^{-4}\ cm}\right) - \sin^{-1}\left(2\frac{5461 \times 10^{-8}\ cm}{2.5 \times 10^{-4}\ cm}\right)$

=

12. $d = \frac{1}{6000/cm} = 1.67 \times 10^{-4}\ cm$

$d \sin\theta_m = n\lambda$; $d \sin\theta_2 = 2\lambda$, for $n = 2$

Let us assume θ_2 is small.

$\sin\theta_2 \approx \tan\theta_2 = \frac{32\ cm}{500\ cm} = \frac{8}{125}$

$d\left(\frac{8}{125}\right) = 2\lambda$

$\therefore \lambda = \frac{1}{2} d\left(\frac{8}{125}\right) = \frac{1}{2} \times 1.67 \times 10^{-4}\ cm\left(\frac{8}{125}\right)$

Work sheet III

Temperature and Expansion

1. The boiling point of oxygen is $-297.35\,^{\circ}F$. What is the corresponding temperature on the Celsius scale? The Rankin Scale? The Kelvin scale?

2. A piece of copper tubing is 6m long at $20^{\circ}C$. How much will it increase in length when heated to a temperature of $80^{\circ}C$? ($\alpha_{Cu} = 1.7 \times 10^{-5}/C^{\circ}$)

3. A silver bar ($\alpha_{Ag} = 2 \times 10^{-5}/C^{\circ}$) is 2ft long at $70\,^{\circ}F$. How much will it increase in length when heated to a temperature of $100^{\circ}C$?

4. The diameter of a hole in a steel plate is 9cm when the temp is $20^{\circ}C$. What will the diameter of the hole be at $200^{\circ}C$? Find the change in area. ($\alpha_{steel} = 1.2 \times 10^{-5}/C^{\circ}$)

5. A rectangular sheet of aluminum measures 6 by 8cm at $23^{\circ}C$. What is its area at $0^{\circ}C$?

6. What is the increase in volume of 26 liters of ethyl alcohol when it is heated from 20 to $50^{\circ}C$? ($\beta_{alcohol} = 11 \times 10^{-4}/C^{\circ}$)

Quantity of Heat

1. What quantity of heat will be released when 40 lb_m of copper cools from 78 to $32\,^{\circ}F$?
2. How much heat is developed by the brakes of a 4000 lb_m car in order to bring it to a rest from a speed of 60 mi/h?
3. When 450 cal of heat is applied to a brass ball, its temperature increases from 20 to $70^{\circ}C$. What is the mass of the ball?
4. A 450 g cylinder of lead is heated to $100^{\circ}C$ and dropped into a 50 g copper calorimeter. The calorimeter contains 100 g of water initially at $10^{\circ}C$. Find the equilibrium temperature of the mixture.

5. Suppose that 200 g of copper at 300°C is dropped into 310 g of water at 15°C contained in a 310 g copper calorimeter cup. Compute the equilibrium temperature.

6. How much heat will be released by converting 0.5 lb_m of steam to ice at 10°F?

7. If 40 g of steam at 100°C is mixed with 20 g of ice at -5°C, what will the equilibrium temperature be?

Ans. ? 8. How many pounds of coal must be burned to melt completely 50 lb_m of ice in a heater that is 60 percent efficient?

Thermal properties of Matter

1. How much air at atmospheric pressure can be stored in a 12 ft^3 tank which can withstand an absolute pressure of 120 lb/in^2.

2. A tank with a capacity of 14 liters contains helium gas at 76°F under a pressure of 400 lb/in^2
   a. What will be the volume of a balloon filled with this gas if the helium expands to an internal pressure of 1 atm and if its temperature drops to -35°F?
   Eventually the whole system returns to the original temp. of 76°F. b. What is the final volume of the balloon?

3. A 6 ft^3 container is filled with a gas at an absolute pressure of 300 lb/in^2 and at a temperature of 27°C. What is the new pressure when the temp. is raised to 127°C?

4. A 600 cm^3 container holds 6 g of a gas under a pressure of 1 atm and a temperature of 57°C. What will be the new pressure if 10 g of the same gas fills a 300 cm^3 container at 7°C?

Phys.

Gemene Alemayehu
Yesfin Wondimu
Teshome Senbeta
Abdulwasse Ahmed
Sisay Melese
Indeshaw Reta

Chem.

Wondimagegne Mamo
Amidetsion Terefe
Kasim Muche
Moges Helmassie
Yilma Beyene
Wolde yimam
Amare yikeyes
Gefachew Zewdu
Abebe Adella
Kalayu G/Igziabhere
Berhe Taferre
Milkyas Gugale
Gember Bekele

where (P_1, V_1, T_1) may be considered coordinates of the initial state and (P_2, V_2, T_2) the coordinates of the final state. In other words, for a given mass the ratio PV/T is constant for an ideal gas.

Note: In using any of the three equations which involve temp. we have to take the absolute value of the temp.) i.e.) We have to use the Kelvin scale.

Examples:

1. Body temp. is normal at 98.6 °F. What is the corresponding temp. on the Celsius scale? The Kelvin scale?

Solution

From the relations between °C & °F,

$$T_{°C} = (T_{°F} - 32)\,5/9$$

Then for $T_{°F} = 98.6°F$,

$$T_{°C} = (98.6 - 32)\,5/9\ {}^{\circ}C$$

$$= 66.6 \times 5/9$$

$$\approx 7.4 \times 5\ {}^{\circ}C$$

$$= 37\ {}^{\circ}C$$

$T°_K = T_{°C} + 273$, Then $T_{°K} = 37 + 273 = 310\ {}^{\circ}C$

2. The boiling point of sulfur is 444.5°C. What is the corresponding temp. on the Fahrenheit scale? The Rankin scale?

Solution

$$T°_F = \tfrac{9}{5} T_{°C} + 32$$

Then for $T_{°C} = 444.5°C$,

$$T_{°F} = (9/5 \times 444.5 + 32)\,{}^{\circ}F = (9 \times 88.9 + 32)\,{}^{\circ}C$$

or $T_{°F} = 832.1\ {}^{\circ}F$

$$T°_R = T_{°F} + 460 = (832.1 + 460)\,{}^{\circ}R = 1292.1\ {}^{\circ}R$$

3. A piece of copper tubing is 6m long at 20°C. How much will it increase in length when heated to a temp. of

80°c ? (α for copper is $1.7 \times 10^{-5}/c^{\circ}$

Solution

From the relation for the change in length Δl of a material

$$\Delta l = \alpha l_0 \Delta T$$

Then for our case

$$\Delta l = (1.7 \times 10^{-5}/c^{\circ})(6m)(80^{\circ}c - 20^{\circ}c)$$

$$= (1.7 \times 10^{-5})(6)(60)m$$

$$= 6.12mm$$

4. A rectangular sheet of aluminium measures 6 by 8cm at 23°c. What is its area at 0°c?

Solution

Here the area before heating, A_0, is asked

Then, $\Delta A = \gamma A_0 \Delta T$

$$A - A_0 = \gamma A_0 \Delta T$$

$$A = A_0 + \gamma A_0 \Delta T$$

$$A = A_0(1 + \gamma \Delta T)$$

$$\Rightarrow A_0 = \frac{A}{1 + \gamma \Delta T}$$

Then since $\gamma = 2\alpha$

$$A_0 = \frac{6cm \times 8cm}{1 + 2 \times 2.4 \times 10^{-5}/c^{\circ} \cdot 23^{\circ}c}$$

$$= \frac{6cm \times 8cm}{1.00}$$

5. What is the increase in volume of 16 litres of ethyl alcohol when it is heated from 20 to 50°c?

Solution

Then, $\Delta V = \beta V_0 \Delta T$

$\Delta V = (11 \times 10^{-5}/c^{\circ})(16 \text{ litres})(30^{\circ}c) = 0.0528 \text{ litres}$.

6. prove that the density of a material changes with temp. so that the new density ρ is given by

$$\rho = \frac{\rho_0}{1+\beta\Delta T}$$

Solution

From the relation for the change in volume of a material,

$$\Delta V = \beta V_0 \Delta T$$

$$\Rightarrow V - V_0 = \beta V_0 \Delta T$$

$$V = V_0 + \beta V_0 \Delta T$$

$$V = V_0(1+\beta\Delta T)$$

But $V = \frac{m}{\rho}$, $V_0 = \frac{m}{\rho_0}$

where m is the mass

ρ_0 = the initial density

ρ = the final density

then

$$\frac{m}{\rho} = \frac{m}{\rho_0}(1+\beta\Delta T)$$

$$\frac{1}{\rho} = \frac{1}{\rho_0}(1+\beta\Delta T)$$

or $\rho = \frac{\rho_0}{1+\beta\Delta T}$, which is the required expression.

Key to phys. 451 Final Exam, September 4

1. The primitive base vectors of the fcc lattice are

$$\vec{a}_1 = \tfrac{1}{2}a(\hat{x}+\hat{y}),\quad \vec{a}_2 = \tfrac{1}{2}a(\hat{y}+\hat{z}),\quad \vec{a}_3 = \tfrac{1}{2}a(\hat{x}+\hat{z})$$

(2) (2) (2)

so that

$$\vec{a}_2\times\vec{a}_3 = \tfrac{1}{4}a^2(\hat{x}+\hat{y}-\hat{z})$$
$$\vec{a}_3\times\vec{a}_1 = \tfrac{1}{4}a^2(-\hat{x}+\hat{y}+\hat{z})$$
$$\vec{a}_1\times\vec{a}_2 = \tfrac{1}{4}a^2(\hat{x}-\hat{y}+\hat{z})$$

(2)

$$V_c = \vec{a}_1\cdot\vec{a}_2\times\vec{a}_3 = a^3.$$

Then

$$\vec{G} = \frac{2\pi}{V_c}(s_1\vec{a}_2\times\vec{a}_3 + s_2\vec{a}_3\times\vec{a}_1 + s_3\vec{a}_1\times\vec{a}_2)$$

$$= \frac{2\pi}{a}\left[(s_1-s_2+s_3)\hat{x} + (s_1+s_2-s_3)\hat{y} + (-s_1+s_2+s_3)\hat{z}\right].$$

(2)

2. a. The dispersion relation for a linear lattice of N atoms with 1 atom in a primitive basis is given by

$$\omega = \sqrt{\frac{4C}{M}}\sin\tfrac{1}{2}ka \equiv \omega_m \sin\tfrac{1}{2}ka$$

where $\omega_m = \sqrt{4C/M}$ (1)

The density of states in 1-D is given by

$$D(\omega) = \frac{L}{\pi}\frac{dk}{d\omega} = \frac{L}{\pi}\frac{1}{d\omega/dk}$$ (2)

But

$$d\omega/dk = \tfrac{1}{2}a\,\omega_m\cos\tfrac{1}{2}ka = \tfrac{1}{2}a\omega_m(1-\sin^2\tfrac{1}{2}ka)^{1/2}$$

or

$$\frac{d\omega}{dk} = \tfrac{1}{2}a(\omega_m^2-\omega^2)^{1/2}.$$

Now,

$$D(\omega) = \frac{L}{\pi}\cdot\frac{2}{a}\frac{1}{(\omega_m^2-\omega^2)^{1/2}}$$ (1)

Since $\frac{L}{a} = N$, no. of atoms we write

$$D(\omega) = \frac{2N}{\pi}\frac{1}{(\omega_m^2-\omega^2)^{1/2}}$$ (1)

b. In 3-D and in $\vec{k}$-space the no. of modes is

Given by

$$N = \left(\frac{L}{2\pi}\right)^3 \cdot \frac{4}{3}\pi k^3 \quad \#1 / 4^3$$

so that the density of states becomes (2)

$$D(\omega) = \frac{dN}{d\omega} = \frac{dN}{dk}\frac{dk}{d\omega} = \frac{dN}{dk}\frac{1}{d\omega/dk}$$

or,

$$D(\omega) = 4\left(\frac{L}{2\pi}\right)^3 \pi k^2 \frac{1}{d\omega/dk}$$

For $\omega(k) = \omega_0 - Ak^2$

$$d\omega/dk = |-2Ak| = 2Ak$$

and

$$D(\omega) = \left(\frac{L}{2\pi}\right)^3 \left(\frac{2\pi}{A}\right) k \quad (1)$$

That is,

$$D(\omega) = \left(\frac{L}{2\pi}\right)^3 \left(2\pi/A\right)\left(\frac{\omega_0-\omega}{A}\right)^{1/2}, \quad \text{for } \omega < \omega_0$$

$$= \left(\frac{L}{2\pi}\right)^3 \left(\frac{2\pi}{A^{3/2}}\right)(\omega_0-\omega)^{1/2}. \quad (1)$$

For $\omega > 0$, k becomes imaginary leading to

$$D(\omega) = 0. \quad (1)$$

3. In the Debye model the heat capacity at Const. volume is given ~~as~~ a function of temp. by

$$C_V = 9Nk_B(T/\theta)^3 \int_0^{x_D} \frac{x^4 e^x}{(e^x-1)^2}\,dx \quad (3)$$

In the high temp. case

$$x_D = \frac{\hbar\omega_D}{T} \ll 1 \quad (2)$$

and hence

$$x = \frac{\hbar\omega}{T} \ll 1.$$

Then,

$$\frac{e^x}{(e^x-1)^2} \approx \frac{1}{(1+x-1)^2} = \frac{1}{x^2} \quad (2)$$

and the integral becomes

$$\frac{1}{3}x_D^3 = \frac{1}{3}(\Theta/T)^3 \quad (2)$$

It follows that

$$C_V = 9Nk_B(T/\Theta)^3 \cdot \frac{1}{3}(\Theta/T)^3$$

or,

$$C_V = 3Nk_B \quad \text{for} \quad T \gg \Theta \quad (1)$$

4. In 2-D the density of states is obtained from

$$D(\omega)d\omega = D(k)dk = \left(\frac{L}{2\pi}\right)^2 2\pi k\,dk = \frac{A}{2\pi}k\,dk \quad (1)\ (2)$$

where $A = L^2$, area of the specimen.

Then,

$$D(\omega) = \frac{A}{2\pi}k\frac{dk}{d\omega} = \frac{A}{2\pi}k\frac{1}{d\omega/dk}$$

From the dispersion relation

$$\omega^2 = \frac{2\pi\beta}{\rho\lambda^3}$$

we find

$$\omega^2 = \frac{\beta}{\rho}k^3, \quad \omega = \sqrt{\frac{\beta}{\rho}}\,k^{3/2}$$

whence

$$d\omega/dk = \frac{3}{2}\sqrt{\beta/\rho}\,k^{1/2} \quad (1)$$

It follows that

$$D(\omega) = \left(\frac{A}{2\pi}\right)\left(\frac{1}{3}\right)\left(\frac{\rho}{\beta}\right)^{1/3}\left(\frac{3}{2}\right)\sqrt{\frac{\rho}{\beta}}\,k^{1/2}\omega^{2/3}$$

or,

$$D(\omega) = \frac{A}{2\pi}k\frac{2}{3}\sqrt{\frac{\rho}{\beta}}\,k^{-1/2} = \frac{A}{3\pi}\sqrt{\frac{\rho}{\beta}}\,k^{1/2}$$

$$D(\omega) = \frac{A}{3\pi}\left(\frac{\rho}{\beta}\right)^{2/3}\omega^{1/3} \quad (1)$$

The thermal energy of this system (lattice in Debye model) becomes

$$U = \int d\omega\, D(\omega)\,n(\omega)\,\hbar\omega = \frac{\pi A}{3\pi}\left(\frac{\rho}{\beta}\right)^{2/3}\int_0^{\omega_D}\frac{\omega^{4/3}}{e^{\hbar\omega/kT}-1}\,d\omega$$

(1)

Let $x = \hbar\omega/\tau$ and $x_D = \hbar\omega_D/\tau$; $\tau = k_B T$.

Then

$$U = \frac{\hbar A}{3\pi}\left(\frac{\rho}{b}\right)^{2/3}\left(\frac{k_B T}{\hbar}\right)^{7/3}\int_0^{x_D}\frac{x^{4/3}}{e^x - 1}\,dx \qquad (2)$$

The Debye cut off frequency ω_D cannot be determined easily unless we know the no. of degrees of freedom associated with the surface. Of course this degree of freedom is of the order of the no. of atoms per unit area of the surface.

For a unit area of the surface the average energy is

$$U = \frac{\hbar}{3\pi}\left(\frac{\rho}{b}\right)^{2/3}\left(\frac{k_B T}{\hbar}\right)^{7/3}\int_0^{x_D}\frac{x^{4/3}}{e^x - 1}\,dx$$

The Debye approximation at low temp.:

$$x_D = \frac{\hbar\omega_D}{k_B\tau} = \frac{\theta_D}{T} \gg 1 \quad \text{or} \quad k_B T \ll \theta_D \qquad (1)$$

The ~~integral~~ energy now becomes

$$U = \frac{\hbar}{3\pi}\left(\frac{\rho}{b}\right)^{2/3}\left(\frac{k_B T}{\hbar}\right)^{7/3}\int_0^{x_D} x^{4/3} e^{-x}\,dx$$

$$= a\,T^{7/3} \qquad (1)$$

where a includes all the constants.

The surface specific heat is

$$C_v = \left(\frac{\partial U}{\partial T}\right)_v = \frac{7}{3}\,a\,T^{4/3} \qquad (1)$$

At high temp.

$$\int_0^{x_D}\frac{x^{4/3}}{e^x - 1}\,dx \simeq \int_0^{x_D} x^{4/3}\,dx = \int_0^{x_D} x^{1/3}\,dx = \frac{3}{4}x_D^{4/3},$$

whence

$$U = \frac{\hbar}{3\pi}\left(\frac{\rho}{b}\right)^{2/3}\left(\frac{k_B T}{\hbar}\right)^{7/3}$$

5. a, $\langle n \rangle = \dfrac{1}{e^{\hbar\omega/k_BT} - 1}$

For a sufficiently long wavelength $\hbar\omega \ll k_BT$ so that (1)

$$n = \frac{1}{e^{\hbar\omega/k_BT} - 1} \approx \frac{1}{1 + \hbar\omega/k_BT - 1} = \frac{k_BT}{\hbar\omega}.$$ (1)

That is

$$U_{av} = n\hbar\omega \simeq k_BT \quad (2)$$

b, of the allowed volume in $\vec{K}$-space the fraction occupied by the excited modes of energy $\leq k_BT$ is given by of the order of

$$(k_T/k_D)^3 \quad (2)$$

where k_T is a thermal wavevector defined by

$$\hbar v k_T = k_BT. \quad (1)$$

Then

$$\left(\frac{k_T}{k_D}\right)^3 = \left[\frac{k_BT/\hbar v}{(6\pi^2 N/V)^{1/3}}\right]^3 = \left(\frac{T}{\theta_D}\right)^3. \quad (1)$$

It means the no. of excited modes with energy k_BT is of the order of

$$3N(T/\theta_D)^3. \quad (1)$$

Thus

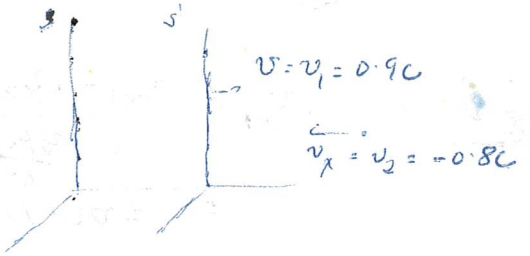
$$U \sim 3N(T/\theta_D)^3 k_BT = 3Nk_BT(T/\theta_D)^3$$

and

$$C_V = \left(\frac{\partial U}{\partial T}\right)_V \sim 12Nk_B(T/\theta_D)^3. \quad (1)$$

The quantity t_0, which is determined by events that occur at the same place in an observer's frame of reference, is called the proper time of the interval between the events.

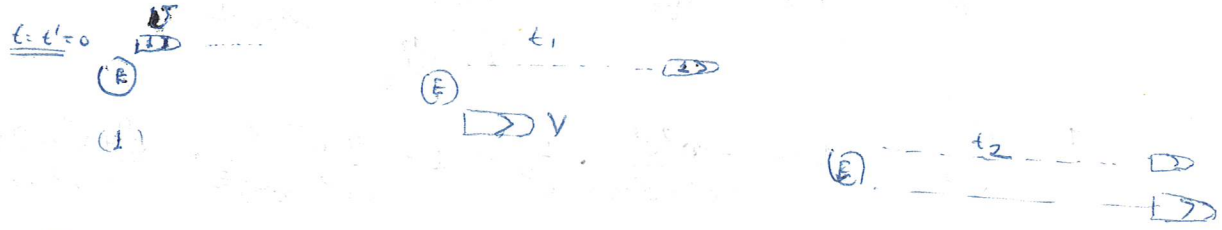
1.



$$v_x' = \frac{v_x - v_M}{1 - \frac{v_x v}{c^2}} = \frac{-0.9c - (-0.8c)}{1 - (-0.9)(0.8)}$$

$$= -\frac{1.7c}{1.72}$$

2.



Dist. moved by the two spaceships when they are abreast of each other is

$$vt_2 = V(t_2 - t_1) \Rightarrow t_2 = \frac{V}{V - v} t_1.$$

a, $(t_1 - 0)$ is a proper time interval

b, $(t_1 - 0) \rightarrow (t_1' - 0)$

$$(t_1 - 0) = (t_1' - 0)/\sqrt{1 - v^2/c^2} \Rightarrow t_1' = t_1\sqrt{1 - v^2/c^2}$$

c, Dist. of the earth from spaceship 1 at time t_1 is

$$x = vt_1$$

d, $(t_2 - 0)$ is a dilated time interval.

e, $(t_2 - 0) \rightarrow (t' - 0)$

$$\text{or, } t_2 = \frac{t'}{\sqrt{1 - v^2/c^2}} \Rightarrow t' = t_2\sqrt{1 - v^2/c^2}.$$

f, $(t_2 - t_1) \rightarrow t'$

$$\text{or, } t_2 - t_1 = t'/\sqrt{1 - v^2/c^2} \Rightarrow t' = (t_2 - t_1)\sqrt{1 - v^2/c^2}.$$

So,

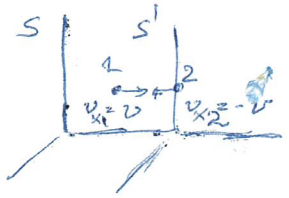
$$x' = vt' = v(t_2 - t_1)\sqrt{1 - v^2/c^2}$$

or,

$$x = v(t_2 - t_1) = \frac{x'}{\sqrt{1 - v^2/c^2}}$$

$$\Rightarrow x' = v(t_2 - t_1)\sqrt{1 - v^2/c^2}$$

4. a),



$$v'_{x1} = \frac{v_{x1}-v_{x2}}{1-\frac{v_{x1}v_{x2}}{c^2}} = \frac{v-(-v)}{1-\frac{v(-v)}{c^2}} = \frac{2v}{1+v^2/c^2}$$

$$= 2vc^2/(c^2+v^2)$$

With this speed relative to S' the K.E. of particle 1 is

$$K' = mc^2 - m_0c^2 = \frac{m_0c^2}{\sqrt{1-\frac{v'^2_{x1}}{c^2}}} - m_0c^2$$

$$1-\frac{v'^2_{x1}}{c^2} = 1-\frac{4v^2c^4}{(c^2+v^2)^2}\cdot\frac{1}{c^2} = 1-\frac{4v^2c^2}{(c^2+v^2)^2} = \frac{c^4+v^4-2v^2c^2}{(c^2+v^2)^2} = \left(\frac{c^2-v^2}{c^2+v^2}\right)^2$$

$$\Rightarrow K' = m_0c^2\left(\frac{1}{\sqrt{1-\frac{v'^2_{x1}}{c^2}}} - 1\right) = m_0c^2\left(\frac{1}{[(c^2-v^2)/(c^2+v^2)]^2}\right)^{1/2} - 1$$

$$= m_0c^2\left(\frac{c^2+v^2}{c^2-v^2} - 1\right) = m_0c^2\cdot\frac{2v^2}{c^2-v^2}$$

or

$$K' = \frac{2m_0v^2}{1-v^2/c^2}$$

b), In Newtonian limits $v \ll c$ and we neglect the 2nd term in the denominator. Or,

$$K' = 2m_0v^2.$$

c),

$$K_1 = mc^2 - m_0c^2 = m_0c^2\left(\frac{1}{\sqrt{1-v^2/c^2}} - 1\right)$$

$$= m_0c^2\left(\frac{1}{\sqrt{1-0.8}} - 1\right) = m_0c^2\left(\frac{1}{\sqrt{0.2}} - 1\right)$$

In the colliding beam mode, the K.E. K_2 of the particles for which $v^2/c^2 = 0.8$ in the laboratory frame is

$$K_2 = K' = \frac{2m_0v^2}{1-v^2/c^2} = \frac{2m_0\times 0.8c^2}{1-0.8} = \frac{1.6\,m_0c^2}{0.2} = 8m_0c^2$$

$$\Rightarrow K_2/K_1 = \frac{8}{\frac{1}{\sqrt{0.2}}-1} = 8/1.19 = 72/11 = 6.54$$

6. [illegible]

8. If the particle moves with a speed v_x in the reference frame s we have

$$E = m_0c^2 / \sqrt{1 - v_x^2/c^2} \quad , \quad p = m v_x$$

Wrt a reference frame s' that moves with a speed v relative to s

$$E' = m_0c^2 / \sqrt{1 - v_x'^2/c^2} \quad , \quad p' = m' v_x' = \frac{m_0}{\sqrt{1 - v_x'^2/c^2}} v_x'$$

where

$$v_x' = (v_x - v) / (1 - v v_x/c^2) .$$

Now,

$$1 - v_x'^2/c^2 = 1 - \left(\frac{v_x - v}{1 - \frac{v v_x}{c^2}}\right)^2 \cdot \frac{1}{c^2} = 1 - \frac{(v_x - v)^2}{(c^2 - v v_x)^2} c^2$$

$$= (c^4 + v^2 v_x^2 - 2c^2 v v_x - v_x^2 c^2 - v^2 c^2 + 2 v v_x c^2) / (c^2 - v v_x)^2$$

$$= (c^4 + v^2 v_x^2 - v_x^2 c^2 - v^2 c^2) / (c^2 - v v_x)^2$$

$$= [c^2 (c^2 - v_x^2) - v^2 (c^2 - v_x^2)] / (c^2 - v v_x)^2$$

$$= (c^2 - v^2)(c^2 - v_x^2) / (c^2 - v v_x)^2$$

$$= (1 - v^2/c^2)(1 - v_x^2/c^2) / (1 - v v_x/c^2)^2$$

$$\sqrt{1 - v_x'^2/c^2} = \sqrt{1 - v^2/c^2}\sqrt{1 - v_x^2/c^2} / (1 - v v_x/c^2)$$

It follows that

$$E' = \frac{m_0c^2}{\sqrt{1 - v_x'^2/c^2}} = \frac{m_0c^2 (1 - v v_x/c^2)}{\sqrt{1 - v^2/c^2}\sqrt{1 - v_x^2/c^2}} = \frac{E (1 - v v_x/c^2)}{\sqrt{1 - v^2/c^2}}$$

$$E' = \frac{E - p v_x / c^2}{\sqrt{1 - v^2/c^2}} = \frac{E - mc^2 \frac{v v_x}{c^2}}{\sqrt{1 - v^2/c^2}} = \frac{E - m v_x v}{\sqrt{1 - v^2/c^2}}$$

or,

$$E' = \frac{E - pv}{\sqrt{1 - v^2/c^2}} .$$

$$p' = \frac{m_0}{\sqrt{1 - v_x'^2/c^2}} v_x' = \frac{m_0 (1 - v v_x/c^2)}{\sqrt{1 - v^2/c^2}\sqrt{1 - v_x^2/c^2}} \cdot \frac{v_x - v}{1 - v v_x/c^2}$$

$$= \frac{m}{\sqrt{1 - v^2/c^2}} (v_x - v) = \frac{m v_x - m v}{\sqrt{1 - v^2/c^2}} = \frac{m v_x - mc^2 \cdot \frac{v}{c^2}}{\sqrt{1 - v^2/c^2}}$$

or,

$$p' = \frac{p - E v/c^2}{\sqrt{1 - v^2/c^2}}$$

13. b. $E - E' = T = E/2 \quad \phi // R' = E/2$

$$\frac{hc}{\lambda} - \frac{hc}{\lambda'} = \frac{hc}{2\lambda}, \quad \frac{hc\,\Delta\lambda}{\lambda\lambda'} = \frac{hc}{2\lambda}, \quad \frac{\Delta\lambda}{\lambda'} = \frac{1}{2}, \quad \lambda' = 2\Delta\lambda$$

or,

$$\lambda + \Delta\lambda = 2\Delta\lambda$$

or,

$$\lambda = \Delta\lambda = \frac{2h}{m_e c} = 2 \times 0.0242\ A^0 = 0.0484\ A^0$$

$$\therefore E = \frac{hc}{\lambda} = \frac{12423\ A^0 eV}{\lambda} = \frac{12423}{0.0484}\ eV$$

c,

$$T = E - E' = \frac{hc}{\lambda} - \frac{hc}{\lambda'} = \frac{hc\,\Delta\lambda}{\lambda\lambda'} = \frac{hc\,\Delta\lambda}{\lambda(\lambda+\Delta\lambda)}$$

$$T = E\frac{\Delta\lambda}{\lambda+\Delta\lambda} = \frac{E\frac{h}{m_0 c}(1-\cos\varphi)}{\lambda + \frac{h}{m_0 c}(1-\cos\varphi)} = \frac{E\frac{h}{\lambda m_0 c}(1-\cos\varphi)}{1 + \frac{h}{\lambda m_0 c}(1-\cos\varphi)}$$

$$T = E\,\frac{hc}{\lambda}\cdot\frac{1}{m_0 c^2}(1-\cos\varphi) \Big/ \left[1 + \frac{hc}{\lambda}\cdot\frac{1}{m_0 c^2}(1-\cos\varphi)\right]$$

or,

$$T = \frac{(E^2/m_0 c^2)(1-\cos\varphi)}{1 + (E/m_0 c^2)(1-\cos\varphi)}$$

Chapter 2

2-1. $s(t) = \frac{1}{2}at^4 - bt^2$

To graph:

- roots (or zeros)

$$\frac{1}{2}at^4 - bt^2 = t^2\left(\frac{a}{2}t^2 - b\right) = 0, \quad t^2 = 0 \text{ or } t^2 = \frac{2b}{a}$$

T.S. $\left\{0, \pm\sqrt{\frac{2b}{a}}\right\}$; $t=0$ is a double root

- Local maxima & minima

$$\frac{ds}{dt} = 2at^3 - 2bt = t(at^2 - b) = 0, \quad t = 0 \text{ or, } t = \pm\sqrt{\frac{b}{a}}$$

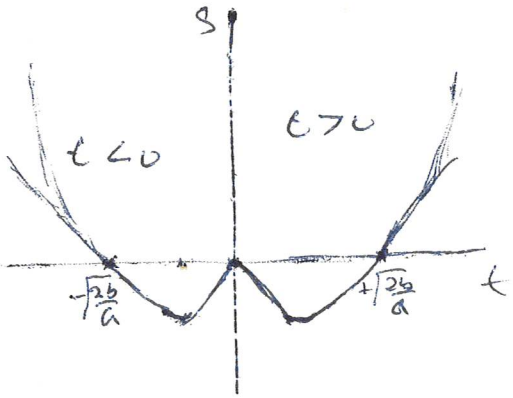
Consider the intervals

$\left[-\sqrt{\frac{2b}{a}}, 0\right]$ & $\left[0, \sqrt{2b/a}\right]$

$$\frac{d^2s}{dt^2} = s'' = 6at^2 - 2b$$

$s''(0) = -2b < 0$, local maximum

$s''\left(\pm\sqrt{\frac{b}{a}}\right) = 6a\frac{b}{a} - 2b = 4b = s''\left(\sqrt{\frac{b}{a}}\right)$, local minima



$$s\left(-\sqrt{\frac{b}{a}}\right) = \frac{1}{2}a\frac{b^2}{a^2} - b\frac{b}{a}$$
$$= \frac{b^2}{2a} - \frac{b}{a}c - \frac{b}{a}$$
$$= s\left(\sqrt{\frac{b}{a}}\right)$$

At $t=0$, the particle starts its motion from the origin and goes to a max. dist of $\sqrt{\frac{b}{a}}$ in the negative x-direction, then reverses its motion in the positive x-direction, crosses the origin, and continues on. So $s(t)$ represents possible positions of the particle for $t \geq 0$.

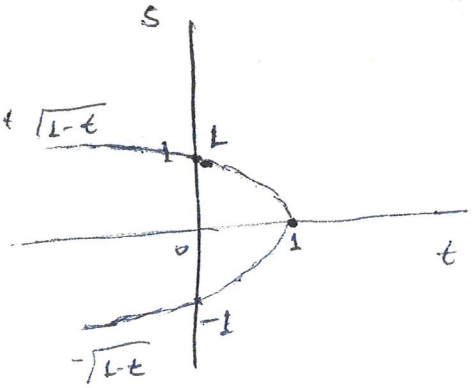
2-2. $s(t) = \pm s_0\sqrt{1+\alpha t}$, $s_0 = 1m$, $\alpha = \pm 1s^{-1}$

$s(t) = \pm\sqrt{1+\alpha t}$

1. $s(t) = \pm\sqrt{1-t}$, $\alpha = -1$

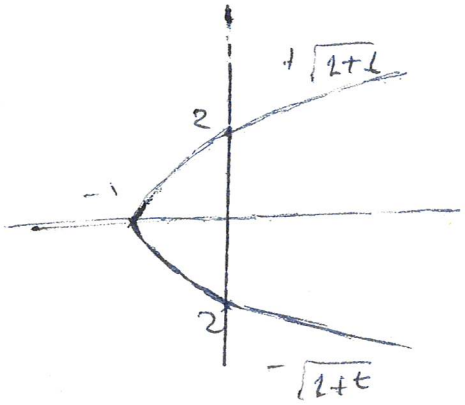
2. $s(t) = \pm\sqrt{1+t}$, $\alpha = 1$

1.



For a physical particle
$t \in [0,1]$
since time develops forward

2.



~~For~~ a physical particle
$t \in [0, \infty)$

Graph:

2-8 $s(t) = t^3 - 9t^2 + 24t = t(t^2 - 9t + 24)$

- roots

$t = 0$ or $t^2 - 9t + 24 = 0$. $b^2 - 4ac < 0$, no other roots

T.S $\{0\}$

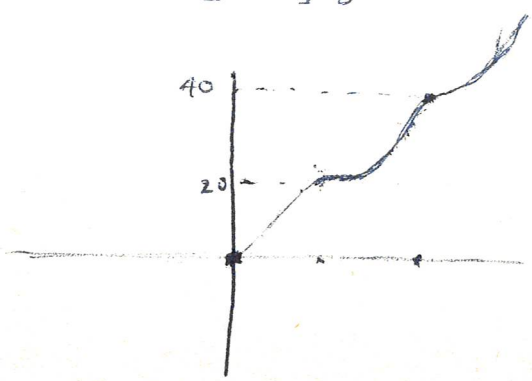
- Extremum points

$\frac{ds}{dt} = 0 \Rightarrow t = 2, 4$

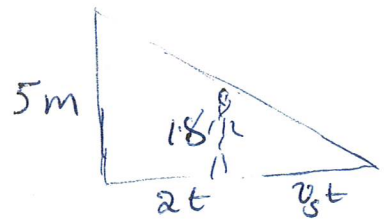
$\frac{d^2s}{dt^2} = s'' = t - 3$; $s''(2) = 2 - 3 = -1 < 0$, max.
$s''(4) = 4 - 3 = 1 > 0$, min

- limits

$t \to \pm\infty$, $s \to \pm\infty$



2-9.

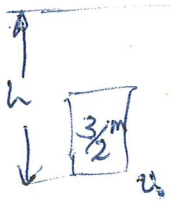


$$\frac{5}{1.8} = \frac{(2+v_s)t}{v_s t} = \frac{2+v_s}{v_s}, \quad 5v_s = 3.6 + 1.8 v_s$$

$$3.2 v_s = 3.6, \text{ or } v_s = \frac{3.6}{3.2} = \frac{9}{8} = 1.125$$

or, $v_s = 1.125$ m/s.

2-28.



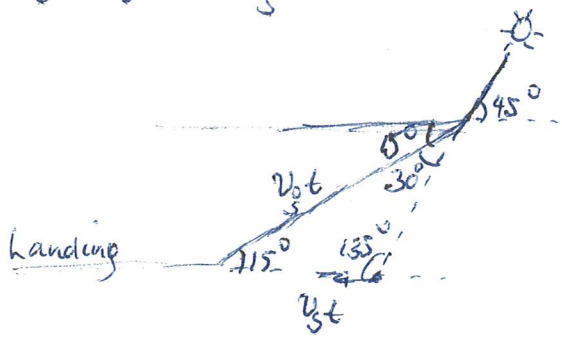
$t_u = t_d = 0.3$, conservation of M.E.

$$\frac{3}{2} = 0.3 v_0 - \frac{1}{2} \times 10 \times (0.3)^2$$

$$v_0 = \frac{1.5 + 0.45}{0.3} \text{ m/s} = \frac{1.95}{0.3} \text{ m/s} = 6.5 \text{ m/s}.$$

$$0 - v_0^2 = -2gh \Rightarrow h = \frac{v_0^2}{2g} = \frac{6.5^2}{2\times 10} \text{ m} = 2.11 \text{ m}.$$

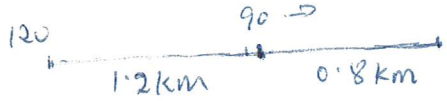
2-36.



$$\frac{v_s t}{\sin 135^\circ} = \frac{v_s t}{\sin 30^\circ}, \quad v_s = \frac{\sin 30^\circ}{\sin 135^\circ} v_0$$

$$v_s = \frac{\sin 30^\circ}{\sin 45^\circ} v_0 \quad v_s = v_0/\sqrt{2}.$$

2-41.



Safe until a dist. of 0.8 km remains between them.

$$v^2 - v_0^2 = 2as, \quad a = \frac{v^2 - v_0^2}{2s} = \frac{90^2 - 120^2}{2\times 1.2} \text{ km/h}^2 = -0.20 \text{ m/s}^2.$$

2-44 For each upward or downward movement

$$h = \frac{1}{2} g t^2 \Rightarrow t = \sqrt{2h/g}$$

The ball strikes the surface every $T = 2t \left(= 2\sqrt{2h/g}\right)$ time later; i.e., the period is $2T$

$\therefore$ no. of strikes per second $= \frac{1}{T} = \frac{1}{2\sqrt{2h/g}} = \frac{1}{2}\sqrt{\frac{g}{2h}}$

$= 3.535 \text{ sec}^{-1} = 212 \text{ min}^{-1}$.

Chapter 4

4-5 $\vec{r}(t) = ct\,\hat{i} + (bt - at^2)\,\hat{j}$

$\Rightarrow x = ct, \quad y = bt - at^2$

(i.e.) $$y = \frac{b}{c}x - \frac{a}{c^2}x^2$$

$$\left.\frac{dy}{dx}\right|_{t=0} = \left.\left(\frac{b}{c} - \frac{2a}{c^2}x\right)\right|_{t=0} = \frac{b}{c}, \text{ since } x=0 \text{ at } t=0$$

But

$$\left.\frac{dy}{dx}\right|_{t=0} = \left.\text{Slope}\right|_{t=0} = \tan 45^\circ = \frac{b}{c}, \quad 1 = \frac{b}{c} \Leftrightarrow b = c \quad \cdots (1)$$

If y attains a max. value at time t_m, then

$$\left.\frac{dy}{dt}\right|_{t_m} = b - 2at_m = 0 \Rightarrow t_m = \frac{b}{2a}.$$

And,

$$y_{ma} = h = bt_m - at_m^2 = b\left(\frac{b}{2a}\right) - a\frac{b^2}{4a^2} = \frac{b^2}{2a} - \frac{b^2}{4a} = \frac{b^2}{4a} \quad \cdots (2)$$

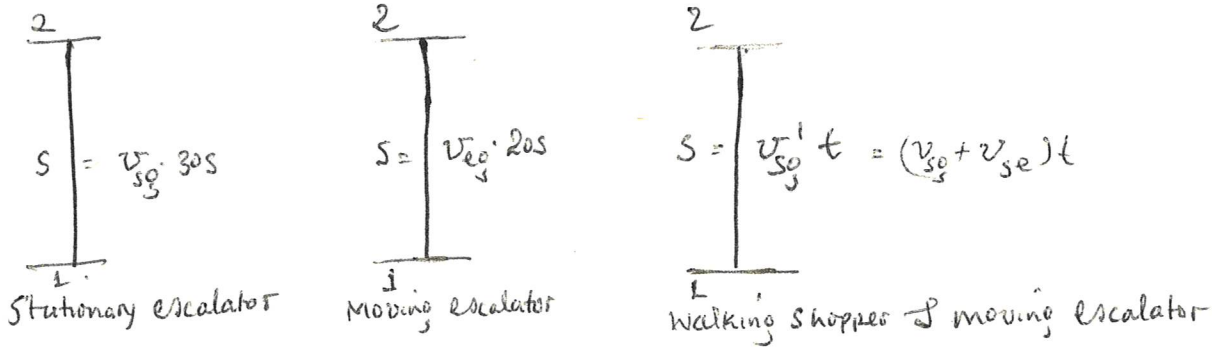
Now, the corresponding value of x is given by

$$x_m = ct_m = c\left(\frac{b}{2a}\right) = \frac{b^2}{2a}, \text{ since } b=c \text{ from } 1$$

Or,

$$x_m = 2h, \text{ from } (2).$$

4.25



$$30\,v_{sg} = 20\,v_{eg} = v'_{sg}t = (v_{sg} + v_{eg})t$$

1. $v_{sg}/v_{eg} = 2/3$

2. $S = (v_{sg} + v_{se})t$ $\quad 20\,v_{eg} = (v_{sg} + v_{eg})t = \left(\frac{2}{3}v_{eg} + v_{eg}\right)t$

Or,

$$20\,v_{eg} = \frac{5}{3}v_{eg}t$$

Or,

$$t_{eg} = 12\,s.$$

4.27

Downstream :

$$v_{bg} = v_{bw} + v_{wg}, \quad v_{bw} = 5\,km/h \qquad \cdots (1)$$

Upstream :

$$v'_{bg} = v'_{bw} - v_{wg}, \quad v'_{bw} = 10\,km/h \qquad \cdots (2)$$

From 1

$$\frac{s}{t} = 5 + v_{wg} \qquad \cdots (3)$$

From 2

$$\frac{s}{1.15t} = 10 - v_{wg} \qquad \cdots (4)$$

Dividing (3) by (4) we find

$$1.15 = (5 + v_{wg}) / (10 - v_{wg})$$

$\Rightarrow$

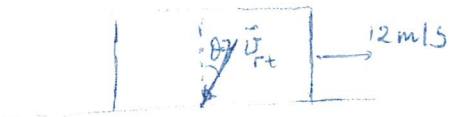
$$11.5 - 1.15\, v_{wg} = 5 + v_{wg}$$

$$6.5 = 2.15\, v_{wg}$$

or,

$$v_{wg} = \frac{6.5}{2.15}\ km/h = 3.02\ km/h.$$

4-28.



$$\vec{v}_{rg} = \vec{v}_{rb} + \vec{v}_{tg}$$

$$v_{rg,x} = -v_{rt,x} + v_{tg,x}, \quad v_{rt,x} = v_{tg,x}$$

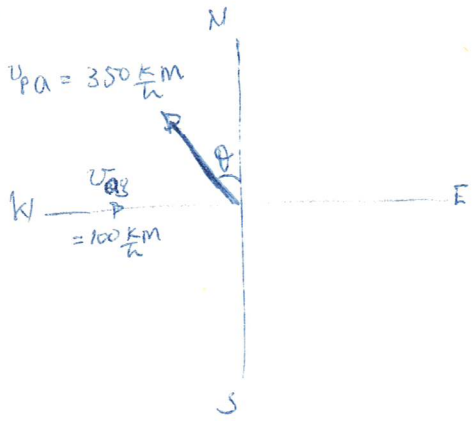
$$v_{rg,y} = v_{rt,y} + v_{tg,y}, \quad v_{tg,y} = v_{rt,y}$$

or,

$$v_{rt,x} = v_{tg,x} \Rightarrow v_{rt} \sin 27^\circ = 12\ m/s \Rightarrow v_{rt} = \frac{12\,m/s}{\sin 27^\circ}$$

$$v_{rt,y} = v_{rt} \cos 37^\circ = \frac{12\,m/s}{\tan 27^\circ}$$

4.29.



$$\vec{v}_{pg} = \vec{v}_{pa} + \vec{v}_{ag}$$

$$v_{pg,x} = v_{pa,x} + v_{ag,x}$$

$$0 = -350\cos\theta + 100$$

$$\theta = \cos^{-1}\left(\frac{100}{350}\right) = 16.6^\circ$$

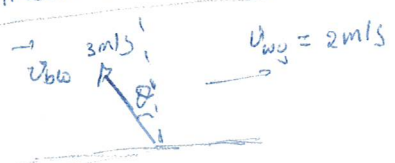
$$v_{pg,y} = v_{pa,y} + \cancel{v_{ag,y}}^{0} = v_{pa,y}$$

$$v_{pg} = v_{pg,y} = v_{pa,y} = 350\cos 16.6^\circ$$

$$= 335 \text{ km/h}.$$

4-30.

upstream Movement:



$$\vec{v}_{bg} = \vec{v}_{bw} + \vec{v}_{wg}$$

$$v_{bg,x} = v_{bw,x} + v_{wg} \quad \text{---(1)}$$

$$v_{bg,y} = v_{bw,y} \quad \text{---(2)}$$

But $$v_{bg,y} = \frac{150\text{m}}{120\text{min}} = 1.25 \text{ m/s} = v_{bw,y}$$

Now,

$$v_{bw,y} = v_{bw}\cos\theta = 3\cos\theta = 1.25$$

$$\theta = \cos^{-1}\left(\frac{1.25}{3}\right) = 66.5^\circ$$

$$v_{bw,x} = 3 \text{ m/s } \sin(66.5^\circ) = 3 \times 0.91 \text{ m/s} = 2.73 \text{ m/s}$$

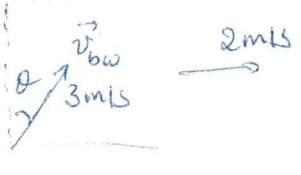
i.e.,

$$v_{bg,x} = 2.73 \text{ m/s} - 2 \text{ m/s} = 0.73 \text{ m/s}$$

Thus the boat reaches the opposite shore at the upstream point

$$x = (0.73 \text{m/s})(120 \text{s}) = 87.6 \text{m}.$$

Downstream Movement:



Using the same calculation as above

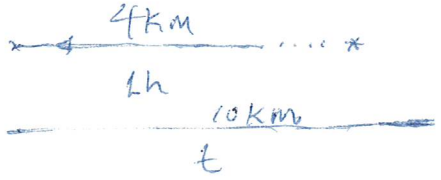
$$v_{bg,x} = 2.73\ m/s + 2m/s = 4.73m/s$$

and the boat reaches the opposite shore at the down-stream point

$$x = (4.73m/s)(120s) = 567.6m.$$

7-43

a,



upstream:

$$v_{bg} = v_{br} - v_{rg}, \quad v_{bg} = 4km/hr \quad \text{---(1)}$$

Downstream

$$v'_{bg} = v_{br} + v_{rg} \quad \text{---(2)}$$

$$10km = v'_{bg}\, t \quad \text{---(3)}$$

$$6km = (1+t)\, v_{rg} \quad \text{---(4)}$$

From (4)

$$t = \frac{6}{v_{rg}} - 1$$

Using this value in (3)

$$10 = v'_{bg}\left(\frac{6}{v_{rg}} - 1\right) \quad \text{or} \quad v'_{bg} = \frac{10\, v_{rg}}{6 - v_{rg}}$$

Now we have

$$4 = v_{br} - v_{rg} \quad \text{from (1)}$$

$$\frac{10\, v_{rg}}{6 - v_{rg}} = v_{br} + v_{rg} \quad \text{from (2) and the expression of } v'_{bg}$$

$$\Rightarrow \frac{10\, v_{rg}}{6 - v_{rg}} - 4 = 2 v_{rg} \Leftrightarrow 10 v_{rg} - 4(6 - v_{rg}) = 12 v_{rg} - 2 v_{rg}^2$$

$$2v_{rg}^2 + 2v_{rg} - 24 = 0$$

$$v_{rg} = \frac{-2 \pm \sqrt{4+192}}{4}$$

. Since we are considering the magnitude of v_{rg} we take the positive value

$$v_{rg} = \frac{-2+14}{4}\ \frac{km}{h} = 3\ km/h$$

b, The velocity of the canoe (boat) in still water (using (2) and the result above) is

$$v_{bg} = v_{bg} + v_{rg} = 4\ km/h + 3\ km/h = 7\ km/h$$

4-44 a, Max upward dist moved by the bolt

$$h = \frac{v_0^2}{2g} = \frac{2^2}{2\times 10} = \frac{1}{5} = 0.2\ m$$

Time required by the bolt to move this dist is found from the following eq.

$$0.2 = 2t - \tfrac{1}{2}\times 10t^2 = 2t - 5t^2$$

$$5t^2 - 2t + 0.2 = 0$$

$$t = \frac{2 \pm \sqrt{4-4}}{10} = \frac{2 \pm 0}{10} = 0.2\ sec.$$

But in this time interval the boat has moved some dist.

$$h = 2\ m/s \times 0.2\ sec = 0.4\ m$$

Now the dist between the bolt and the elevator floor is

$$(3 + 0.2 - 0.4)\ m = 2.8\ m$$

And,

$$2.8\ m = 2t + \tfrac{1}{2}\times 10t^2 = 2t + 5t^2$$

$$5t^2 + 2t - 2.8 = 0$$

$$t = \frac{-2 \pm \sqrt{2^2 + 4 \times 5 \times 2 \cdot 8}}{10} \text{ sec.} = \frac{-2 \pm \sqrt{60}}{10} \text{ sec} = 0 \cdot 57 \text{ sec.}$$

i.e, The time of flight of the bolt is

$$T = 0 \cdot 2 \text{ sec} + 0 \cdot 57 \text{ sec.} = 0 \cdot 77 \text{ sec.}$$

b, v_{be} (while striking the elevator floor) $= -gt = -10 \text{ m/s}^2 \times 0 \cdot 57 \text{ s} = -5 \cdot 7 \text{ m/s}$

c, $v_{bg} = v_{be} + v_{eg} = -5 \cdot 7 \text{ m/s} + 2 \text{ m/s} = -3 \cdot 7 \text{ m/s}$

d, $s_{be} = \frac{1}{2} g t^2$, because the elevator observer sees only the downward movement of the bolt

$$= \frac{1}{2} \times 10 \times 0 \cdot 77^2 \text{ m} = 2 \cdot 9645 \text{ m}$$

$$s_{bg} = 0 \cdot 2 \text{ m} + 1 \cdot 6245 \text{ m} \; (+ \tfrac{1}{2} \times 5 \times 0 \cdot 57^2) = 1 \cdot 8245 \text{ m.}$$

Kotebe College of Teacher Education

Phys.202 Final Exam.

Name_____

ID.No._____

Choose the best answer and write the letter of your choice on the space provided corresponding to each question.

_____1. Each of two small spheres is charged positively, the combined charge being 5×10^{-5}c. If each sphere is repelled from the other by a force of 1nt.
When the spheres are 2m apart, how is the total charge distributed ?

a) $q_1 = 10^{-4}$c, $q_2=4\times10^{-5}$c b) $q_1= 2\times10^{-5}$, $q_2= 3\times10^{-5}$c c) $q_1= q_2 = 2.5\times10^{-5}$c d) none

_____2. A uniform electric filed E is established in the space between two large parallel plates. In this field one suspends a small conducting sphere of mass m from a string of length l. What is the period of this pendulum when the sphere is given a charge ~~+q~~. ~~Assume the~~ lower plate is charged negatively.

a) $T = \frac{1}{2\pi}\sqrt{\frac{ml}{qE}}$ b) $T = 2\pi\sqrt{\frac{ml}{qE}}$ c) $T = 2\pi\frac{ml}{qE}$) d) none.

_____3. A certain closed surface encloses an electric dipole. What is for this surface.

a) $\Phi_E = 2\,q/\epsilon_o$ b) $\Phi_E = -2\,q/\epsilon_o$ c) $\Phi_E = q/\epsilon_o$ d) $\Phi_E = 0$.

_____4. What is the charge density on the surface of a conducting sphere of radius 0.15m whose potential is 200 volts ?

a) 2×10^{-3} c/m^3 b) 1.2×10^{-8} c/m^2 c) 4.4×10^{-2} c/m^3 d) 5×10^{-6} c/m^2

_____5. Two capacitors (2μf and 4μf) are connected in paralled across a 300V p.d. Calculate the total energy stored in the system.

a) 0 ~~= Q~~ J b) 11 J c) 0.27V d) none

Kotebe College of Teacher Education

Phys.202 Final Exam.

Name_____

ID.No._____

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_____2. A uniform electric filed E is established in the space between two large parallel plates. In this field one suspends a small conducting sphere of mass m from a string of length l. What is the period of this pendulum when the sphere is given a charge ~~+q~~. ~~Assume the~~ lower plate is charged negatively.

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_____5. Two capacitors (2μf and 4μf) are connected in paralled across a 300V p.d. Calculate the total energy stored in the system.

a) 0 ~~= Q~~ J b) 11 J c) 0.27V d) none

6.

_____6. A light bulb designed to consume power P(100 walts) when connected to a source V (100 volts) is actually connected to a source V^1 (50 volts). What power does it consume ?

a) 100 W b) 50 W c) 25 W d) none

_____7. A wire of 1m long carries a current of A and makes an angle of 30^o with a uniform magnetic field with B = 1.5 T. Calculate the magnitude and direction of the force on the wire.

a) 7.5 N perpendicular to both the wire and B

b) 2 N " " " " " "

c) No force acts

d) none

_____8. A square loop of wire of edge a carries a current i. What in the value of B at the center of the of loop ?

a) $2\sqrt{2}\mu_o i/\pi a$ b) $4\mu_o i/\pi a$ c) $2\mu_o i/\pi a$ d) none

_____9. A circular loop 10 cm in radius is placed with its plane at right angles to a uniform 0.80 T magnetic field. By a certain mechanism not specified the radius of the loop is caused to shrink at an instantaneous rate of 80 cm/s. What emf is induced in the loop?

a) 8V b) 5V c) 0V d) 0.40V

_____10. What is the SI unit of self - inductance L ?

a) Henry b) V.S /A c) Wb /A d) J/A^2 e) none.

II. Solve the following problems with clear steps. Neatness is considered.

1. A solid insulating sphere carries a uniform charge per unit volume of ρ. Let $\bar{r}$ be the vector from the center of the sphere to a general point p within the sphere. a) show that the electric field at p is given by $E = \rho \bar{r} / 3\epsilon_0$. b) A spherical cavity is removed from the above sphere, as shown in the figure below. Using superposition (Vector addition) concepts show that the electric field at all points within the cavity is $E = \rho \bar{a}/3\epsilon_0$ (uniform field), where $\bar{a}$ is the vector connecting the center of the sphere with the center of the cavity.

Kotebe College of Teacher Education

Department of Physics

Name_____ 12 June 1990

ID.No._____ Time allowed 2 hrs

Instructions

Solve any ten of the following twelve problems on the spaces provided after each problem. Neatness is considered.

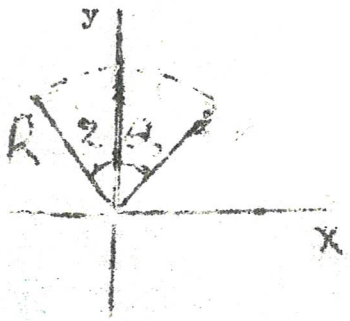
In Case you need you may use:

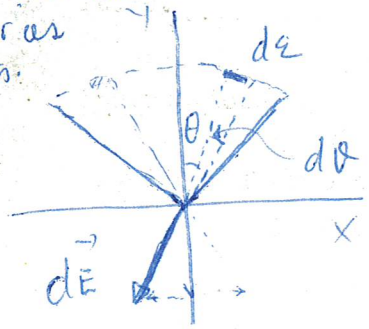
$K = 9 \times 10^9$ N m^2/C^2 , $\mu_0 = 4\pi \times 10^{-7}$ Wb/m A.

- Use of Correct formulas, principles, rules 3
- Explanation, Steps, 4
- Exact Answer 3

Except No 11 for which each item a,b... Carries 2.5 points.

1. A uniformly charged rod with change per unit length λ is bent into the shape of a circular arc of radius R as shown below. The arc subtends an angle $2\theta_0$ at the center of the circle. Show that the electric field at the center of the circle is in the y direction with a magnitude given by $E_y = \lambda \sin\theta_0 / 2\pi\epsilon_0 R$.





consider the charge element on the right of the y-axis

$$d\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{dq}{R^2}\vec{e}_r$$

This has two components dE_x and dE_y. But the x-comp. will be cancelled by an oppositely directed comp. of the charge element on the left side of the y-axis. That is the effective comp. is only the y-comp. which is given by

$$dE_y = \frac{1}{4\pi\epsilon_0}\frac{dq}{R^2}\cos\theta$$

or,

$$dE_y = \frac{1}{4\pi\epsilon_0}\frac{\lambda R d\theta}{R^2}\cos\theta$$

$$= \frac{\lambda}{4\pi\epsilon_0 R}\cos\theta\, d\theta$$

and

$$E_y = \frac{\lambda}{4\pi\epsilon_0 R}\int_{-\theta_0}^{\theta_0}\cos\theta\, d\theta$$

$$= \frac{\lambda}{4\pi\epsilon_0 R}\sin\theta\Big|_{-\theta_0}^{\theta_0}$$

$$= \frac{\lambda}{4\pi\epsilon_0 R}[\sin\theta_0 - (\sin(-\theta_0))]$$

or,

$$E_y = \frac{\lambda \sin\theta_0}{2\pi\epsilon_0 R}.$$

2. Each of two small spheres is charged positively, the combined charge being 5×10^{-5} c. If each sphere is repelled from the other by a force of 1N when the spheres are 2m apart, how is the total charge distributed between the spheres ?

q_1 ○ ○ q_2

$q_1 + q_2 = q, \quad q = 5 \times 10^{-5}$ c

$$1 = \frac{9\times10^9 \, q_1 q_2}{4} = \frac{9\times10^9}{4} q_1 (q - q_1)$$

$$\frac{4}{9}\times10^{-9} = q_1 q - q_1^2$$

$$q_1^2 - q q_1 + \frac{4}{9}\times10^{-9} = 0$$

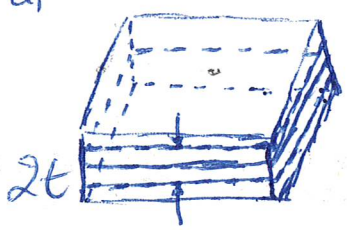
or,

$$q_1 = \frac{q \pm \sqrt{q^2 - \frac{16}{9}\times10^{-9}}}{2} = 5\times10^{-5} \pm \frac{\sqrt{25\times10^{-10} - \frac{16}{9}\times10^{-9}}}{2} \text{ C}$$

Taking the "+" sign (which is equivalent to taking the minus sign for this prob.) we get $q_1 = 4.64\times10^{-5}$ C and $q_2 = 0.36\times10^{-5}$ C

3. A nonconducting plane slab of thickness d has a uniform charge density ρ. Find the magnitude of the electric field at all points in space both a, inside and b, outside the slab.

a,



consider a Gaussian surface of thickness $2t$ inside the slab which encloses a charge say q'.

$$\epsilon_0 \oint \vec{E}\cdot d\vec{s} = q'$$

$$\epsilon_0 EA + \epsilon_0 EA = q' = 2\rho A t$$

(or) $E = \frac{\rho t}{\epsilon_0}$.

b, considering a Gaussian surface with thickness t and we have

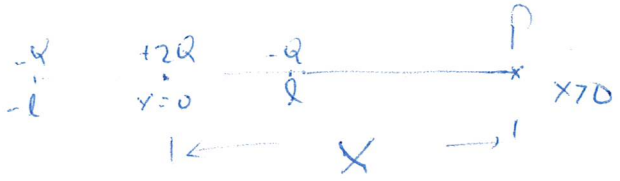
$$\epsilon_0 \oint_s \vec{E}\cdot d\vec{s} = q = \rho A d$$

$$2\epsilon_0 E A = \rho A d$$

or,

$$E = \frac{\rho d}{2\epsilon_0}.$$

4. Three point charges are located as follows: $-Q$ at $x = -1$, $+2Q$ at $x = 0$, and $-Q$ at $x = +1$. Calculate the potential along the x - axis in the distant region ($|x| >> 1$).



$$V_P = \frac{-q}{4\pi\epsilon_0(x+l)} + \frac{2q}{4\pi\epsilon_0 x} - \frac{q}{4\pi\epsilon_0(x-l)}$$

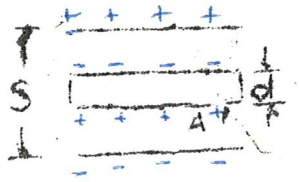
$$= \frac{q}{4\pi\epsilon_0}\left[-\frac{1}{x}(1+l/x)^{-1} + \frac{2}{x} - \frac{1}{x}(1-l/x)^{-1}\right]$$

$$= \frac{q}{4\pi\epsilon_0}\left[-\frac{1}{x}\left(1 - \frac{l}{x} + \frac{l^2}{x^2}\right) + \frac{2}{x} - \frac{1}{x}\left(1 + \frac{l}{x} + \frac{l^2}{x^2}\right)\right]$$

or,

$$V_P = \frac{q}{4\pi\epsilon_0}\frac{-2l^2}{x^3} = -\frac{ql^2}{2\pi\epsilon_0 x^3}.$$

5. A conducting slab with a thickness d and area A is inserted into the space between the plates ofoparallel - plate capacitor with spacing S and area A, as shown in the diagram. What is the value of the capacitance of the system ?



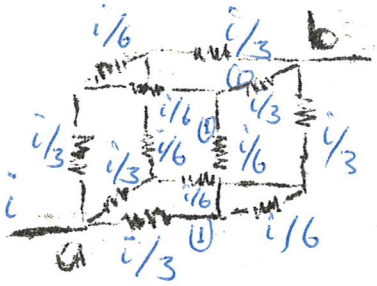
The charge induced on the conductor is the same in magnitude as that magnitude on the plate of the capacitor. Moreover, the electric field inside a conductor is zero. Thus, in effect, we have two capacitor in series. That is

$$\frac{1}{C} = \frac{1}{\frac{\varepsilon_0 A}{(S-d)/2}} + \frac{1}{\frac{\varepsilon_0 A}{(S-d)/2}} = 2\left(\frac{S-d}{2\varepsilon_0 A}\right) = \frac{S-d}{\varepsilon_0 A}$$

or,

$$C = \varepsilon_0 A/(S-d).$$

6. Twelve resistors each having a resistance R are connected to form a cube as shown below. Find the equivalent resistance between the terminals a and b.



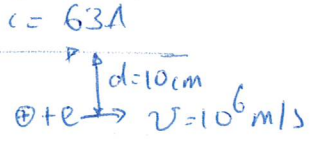
Choosing path 1 for the movement from a to b we find that

$$Re\, i = R\frac{i}{3} + R\frac{i}{6} + R\frac{i}{3} = \frac{5}{6}Ri$$

i.e.)

$$Re = \frac{5}{6}R.$$

7. A long straight wire carries a current i (63 A). A proton is projected parallel to the wrie and in the direction of the current a distance d(10cm) and speed v (10^6 m/S). What force acts oh the proton



The magnetic field at the site of the proton is directed into the plane of the fioure so that it ⊥ to the velocity of the proton which is moving parallel to the wire.

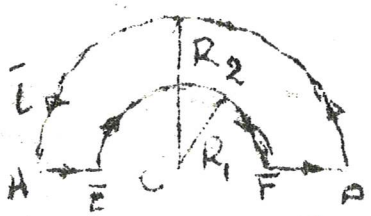
Thus,

$$F = evB = ev\frac{\mu_0 i}{2\pi d}$$

or,

$$F = \frac{1.6\times10^{-19}\,C \times 10^{6}\,m/s \times 4\pi\times10^{-7}\,\frac{Wb}{A\cdot m} \times 63A}{2\pi \times 0.1m} = 2.02\times10^{-17}\,N.$$

8. In the figure below suppose that the positive X axis points outward from the plane of the figure. a) Use the Biot – Savart law to calculate the magnetic field $\vec{B}$ at C, the common center of the semicircular arcs AD and EF, of radii R_2 and R_1 respectively, forming part of the circuit A E F D A and carrying current i as shown. b) Calculate magnetic dipole moment $\vec{M}$ of this circuit.



~~the~~ the +x direction whereas that due to the smaller one is directed in the -ve x direction. For example taking a small current element for the larger arc $dB_{R_2} = \frac{\mu_0 i}{4\pi R_2} \frac{dl}{R^2} \Rightarrow B_{R_2} = \frac{\mu_0 i}{4R_2}$

a) One can see from the right hand rule that the magnetic field due to the larger arc is directed in

Similarly, $B_{R_1} = -\mu_0 i / 4R_1$

$\therefore \vec{B}_T = \frac{\mu_0 i}{4}\left(\frac{1}{R_2} - \frac{1}{R_1}\right)\vec{i}$.

b) $\vec{M} = iA = i\left[\frac{\pi R_2^2}{2} - \frac{\pi R_1^2}{2}\right]\vec{i} = \frac{i\pi}{2}(R_2^2 - R_1^2)\vec{i}$.

9. Prove that the electric field E ~~in~~ a changed parallel – plate capacitor cannot drop abrupiey to zero as one moves at right angles to it, as suggested by the arrow in the figure below. (see point a). In actual capacitors fringing of the lines of force always occurs, which means that E approaches zero in a continuous and gradual way. (Hint: Apply Faradays law to the rectangular path shown by the dashed lines.)

From Faradays law

$\oint_C \vec{E}\cdot d\vec{l} = -\frac{d\phi_B}{dt} = 0$, since there is no induced emf.

$\Rightarrow \oint_C \vec{E}\cdot d\vec{l} = EL + 0 + 0 + 0 = 0$

(i.e) $EL = 0$

This means that $E = 0$ to the left of point a. This is contrary to the prob. and hence $\vec{E}$ cannot fall abruptly to zero to the left of point a.

10. Prove that if the flux of a magnetic field $\vec{B}$ through a coil of N turns changes in any way from Φ_1 to Φ_2, then the charge Q that flows through the circuit of the coil of total resistance R is given by

$$Q = \frac{N(\Phi_2 - \Phi_1)}{R}$$

$$i = \frac{V_L}{R} = -\frac{1}{R}\frac{d(N\Phi_B)}{dt} = -\frac{N}{R}\frac{d\Phi_B}{dt}$$

The magnitude of this current is

$$i = \frac{N}{R}\frac{d\Phi_B}{dt}$$

But $i = \frac{dQ}{dt}$ so that

$$\frac{dQ}{dt} = \frac{N}{R}\frac{d\Phi_B}{dt}$$

$$dQ = \frac{N}{R}d\Phi_B$$

Integrating we have

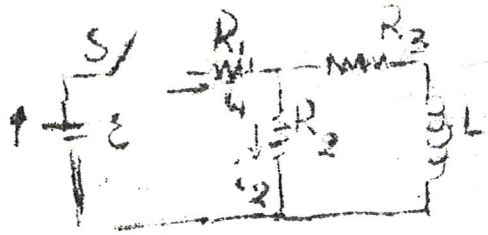
$$Q = \int_{\Phi_1}^{\Phi_2} \frac{N}{R} d\Phi_B$$

$$= \frac{N}{R}\Phi_B \Big|_{\Phi_1}^{\Phi_2}$$

or,

$$Q = \frac{N(\Phi_2 - \Phi_1)}{R}$$

11. In the circuit shown below were = 100 volts, R_1 = 10 Ω, R_2 20 Ω R_3 = 30 Ω, and L = 2H, find the values of i_1 and i_2 a) immediately after S is closed b) a long time later c) immediately after S is opened d) a long time later.



a)

When S is closed Kirchhoff's second rule for the left and right loops gives

$$\mathcal{E} = R_1 i_1 + R_2 i_2$$

$$\mathcal{E} = R_1 i_1 + i_L R_3 + L\frac{di_L}{dt}$$

Also,

$$i_1 = i_2 + i_L$$

$$\Rightarrow \mathcal{E} = R_1(i_2 + i_L) + R_2 i_2$$

$$= (R_1 + R_2) i_2 + R_1 i_L$$

$$i_2 = \frac{\mathcal{E} - R_1 i_L}{R_1 + R_2}$$

Then,

$$\mathcal{E} = R_1\left(\frac{\mathcal{E} - R_1 i_L}{R_1 + R_2} + i_L\right) + R_3 i_L + L\frac{di_L}{dt}$$

$$\frac{R_2 \mathcal{E}}{R_1 + R_2} = \left(\frac{R_1 R_2}{R_1 + R_2} + \frac{R_1 R_3}{R_1 + R_2} + \frac{R_2 R_3}{R_1 + R_2}\right) i_L + L\frac{di_L}{dt}$$

We find

$$\mathcal{E}' = \frac{R_2}{R_1 + R_2}\mathcal{E}, \quad R' = \frac{R_1 R_2 + R_1 R_3 + R_2 R_3}{R_1 + R_2}$$

Whence

$$\mathcal{E}' = R' i_L + L\frac{di_L}{dt}.$$

Separating variables and integrating we obtain

$$i_L = \frac{\mathcal{E}'}{R'} + C e^{-\frac{R'}{L}t}$$

But $i_L(0) = 0 \Rightarrow C = -\frac{\mathcal{E}'}{R'}$.

or,

$$i_L = \frac{\mathcal{E}'}{R'}\left(1 - e^{-\frac{R'}{L}t}\right).$$

Also,

$$i_2 = \frac{\mathcal{E} - R_1 i_L}{R_1 + R_2}$$

and,

$$i_1 = i_2 + i_L = \frac{\mathcal{E} + R_2 i_L}{R_1 + R_2}.$$

Immediately after S is closed $i_L = 0$ so that

$$i_1 = i_2 = \frac{100\,V}{10\,\Omega + 20\,\Omega} = 3.33\,A.$$

b,

$$i_1 = \frac{\mathcal{E} + R_2(\mathcal{E}'/R')}{R_1 + R_2}$$

$$= \frac{\mathcal{E} + R_2(R_2\mathcal{E}/(R_1R_2 + R_1R_3 + R_2R_3))}{R_1 + R_2}$$

$$= \frac{(R_1R_2 + R_1R_3 + R_2R_3 + R_2^2)\mathcal{E}}{(R_1 + R_2)(R_1R_2 + R_1R_3 + R_2R_3)}$$

$$= \frac{R_2\mathcal{E}}{R_1R_2 + R_1R_3 + R_2R_3} + \frac{R_3\mathcal{E}}{R_1R_2 + R_1R_3 + R_2R_3}$$

$$= \frac{(R_2 + R_3)\mathcal{E}}{R_1R_2 + R_1R_3 + R_2R_3}$$

$$= \frac{20 + 30}{10\times20 + 10\times30 + 20\times30} \times 100\,A$$

$$= \frac{50}{11}\,A = 4.55\,A.$$

Similarly,

$$i_2 = \frac{R_3\mathcal{E}}{R_1R_2 + R_1R_3 + R_2R_3}$$

$$= \frac{30 \times 100}{10\times20 + 10\times30 + 20\times30}\,A$$

$$= 2.73\,A.$$

c, When S is opened current appears only in the right loop and Kirchhoff's loop rule becomes

$$L\frac{di_L}{dt} + (R_2 + R_3)i_L = 0.$$

Separating variables and solving for i_L we obtain

$$i_L(t) = C' e^{-\frac{R_2 + R_3}{L}t}$$

But $i_L(0) = \mathcal{E}'/R'$, the peak value of the current that appears when S is closed. Thus,

$$i_L(t) = \frac{\mathcal{E}'}{R'} e^{-\frac{R_2 + R_3}{L}t}.$$

The moment when S is opened ($t = 0$ for this case) $i_1 = 0$,

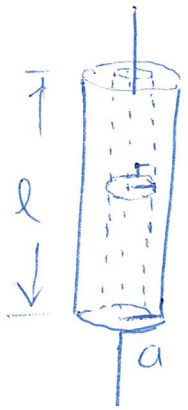
$$i_2 = i_L(0) = \frac{\mathcal{E}'}{R'} = \frac{R_2\mathcal{E}}{R_1R_2 + R_1R_3 + R_2R_3}$$

or,

$$i_2 = \frac{20 \times 100}{10\times20 + 10\times30 + 20\times30}\,A = \frac{20}{11}\,A = 1.82\,A.$$

d, $t \to \infty$, $i_1 = 0$ as before and $i_2 = 0$, since its dependence on time decreases exponentially.

12. Show that the self - inductance for a length l of a long wire associated with the flux inside the wire is 1/8π, independent of the diameter. Assume a uniform current density.



$\int \vec{B} \cdot d\vec{l} = \mu_0 i'$, i' the current enclosed by the loop of radius r.

$B 2\pi r = \mu_0 i'$

or, $B = \dfrac{\mu_0 i'}{2\pi r}$

But if the total current flowing through the wire is i,

$i' = i \dfrac{\pi r^2}{\pi a^2}$, a the radius of the wire.

i.e.,

$B = \dfrac{\mu_0 i r}{2\pi a^2}$

Now,

$u_B = \dfrac{B^2}{2\mu_0} = \dfrac{\mu_0 i^2 r^2}{8\pi^2 a^4}$

whereas the ~~total~~ energy dU_B stored in the cylindrical portion of thickness dr is

$dU_B = u_B dV = u_B (2\pi r dr) l$

$= \dfrac{\mu_0 i^2 r^2}{8\pi^2 a^4} \cdot 2\pi r l \, dr$

$= \dfrac{\mu_0 i^2 l r^3}{4\pi a^4} dr$

or,

$U_B = \dfrac{\mu_0 i^2 l}{4\pi a^4} \int_0^a r^3 dr$

$= \dfrac{\mu_0 i^2 l}{16\pi}$

Thus,

$\frac{1}{2} L i^2 = \dfrac{\mu_0 i^2 l}{16\pi}$

i.e.,

$L = \dfrac{\mu_0 l}{8\pi}$.

2. A long wire carries a current of uniform density. Let i be the total current carried by the wire and show that the magnetic energy per unit length stored within the wire equals $\mu_0 i^2 / 16\pi$. What will be the inductance of a length l of such a wire?

I

1. Suppose it is two. Let us then consider two unequal forces $\bar{a}$ and $\bar{b}$ with these vectors making angles α and β with the x-axis resp.

Their sum has comps., say

$$C_x = a\cos\alpha + b\cos\beta$$

$$C_y = a\sin\alpha + b\sin\beta$$

But $C_x = 0$ and $C_y = 0$ i.e.,

$$b\cos\beta = -a\cos\alpha$$
$$b\sin\beta = -a\sin\alpha \quad \} \ldots (1)$$

Also $C_x^2 + C_y^2$

$$= (a\cos\alpha + b\cos\beta)^2$$
$$+ (a\sin\alpha + b\sin\beta)^2 = 0$$

$$\Rightarrow a^2 + b^2 + 2ab\cos\alpha\cos\beta + 2ab\sin\alpha\sin\beta = 0$$

Using (1),

$$a^2 + b^2 - 2a^2\cos^2\alpha - 2a^2\sin^2\alpha = 0$$

$$a^2 + b^2 - 2a^2 = 0$$

$$b^2 - a^2 = 0$$

i.e., $\boxed{b = a}$ That means $\bar{a}$ and $\bar{b}$ have to be vectors of the same magnitude.

We then get (from 1)

$$\cos\beta = -\cos\alpha = \cos(\alpha + \pi)$$

$$\sin\beta = -\sin\alpha = \sin(\alpha + \pi)$$

$$\Rightarrow \boxed{\beta = \alpha + \pi}$$

i.e., the vectors have to be oppositely directed.

So if the sum of two vectors is zero, they must have the same magnitude and be oppositely directed.

For three vectors $\bar{a}, \bar{b}$ & $\bar{c}$ Let $\bar{d}$ be the sum

$$d_x = a\cos\alpha + b\cos\beta + c\cos\gamma = 0$$

$$d_y = a\sin\alpha + b\sin\beta + c\sin\gamma = 0 \quad \ldots\ldots (1)$$

Also, $(d_x^2 + d_y^2) = 0 \quad \ldots (2)$

These two eqs give if the sum of three vectors is zero, then

$$d\,\bar{c}^2 = (\bar{a} + \bar{b})^2$$

i.e., the vectors are unequal and thus the min. no. of unequal vectors whose sum is zero is three.

2. a, If they are ~~oppositely directed~~ parallel the sum is ~~zero~~ two.

b, Let the two vectors make angles α & β with the x-axis resp. Now if they add to ~~zero~~ give a vector of mag. 2, we have

~~$4\cos\alpha + 2\cos\beta = 0$~~

~~$4\sin\alpha + 2\sin\beta = 0$~~

~~and~~ $(2\cos\alpha + 2\cos\beta)^2$

$+ (\sin\alpha + 2\sin\beta)^2 = 4$

$1 + 4 + 4\cos\alpha\cos\beta$

$+ 4\sin\alpha\sin\beta = 4$

$1 + 4\cos(\alpha - \beta) = 0$

$\cos(\alpha - \beta) = -1/4 = -0.25$

$\Rightarrow \alpha - \beta \approx 104^\circ 5$

$\alpha = \beta + 104^\circ 5$

so if the angle between the two vectors is about 104° they will add to give a vector of magnitude 2.

c) Let the angles these vectors make with the x-axis γ_1 and γ_2 resp. So

$$(\cos\gamma_1 + 3\cos\gamma_2)^2 + (\sin\gamma_1 + 3\sin\gamma_2)^2 = 4$$

$$\cos^2\gamma_1 + 9\cos^2\gamma_2 + 6\cos\gamma_1\cos\gamma_2 + \sin^2\gamma_1 + 9\sin^2\gamma_2 + 6\sin\gamma_1\sin\gamma_2 = 4$$

$$1 + 9 + 6\cos(\gamma_1 - \gamma_2) = 4$$

$$\text{or } \cos(\gamma_1 - \gamma_2) = -1$$

$$\Rightarrow \gamma_1 - \gamma_2 = \pi$$

$$\text{or } \gamma_1 = \gamma_2 + \pi$$

i.e., their sum is 2 if these vectors are anti parallel.

d)

vector	angle
1	α_1
4	γ_2

$$(\cos\alpha_1 + 4\cos\gamma_2)^2 + (\sin\gamma_1 + 4\sin\gamma_2)^2 = 4$$

$$\Rightarrow 1 + 16 + 8\cos(\gamma_1 - \gamma_2) = 4$$

$$\text{or } \cos(\gamma_1 - \gamma_2) = -\frac{13}{8} \leq -1$$

So the 1m & 4m vectors cannot be added to give a vector of magnitude 2m.

3. c) Let these vectors make angles α, β & γ with the x-axis. If they add up to zero, then

$$10\cos\alpha + 10\cos\beta + 10\cos\gamma = 0$$
$$10\sin\alpha + 10\sin\beta + 10\sin\gamma = 0$$

$$\Rightarrow \left.\begin{aligned}\cos\alpha + \cos\beta + \cos\gamma &= 0\\ \sin\alpha + \sin\beta + \sin\gamma &= 0\end{aligned}\right\} \ldots 1$$

$$100\cdot(\cos\alpha + \cos\beta + \cos\gamma)^2 + 100\cdot(\sin\alpha + \sin\beta + \sin\gamma)^2 = 0$$

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + 2\cos\alpha\cos\beta + 2\cos\alpha\cos\gamma + 2\cos\beta\cos\gamma + \sin^2\alpha + \sin^2\beta + \sin^2\gamma + 2\sin\alpha\sin\beta + 2\sin\alpha\sin\gamma + 2\sin\beta\cdot\sin\gamma = 0$$

using (1) and simplifying,

$$3 - 2\cos\alpha(\cos\alpha + \cos\beta) + 2\cos\alpha\cos\beta - 2\cos\beta(\cos\alpha + \cos\beta) + 2\sin\alpha\sin\beta - 2\sin\alpha(\sin\alpha + \sin\beta) - 2\sin\beta(\sin\alpha + \sin\beta) = 0$$

$$3 - 2 - 2 - 2\cos(\alpha - \beta) = 0$$

$$\text{or } \cos(\alpha - \beta) = -\tfrac{1}{2}$$

i.e., $\alpha - \beta = 2\pi/3$

$$\therefore \beta = \alpha - 2\pi/3$$

$$\cos\gamma = -\cos\alpha - \cos\beta = -\tfrac{1}{2}(\cos\alpha + \sqrt{3}\sin\alpha)$$

$$\sin\gamma = -\sin\alpha - \sin\beta = \tfrac{1}{2}(\sqrt{3}\cos\alpha - \sin\alpha)$$

$$\therefore \tan\gamma = \frac{\tan\alpha - \sqrt{3}}{\sqrt{3}\tan\alpha + 1}$$

zero

So these vectors add^ for the above values of α, β & γ. Infact α is arbitrary. For example take $\alpha = 0$, then $\beta = -2\pi/3$, $\gamma = 2\pi/3 = -\frac{\pi}{3}$

Let us check if indeed the sum is zero.

~~X~~- comp.

$$10\cos 0 + 10\cos(-2\pi/3) + 10\cos(2\pi/3)$$
$$= 10 + 10(-1/2) + 10(-1/2)$$
$$= 0$$

y- comp.

$$10\sin 0^\circ + 10\sin(-2\pi/3) + 10\sin(2\pi/3)$$
$$= 0 - 10\sin 2\pi/3 + 10\sin 2\pi/3$$
$$= 0$$

∴ 10, 10, 10 add up to zero

b, If the two magnitude 10 vectors are directed opposite to the magnitude 20 vector, the sum is zero.

c, Again let the angles of the three vectors be α, β & γ resp. Then

$$10\cos\alpha + 20\cos\beta + 2\cos\gamma = 0$$
$$10\sin\alpha + 20\sin\beta + 20\sin\gamma = 0$$

or $\cos\alpha + 2\cos\beta + 2\cos\gamma = 0$
$\sin\alpha + 2\sin\beta + 2\sin\gamma = 0$
$\ldots\ldots\ldots\ldots (1)$

Also

$$(\cos\alpha + 2\cos\beta + 2\cos\gamma)^2 + (\sin\alpha + 2\sin\beta + 2\sin\gamma)^2 = 0$$

$$\cos^2\alpha + 4\cos^2\beta + 4\cos^2\gamma + 4\cos\alpha\cos\beta + 4\cos\alpha\cos\gamma + 8\cos\beta\cos\gamma + \sin^2\alpha + 4\sin^2\beta + 4\sin^2\gamma + 4\sin\alpha\sin\beta + 4\sin\alpha\sin\gamma + 8\sin\beta\sin\gamma = 0$$

Using (1) and simplifying

$$1 + 4 + 4 + \cancel{4\cos\alpha\cos\beta} + 4\cos\alpha\cos\beta - 4\cos\alpha(\cos\alpha + 2\cos\beta) - 8\cos\beta(\cos\alpha + 2\cos\beta) + 4\sin\alpha\sin\beta - 4\sin\alpha(\sin\alpha + 2\sin\beta) - 8\sin\beta(\sin\alpha + 2\sin\beta) = 0$$

$$9 - 4\cos\alpha\cos\beta - 4\cos^2\alpha - 8\cos\alpha\cos\beta - 8\cos\alpha\cos\beta - 16\cos^2\beta + 4\sin\alpha\sin\beta - 4\sin^2\alpha - 8\sin\alpha\sin\beta - 8\sin\alpha\sin\beta - 16\sin^2\beta = 0$$

$$9 - 4\cos(\alpha-\beta) - 4 - 16\cos(\alpha-\beta) - 16 = 0$$
$$-11 - 20\cos(\alpha-\beta) = 0$$
$$\cos(\alpha-\beta) = -11/20$$
$$\Rightarrow \alpha - \beta \approx 2\pi/3$$
$$\beta = \alpha - 2\pi/3$$

Next we find γ using (1) and we can really see that for arbitrary α the three vectors add up to zero.

d, The angles are in (c) are α, β & γ resp.

$10\cos\alpha + 20\cos\beta + 40\cos\gamma = 0$

$10\sin\alpha + 20\sin\beta + 40\sin\gamma = 0$

or $\cos\alpha + 2\cos\beta + 4\cos\gamma = 0$

$\sin\alpha + 2\sin\beta + 4\sin\gamma = 0$ ... (1)

$(\cos\alpha + 2\cos\beta + 4\cos\gamma)^2$

$+ (\sin\alpha + 2\sin\beta + 4\sin\gamma)^2 = 0$

$1 + 4 + 16$

$+ 4\cos\alpha\cos\beta + 8\cos\alpha\cos\gamma$

$+ 16\cos\beta\cos\gamma$

$+ 4\sin\alpha\sin\beta + 8\sin\alpha\sin\gamma$

$+ 16\sin\beta\sin\gamma = 0$

Using (1)

$21 + 4\cos\alpha\cos\beta$

$+ 8\cos\alpha(\cos\alpha + 2\cos\beta)$

$- 16\cos\beta(\cos\alpha + 2\cos\beta)$

$+ 4\sin\alpha\sin\beta$

$- 8\sin\alpha(\sin\alpha + 2\sin\beta)$

$- 16\sin\beta(\sin\alpha + 2\sin\beta) = 0$

Simplifying,

$21 + 4\cos(\alpha-\beta) - 8$

$- 16\cos(\alpha-\beta) - 32$

$- 16\cos(\alpha-\beta) = 0$

$-19 - 28\cos(\alpha-\beta) = 0$

$\cos(\alpha-\beta) = -\frac{19}{28}$

We continue to find γ and for arbitrary α the sum will be zero.

4. At an angle of π i.e., when the vectors are antiparallel.

5. No, the result in prob. 1.

6. No. The magnitude is given by the sum of the squares of the comps. which is diff. from zero for nonzero comps.

II solution to problems

1. Let the magnitude of the unknown comp. be a. Then

$40^2 = 25^2 + a^2$

$\Rightarrow a^2 = 40^2 - 25^2$

$= (8\times5)^2 - (5\times5)^2$

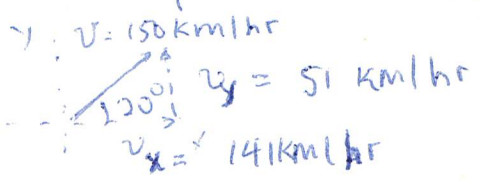
$= (8^2 - 5^2)5^2$

$= 39 \cdot 5^2$

i.e., $a = 5\sqrt{39}$

2. These forces are the comps. of the 10 N force along the x- & y-directions. So that the horizontal comp. is $(10\cos40^\circ)$ N whereas the vertical comp. is $(10\sin40^\circ)$ N

3.



$v_x = 150\ \text{km/hr}\cos20^\circ =$

Thus

$y(15) = (51\ \text{km/hr})\frac{1}{60}\ \text{hr}$

$= 0.85\ \text{km}$

$x(15) = (141\ \text{km/hr})\frac{1}{60}\ \text{hr}$

$= 2.35\ \text{km}$

4. The angle that the velocity

$\vec{V}_1 = (i + 2j)$ m/s

makes with the x-axis is

$\theta = \tan^{-1}(2) = 63.4°$.

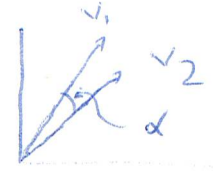
Similarly,

$\vec{V}_2 = (2i + 3j)$ m/s

makes an angle of

$\theta = 56.3°$

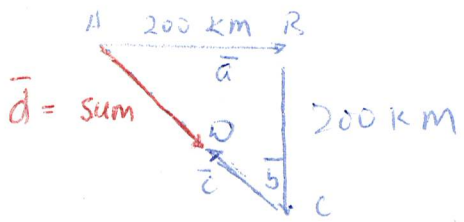
with the x-axis.



so the angle between them is $\alpha = 63.4° - 56.3°$

$= 8.1°$

5.



To find the dist. from A to O we add the three vectors.

vector	x-comp.	y-comp.
$\vec{a}$	200	0
$\vec{b}$	0	-200
$\vec{c}$	$\frac{-100}{\sqrt{2}}$	$\frac{100}{\sqrt{2}}$
$\vec{d}$	$200 - \frac{100}{\sqrt{2}}$	$\frac{100}{\sqrt{2}} - 200$

$d^2 = (129.3)^2 + (-129.3)^2$ km^2

or $d = 182.9$ km is the dist.

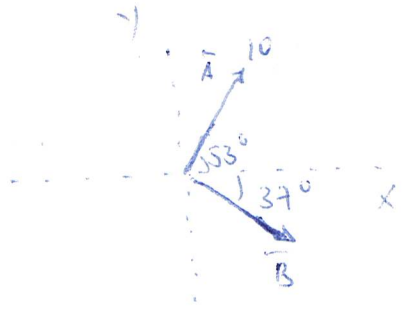
$(\vec{A}-\vec{B})_x = -2$cm, $(\vec{A}-\vec{B})_y = 14$cm, $|\vec{A}-\vec{B}| = 14.14$cm, $\theta = +81.9°$ north of west

$(\vec{B}-\vec{A})_x = 2$cm, $(\vec{B}-\vec{A})_y = -14$cm, $|\vec{B}-\vec{A}| = 14.14$cm, $\theta = -81.9°$,

from A to D.

The airplane has to head northwest to return to A from D.

6.



$\vec{A}$ & $\vec{B}$ are then perpendicular to each other.

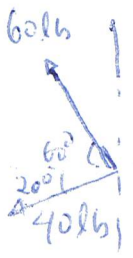
vector	x-comp.	y-comp.
$\vec{A}$	6cm	8cm
$\vec{B}$	8cm	-6cm
Res. $\vec{R}$	14cm	2cm

$R = \sqrt{14^2 + 2^2}$ cm $= 14.14$ cm

$\theta = \tan^{-1}(2/14) = 8.1°$

counterclockwise from the + X direction.

7.



Let the comps of the third vector be C_x and C_y. Then

$\Sigma F_x = 0 \Rightarrow -60 \cdot \frac{1}{2} - 40 \times 0.94 + C_x = 0$

i.e., $C_x = 67.6$ lb

$\Sigma F_y = 0 \Rightarrow 60 \cdot \frac{\sqrt{3}}{2} - 40 \times 0.34 + C_y = 0$

i.e., $C_y = 38.4$ lb

The magnitude of the force is

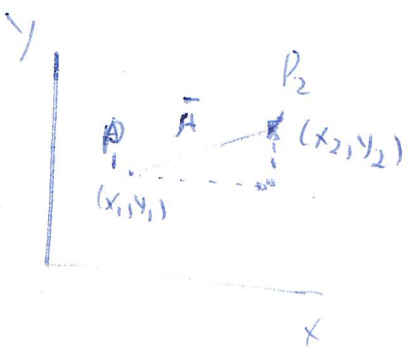
$$C = \sqrt{C_x^2 + C_y^2}$$

$$= \sqrt{67.6^2 + 38.4^2}\ lb = 77.7\ lb$$

The angle it makes with the x-axis is

$$\theta = \tan^{-1}\left(\frac{38.4}{67.6}\right) = 29.6^\circ$$

8.



As seen from the figure.

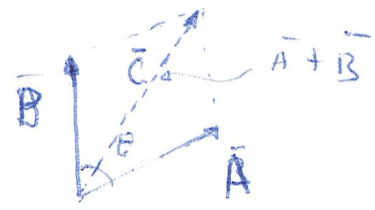
$$A_x = x_2 - x_1$$

$$A_y = y_2 - y_1$$

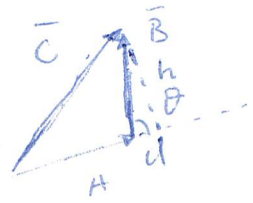
$$A = \sqrt{A_x^2 + A_y^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$\theta = \tan^{-1}\left(\frac{y_2 - y_1}{x_2 - x_1}\right)$ is the angle made with the x-axis.

9.



This can be represented as



$$C^2 = (A + d)^2 + h^2$$

or

$$C^2 = (A + B\cos\theta)^2 + B^2\sin^2\theta$$

$$= A^2 + B^2\cos^2\theta + 2AB\cos\theta + B^2\sin^2\theta$$

$$= A^2 + B^2 + 2AB\cos\theta$$

i.e., $$C = \sqrt{A^2 + B^2 + 2AB\cos\theta}$$

10. a, For this apply a suitable scale

b,

vector	x-comp.	y-comp.
$\bar{A}$	$-300 \cdot \frac{1}{\sqrt{2}}$	$300 \cdot \frac{1}{\sqrt{2}}$
$\bar{B}$	$200 \cdot \frac{\sqrt{3}}{2}$	$200 \cdot \frac{1}{2}$
$\bar{C}$	$-155 \cdot \frac{1}{\sqrt{2}}$	$-155 \cdot \frac{1}{\sqrt{2}}$
Resultant, $\bar{R}$	-148.52	202.53

$$R = 251.15\ N$$

Angle with the x-axis

$$\theta = \tan^{-1}\left(-\frac{202.53}{148.52}\right) = 53.75^\circ$$

north of west.

c, Apply the polygon method for two vectors

d, Apply the Parall. M.

e, Apply the comp. M.

11. a, $\bar{A} \cdot \bar{B} = AB\cos\theta$

θ angle between $\bar{A}$ & $\bar{B}$.

$\Rightarrow \bar{i}\cdot\bar{i} = 1\cdot 1\cos 0^\circ = 1$

This can be because we are dotting a unit vector with itself and $\theta = 0^\circ$.

similarly,

$\bar{j}\cdot\bar{j} = \bar{k}\cdot\bar{k} = 1.$

b, $\bar{i}$, $\bar{j}$ and $\bar{k}$ are $\perp$ to each other. Thus

$\bar{i}\cdot\bar{j} = 1\cdot 1\cos 90^\circ = 0.$

$\bar{i}\cdot\bar{k} = 1\cdot 1\cos 90^\circ = 0$

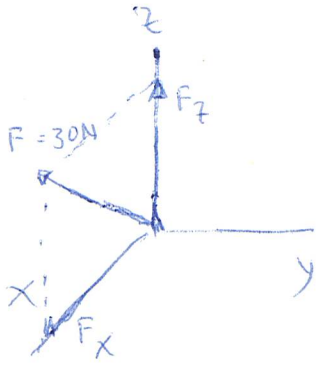
$\bar{j}\cdot\bar{k} = 1\cdot 1\cos 90^\circ = 0$

c, $\bar{A}\cdot\bar{B} = (A_x\bar{i} + A_y\bar{j} + A_z\bar{k}) \cdot (B_x\bar{i} + B_y\bar{j} + B_z\bar{k})$

$= A_xB_x + A_yB_y + A_zB_z$

by (b).

12.



a, $F_x = 30N\cdot\cos 45^\circ = 21.2N$

$F_y = 30N\cdot\cos 90^\circ = 0$

$F_z = 30N\cos 45^\circ = 21.2N$

b, $\bar{F} = (21.2)\bar{i} + (21.2)\bar{k}$ in Newton.

c, $\bar{F}_x\cdot\bar{F}_z$

$= F_x\cdot F_z\cos 90^\circ = 0$

d, $\bar{F}\cdot\bar{F}_x = F\cdot F_x\cos 45^\circ$

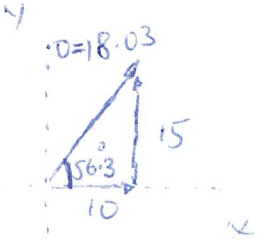
$= 30\times 21.2\times\frac{1}{\sqrt{2}}$

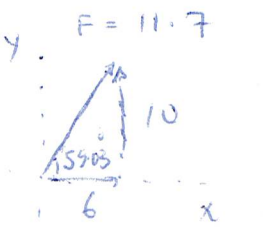
$= 450\ N$

13. a, $\bar{v} = 10\bar{i} + 15\bar{j}$

$\bar{F} = 6\bar{i} + 10\bar{j}$

b,





14. Scalar multiplication is dot product whereas multiplication of a vector by a scalar is the product of a scalar and a vector.

1- Technische Universität Berlin:
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Tel.: 030/314-0; Telefax: 030/2548 02-31
Telex: 184262

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August-schmidt-Straße 4, 4600 Dortmund
Tel.: 0231/755-1; Telefax: 0231/75-1532; Telex
922465

3. Johann Wolfgang Goethe-Universität Frankfurt am Main:
Senckenberganlage 31, 6000 Frankfurt am Main
Tel.: 069/798-1; Telefax: 069/798-8383, 069/798-8530 (präs.); Telex: 413932

4. Universität zu Köln
Albertus-Magnus-Platz, 5000 Köln 41,
Tel.: 0221/470-1; Telefax: 0221/470-5151;
Telex: 8882291

5. Ludwigs-Maximilians-Universität München
Geschwister-Scholl-Platz 1, 800 München 22
Tel.: 089/2180-0, Telefax: 089/33 8297;
Telex: 529860 Unium

Deadline to the German Embass November
Send 88. transcript to the embassy.

(2)

1. Director of Student Awards
University of Alberta
Edmonton, Alberta T6G 2E1
Canada.

(3)

2. University of British Columbia
Faculty of Graduate Studies
235 - 2075 Wesbrook Mall
Vancouver, British Columbia V6T 1Z3
Canada

3. University Arkansas
Fayetteville, Arkansas 72701
USA

4. New Mexico Institute of Mining and Technology
Socorro, NM 87801
USA

5. University of Missouri - Kansas City
5100 Rockhill Road
Kansas City, Mo 64110
USA

6. Director of Admissions
Lafayette College
Easton, Pennsylvania 18042
USA

7. Office of Admissions and Financial Aid
American University
Washington, DC 20016
USA

8. Department of Physics
Clemson University
Clemson, South Carolina 29631
USA

9. Office of Graduate Admissions
Stanford University
Stanford, California 94305

10. Foreign Graduate Admissions
Ohio State university
Columbus, Ohio 43210

11. Department of Physics
Southern Illinois University at Carbondale
Carbondale, Illinois 62901
USA

12. Massachusetts Institute of Technology
Office of Admissions
Room 3-103
Cambridge, Massachusetts 02139

13 College of arts and science
Department of Physics and Astronomy
University of Delaware
Newark, Delaware 19716

14 UCLA
Department of Physics
405 Hilgard Avenue
Los Angeles, CA 90024

27 Nov. 1990

1. Cornell University
Graduate school, Sage Graduate Center
Ithaca, New York 14853-6201
USA

2. The Ohio State University
No Grad. Programme
Admissions office, Lincoln Tower
1800 Cannon Drive
Columbus, OH 43210-1200

3. Graduate Admissions office
Lederle Graduate Research Center
University of Massachusetts
Amherst, MA 01003

7. Admissions office, The Graduate school
2107 Lee Building
The university of Maryland, college par
College park, Maryland 20742 USA.

5. The University of Connecticut
Storrs, connecticut 06269, USA.

6. Department of physics Clemson university
Clemson, South Carolina 29631, USA.

7. Department of physics, Southern Illinois University
at Carbondale, Carbondale Illinois 62901, USA.

8. University of Missouri, Kansas city
No Grad.
5100 Rochill Road, Kansas city, MO 64110, USA

9. Office of Admissions and financial aicle
American university, Washington DC 20016, USA.

10. University of Arkansas, Fayetteville, Arkansas 72701 USA.

11. Director of Students Awards. University of Alberta Edmonton, Alberta, T6G 2E1, Canada.

12. University of British Columbia, Faculty of Graduate Studies 235-2075, Wesbrook Mall Vancouver, British Columbia V6T 1Z3

Testing Centers Address

1. Graduate Record Examination
Educational Testing Services, CN6004
Princeton, New Jersey 08540 USA

USA

I Information services:

1. Institute of International Education (IIE), 809 United Nations Plaza, New York, N.Y. 10017 (about study facilities, financial and other assistance available).

II Universities:

1. Colorado State University
   Fort Collins, Colorado 80523 (not much promising, time)
2. Purdue University
   West Lafayette, Indiana 47907
3. State University of New York
   Central office of International programs, 99 Washington Avenue, Albany, New York 12246
   Application: by 1 February for semester beginning 1 Sept.
4. Syracuse University
   Syracuse, New York 13210 (limited time)
5. United States Agency for Information and Cultural Exchange (USAICE), Washington D.C. 20520
6. University of Illinois, Urbana, Illinois 61801
   Application: by 1 March.
7. University of Mississippi
   University, Mississippi 38677
   Application: by April
8. University of Washington
   Seattle, Washington 98105
9. University of Wisconsin
   River Falls, Wisconsin 54022
10. Virginia Polytechnic Institute and State University
    and 105 Patton Hall Blacksburg, Virginia 24061
    Application: 15 Feb. to the Dean, Graduate school, at the above address.
11. Washington University (private)
    St. Louis, Missouri 63130

1. Addresses

Home : Biru Tregaye, p.o. Box 31892, Addis Ababa, Ethiopia

2. Institution : Biru Tregaye, Kotebe College of Teacher Education, p.o. Box 31248.

- phone : 61 15 55

3. Person to notify : Telele Tregaye, Gondar agricultural office, p.o Box 80, Ethiopia.

- phone : 110154, 110649
- Relationship : Brother

4. Education

From	to	Name, city, ...	Degree
1981	1984	Addis Ababa University Addis Ababa, Ethiopia	B.Sc.
1987	1988	Addis Ababa University Addis Ababa, Ethiopia	B.Sc.

- Courses

5. Solid State Physics I & II (phy. 451 & 452)
2. Classical Mechanics (phy. 581)
3. Quantum Mechanics I (phy 542)
4. Electrodynamics (phy. 505)
5. Advanced Lab. (phy. 501) on quantum mechanics, Nuclear physics, Solid State physics etc.
6. Statistical Mechanics (phy. 522)
7. Quantum Mechanics II (phys 641)
8. Solid State Physics I (phy. 651)

Text Books

1. Physics for Science and Engineering by Marion
2. Classical Mechanics by Herbert Goldstein
3. Introduction Fundamentals of Quantum Mechanics and Statistical physics by Alonso
4. Fundamentals of Electromagnetic Theory by Jon R. Reitz, Frederick J Milford, Robert W. Christy

13. Statistical Mechanics (A set of lectures) A state of matter by R. Kikuchi and H.A. Feiveson.

5. Classical Electrodynamics by John D. Jackson
6. Fundamentals of Statistical and thermal physics by F. Reif
7. Statistical physics by L.D. Landau and E.M. Lifshitz
8. Quantum Mechanics by Leonard D. Schiff
9. Principles of Quantum Mechanics by Shankar
10. Introduction to Solid State physics Charles Kittel
11. Quantum Theory of solids Charles Kittel

6. Title of Thesis
Path Integrals in Quantum and Statistical Mechanics

Feynman Path Integrals and Its Exact Results and Applications by D.C. Khandekar and S.V. Lawande

7. Seminars, summer schools ... - No

8.

	period of duty from	to	Acad. Resp.
Research inst. or univ.			
Kotake College of Teacher Education	1984	1992	Head of physics department.

9. I am interested to pursue my Ph.D. in the field of condensed matter physics and this course will help me to strengthen my background for that purpose.

10. English Lan. Prof.
Reading - good Writing - good Speaking - Average.

11. Financial Sup.
Local support and travel support

Date ____

University of ____
Department of Physics

Biru Tregaye
P.O.BOX 31392
Addis Ababa, Ethiopia

Dear Sir:

First of all I would like to forward my greetings to you. I am a lecturer at College, namely, Kotebe College of Teacher Education, in Addis Ababa, Ethiopia.

I learn that your department offers Teaching and/or graduate assistantships. So to pursue my Ph.D. in physics I am interested to apply to your department as a teaching assistant. I have registered to take the TOEFL and GRE tests in November and December, respectively. I hope you will receive the results of these examinations in time. I, therefore, kindly request you to send me application forms and brochure that will describe your graduate program

yours sincerely

Biru Tregaye.